1. Introduction

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THE GELFAND-NAIMARK THEOREMS FOR C*-ALGEBRAS

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1. INTRODUCTION

Many of the Banach spaces which attract attention are at the same time algebras under some multiplication. In spite of this fact their study from this richer point of view was taken up only after the publication in 1932 of Banach's book [6]. One of the early fundamental results in the general theory of Banach algebras was a generalization of the classical theorem of Frobenius that any finite dimensional division algebra over the complex field is isomorphic to the field of complex numbers. S. Mazur [35] announced in 1938 that every complex normed division algebra is isomorphic to the field of complex numbers. Since the first published proof was given by I. M. Gelfand [22] this result is often called the Mazur-Gelfand theorem [43], [55]. As an immediate consequence one obtains the following beautiful characterization of the complex field among normed algebras: any normed algebra satisfying the norm condition $\| xy \| = \| x \| \cdot \| y \|$ for all elements x and y is isometrically isomorphic to the field of complex numbers.

Many important Banach algebras carry a natural involution. In the case of an algebra of functions the involution is the operation of taking the complex-conjugate and in the case of an algebra of operators on a Hilbert space it is the operation of taking the adjoint operator. Motivated by these observations the Soviet mathematicians Israel M. Gelfand and Mark A. Naimark [23] proved, under some additional assumptions, the following two theorems:

THEOREM I. *Let* A *be a commutative Banach algebra with involution satisfying* $\| x^*x \| = \| x^* \| \cdot \| x \|$ *for all* x *in* A. *Then* A *is isometrically *-isomorphic to* $C_0(X)$, *the algebra of all continuous complex-valued functions which vanish at infinity on some locally compact Hausdorff space* X.

THEOREM II. *Let* A *be a Banach algebra with involution satisfying* $\| x^*x \| = \| x^* \| \cdot \| x \|$ *for all* x *in* A. *Then* A *is isometrically *-iso-*

*morphic to a norm-closed *-subalgebra of bounded linear operators on some Hilbert space.*

The purpose of this paper is to present a thorough discussion of these two representation theorems. We shall trace, as carefully as we have been able, the interesting and rather tangled history which led to their present form. Then proofs of the theorems will be given. Finally, we shall survey some recent developments inspired by the theorems.

2. Definitions and motivation

A **-algebra* is a complex associative linear algebra A with a mapping $x \to x^*$ of A into itself such that for all $x, y \in A$ and complex λ: (a) $x^{**} = x$; (b) $(\lambda x)^* = \bar{\lambda} x^*$; (c) $(x+y)^* = x^* + y^*$; and (d) $(xy)^* = y^* x^*$. The map $x \to x^*$ is called an *involution*; because of (a) it is clearly bijective. A subalgebra B of A is called a **-subalgebra* if $x \in B$ implies $x^* \in B$.

An algebra which is also a Banach space satisfying $\| xy \| \leqslant \| x \| \cdot \| y \|$ for all x and y is called a *Banach algebra.* A Banach algebra which is also a *-algebra is called a *Banach *-algebra.* The involution in a Banach *-algebra is said to be *continuous* if there is a constant M such that $\| x^* \| \leqslant M \| x \|$ for all x; the involution is *isometric* if $\| x^* \| = \| x \|$ for all x.

A norm on a *-algebra is said to satisfy the B**-condition* if $\| x^* x \| = \| x^* \| \cdot \| x \|$ for all x; a B**-algebra* is a Banach *-algebra whose norm satisfies the B*-condition. A B*-algebra with isometric involution clearly satisfies the condition $\| x^* x \| = \| x \|^2$. On the other hand, if A is a Banach *-algebra satisfying $\| x \|^2 \leqslant \| x^* x \|$ (in particular if equality holds), then A is easily seen to be a B*-algebra with isometric involution.

The Banach space $C(X)$ of continuous complex-valued functions on a compact Hausdorff space is a commutative B*-algebra under point-wise multiplication $(fg)(t) = f(t)\, g(t)$, involution $f^*(t) = \overline{f(t)}$, and sup-norm. Similarly, the algebra $C_0(X)$ of continuous complex-valued functions which vanish at infinity on a locally compact Hausdorff space is a commutative B*-algebra.

Examples of noncommutative B*-algebras are provided by the algebra $B(H)$ of bounded linear operators on a Hilbert space H. Multiplication in $B(H)$ is operator composition, the involution $T \to T^*$ is the usual adjoint operation, and the norm is the operator norm $\| T \| = \sup \{ \| T\xi \| : \| \xi \| \leqslant 1, \xi \in H \}$. A norm-closed *-subalgebra of $B(H)$ is called a C**-algebra*; clearly, every C*-algebra is a B*-algebra.