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$$\begin{array}{ccccc}
 C_\eta & \xleftarrow{\approx} & D_\eta & \xrightarrow{\Phi_\eta} & \mathbf{P}^N \times \operatorname{Spec} k((t)) \\
 \cap & & \cap & & \cap \\
 \mathcal{C} & \xleftarrow{\quad} & \mathcal{D} & \xrightarrow{\Phi} & \mathbf{P}^N \times \operatorname{Spec} k[[t]] \\
 \Phi_\eta^*(\mathcal{O}_{\mathbf{P}^N}(1)) & = & \omega_{D_\eta/k((t))}^{\otimes n} & .
 \end{array}$$

Let  $L = \mathcal{O}_{\mathcal{D}}(1)$ . It follows that  $L \cong \omega_{\mathcal{D}/k[[t]]}^{\otimes n} (-\sum r_i D_i)$ , where  $D_i$  are the components of  $D_0$ . Multiplying the isomorphism by  $t^{\min(r_i)}$ , we can assume  $r_i \geq 0$ ,  $\min r_i = 0$ . Let  $D_1 = \bigcup_{r_i=0} D_i$ ,  $D_2 = \bigcup_{r_i>0} D_i$ . If  $f$  is a local equation of  $\sum r_i D_i$ , then  $f \not\equiv 0$  in any component of  $D_1$  since  $r_i = 0$  on all these while  $f(x) = 0$ , all  $x \in D_1 \cap D_2$ , so

$$\#(D_1 \cap D_2) \leq \deg_{D_1}(\mathcal{O}_{\mathcal{D}}(\sum r_i D_i)).$$

But this last degree equals  $(\deg D_1 - n \deg_{D_1}(\omega_{D_0}))$  which contradicts iii) of Proposition 5.5 unless all  $r_i$  are zero. Hence  $L = \omega_{\mathcal{C}}^{\otimes n}$  which shows  $\mathcal{D} = \mathcal{C}$ .

## LINE BUNDLES ON THE MODULI SPACE

For the remainder of this section we examine  $\operatorname{Pic}(\bar{\mathcal{M}}_g)$ . We fix a genus  $g \geq 2$  and an  $e \geq 3$ . Then for all stable  $C$ ,  $\omega_C^{\otimes e}$  is very ample and in this embedding  $C$  has degree  $d = 2e(g-1)$ , the ambient space has dimension  $v-1$  where  $v = (2e-1)(g-1)$  and  $C$  has Hilbert polynomial  $P(X) = dX - (g-1)$ . Let  $H \subset \operatorname{Hilb}_{\mathbf{P}^{v-1}}^P$  be the locally closed smooth subscheme of  $e$ -canonical stable curves  $C$ , let  $C \subset H \times \mathbf{P}^{v-1}$  be the universal curve and let

$$\operatorname{ch} : H \rightarrow \operatorname{Div} = \operatorname{Div}^{d,d} = \left\{ \begin{array}{l} \text{projective space of bihomogeneous forms} \\ \text{of bidegree } (d, d) \text{ in dual coordinates} \\ u, v \text{ (cf. § 1).} \end{array} \right\}$$

be the Chow map. These are related by the diagram

$$\begin{array}{ccccc}
 & & C & & \\
 & & \downarrow \pi & & \\
 \operatorname{Div} & \xleftarrow{\operatorname{ch}} & H & \xrightarrow{\rho} & \bar{\mathcal{M}}_g = H/PGL(v)
 \end{array}$$

If  $\operatorname{Pic}(H, PGL(v))$  is the Picard group of invertible sheaves on  $H$  with  $PGL(v)$ -action, we have a diagram

$$\mathrm{Pic}(\bar{\mathcal{M}}_g) \xrightarrow{\rho^*} \mathrm{Pic}(H, \mathrm{PGL}(v)) \xrightarrow{\alpha} \mathrm{Pic}(H)^{\mathrm{PGL}(v)} \subset \mathrm{Pic}(H).$$

In this situation, we have:

LEMMA 5.8. *In the sequence above,  $\rho^*$  is injective with torsion cokernel and  $\alpha$  is an isomorphism.*

*Proof.*  $\alpha$  is an isomorphism by Prop. 1.4 [14];  $\rho^*$  injective is easy; coker  $\rho^*$  torsion can be proved, for instance, using Seshadri's construction, Th. 6.1 [19].

This lemma allows us to examine  $\mathrm{Pic}(\bar{\mathcal{M}}_g)$  by looking inside  $\mathrm{Pic}(H)^{\mathrm{PGL}(v)}$  which is a much easier group to come to grips with.

DEFINITION 5.9. *Let  $\Delta \subset H$  be the divisor of singular curves,  $\delta = \mathcal{O}_H(\Delta)$  and  $\lambda_n = \Lambda^{\max}(\pi_*(\omega_{C/H}^{\otimes n}))$ , ( $n \geq 1$ ). We write  $\lambda$  for  $\lambda_1$ .*

The sheaves  $\lambda_n$  and  $\delta$  are the most obviously interesting invertible sheaves on  $H$  from a moduli point of view. The next theorem expresses all of these in terms just involving  $\lambda$  and  $\delta$ .

$$\text{THEOREM 5.10. } \lambda_n = \mu^{\binom{n}{2}} \otimes \lambda \text{ where } \mu = \lambda^{12} \otimes \delta^{-1}.$$

*Proof.* The proof is based on Grothendieck's relative Riemann-Roch theorem (see Borel-Serre [4]), which we will briefly recall.

Let  $X$  and  $Y$  be complete smooth varieties over  $k$ ,  $A(X)$  be the Chow ring of  $X$  and  $\mathcal{F}$  be a coherent sheaf on  $X$ . Let  $c_i(\mathcal{F}) \in A(X)$  denote the  $i^{\text{th}}$  Chern class of  $\mathcal{F}$ ,  $\text{Chern}(\mathcal{F}) \in A(X) \otimes \mathbf{Q}$  its Chern character and  $\mathcal{T}(\mathcal{F}) \in A(X) \otimes \mathbf{Q}$  its Todd genus. These are related by:

$$(5.11) \quad \text{Chern}(\mathcal{F}) = rk \mathcal{F} + c_1(\mathcal{F}) + \frac{c_1(\mathcal{F})^2}{2} - c_2(\mathcal{F}) \\ + \text{terms of higher codimension,}$$

$$\mathcal{T}(\mathcal{F}) = 1 - \frac{c_1(\mathcal{F})}{2} + \frac{c_1(\mathcal{F})^2 + c_2(\mathcal{F})}{12} \\ + \text{terms of higher codimension.}$$

Let  $K(Y)$  be the Grothendieck group of  $Y$ ,  $f: X \rightarrow Y$  be a proper map, and  $f_!(\mathcal{F}) = \sum (-1)^i [\mathbf{R}^i f_* \mathcal{F}] \in K(Y)$ . The relative Riemann-Roch theorem expresses the Chern character of  $f_!(\mathcal{F})$ , modulo torsion as

$$\text{Chern}(f_! \mathcal{F}) = f_*(\text{Chern} \mathcal{F} \cdot \mathcal{T}(\Omega_{X/Y}^1))$$

which using (5.11) gives:

$$(5.12) \quad rk f_! \mathcal{F} + c_1(f_! \mathcal{F}) + \dots$$

$$= f_* \left[ \left( rk(\mathcal{F}) + c_1(\mathcal{F}) + \frac{c_1(\mathcal{F})^2}{2} - c_2(\mathcal{F}) \right) \cdot \left( 1 - \frac{c_1(\Omega_{X/Y}^1)}{2} + \frac{c_1(\Omega_{X/Y}^1)^2 + c_2(\Omega_{X/Y}^1)}{12} \right) \right]$$

For the time being, we work implicitly modulo torsion.

Now suppose  $\mathcal{F}$  is a line bundle such that  $R^i f_*(\mathcal{F}) = 0$ ,  $i > 0$  and suppose  $\dim X = \dim Y + 1$ . Then the codimension 1 term on the left of (5.12) (i.e. on  $Y$ ) corresponds to the codimension two term on the right (i.e. on  $X$ ). Since  $c_2(\mathcal{F}) = 0$ , this gives

$$(5.13) \quad c_1(f_* \mathcal{F}) = c_1(f_! \mathcal{F})$$

$$= f_* \left[ \frac{c_1(\Omega_{X/Y}^1)^2 + c_2(\Omega_{X/Y}^1)}{12} - \frac{c_1(\mathcal{F})c_1(\Omega_{X/Y}^1)}{2} + \frac{c_1(\mathcal{F})^2}{2} \right]$$

In case  $f: C \rightarrow S$  is a moduli-stable curve over  $S$ ,  $X = C$  and  $Y = S$ , we can simplify this. Indeed I claim that if  $\text{Sing } C$  is the singular set on  $C$  and  $I_{\text{sing}}$  is its ideal, then

- i)  $\text{codim } \text{Sing } C = 2$
- ii) the canonical homomorphism  $\Omega_{C/S}^1 \rightarrow \omega_{C/S}$  induces an isomorphism  $\Omega_{C/S}^1 = I_{\text{sing}} \cdot \omega_{C/S}$ .

We certainly have the isomorphism of ii) off  $\text{Sing } C$ . At a singular point  $C$  has a local equation of the form  $xy = t^n$ , where  $t$  is a parameter on  $S$ ,  $x$  and  $y$  are affine coordinates on the fibre. Moreover locally  $C$  is singular only at the points  $(0, 0)$  in the fibres where  $t = 0$ , so  $\text{Sing } C$  has codimension 2. Near the singular point

$$\Omega_{C/S}^1 = (\mathcal{O}_C dx + \mathcal{O}_C dy) / (x dy + y dx) \mathcal{O}_C$$

while  $\omega_{C/S}$  is the invertible sheaf generated by the differential  $\zeta$  which is given by  $dx/x$  outside  $x = 0$  and by  $-dy/y$  outside  $y = 0$ . Thus

$$\Omega_{C/S}^1 = \mathcal{M}_{(0,0),C} \cdot \zeta = \mathcal{M}_{(0,0),C} \cdot \omega_{C/S}.$$

Recall the following corollary to Riemann-Roch: if  $X$  is a smooth variety,  $Y \subset X$  a subvariety of codim  $r$  and  $\mathcal{F}$  is coherent on  $Y$ , then considering  $\mathcal{F}$  as a sheaf on  $X$



$$c_i(\mathcal{F}) = \begin{cases} 0, & 1 \leq i \leq r-1 \\ ((-1)^{r-1} (r-1)! rk \mathcal{F}) Y, & i = r \end{cases}$$

Set  $X = C$ ,  $Y = \text{Sing } C$  and  $\mathcal{F} = \Omega_{C/S}^1$ . The Whitney product formula applied to the chern classes of the exact sequence

$$0 \rightarrow \Omega_{C/S}^1 \rightarrow \omega_{C/S} \rightarrow \omega_{C/S} \otimes \mathcal{O}_{\text{Sing } C} \rightarrow 0$$

gives, taking account of the corollary

$$\begin{aligned} 1 + c_1(\omega_{C/S}) \\ = (1 + c_1(\Omega_{C/S}^1) + c_2(\Omega_{C/S}^1) + \dots) \cdot (1 + 0 - [\text{Sing } C] + \dots) \end{aligned}$$

Equating terms of equal codimension, we see that  $c_1(\Omega_{C/S}^1) = c_1(\omega)$  and  $c_2(\Omega_{C/S}^1) = [\text{Sing } C]$  so that (5.13) becomes

$$c_1(f_* \mathcal{F}) = f_* \left[ \frac{c_1(\omega_{C/S})^2 + [\text{Sing } C]}{12} - \frac{c_1(\mathcal{F}) c_1(\omega_{C/S})}{2} + \frac{c_1(\mathcal{F})^2}{2} \right]$$

Applying this to the map  $\pi: C \rightarrow H$ , when  $\mathcal{F} = \omega_{C/H}^{\otimes n}$  gives

$$\begin{aligned} \lambda_n &= \Lambda^{\max}(\pi_* \omega_{C/H}^{\otimes n}) = c_1(\pi_* \omega_{C/H}^{\otimes n}) \\ &= \pi_* \left[ \frac{c_1(\omega_{C/H})^2 + [\text{Sing } C]}{12} - \frac{c_1(\omega_{C/H}^{\otimes n}) c_1(\omega_{C/H})}{2} + \frac{c_1(\omega_{C/H}^{\otimes n})^2}{2} \right] \\ &= \binom{n}{2} \pi_*(c_1(\omega_{C/H})^2) + \frac{\pi_*(c_1(\omega_{C/H})^2) + [\Delta]}{12} \end{aligned}$$

Setting<sup>1)</sup>  $n = 1$ , we see that  $\lambda = \left[ \frac{\pi_*(c_1(\omega_{C/H})^2) + [\Delta]}{12} \right]$  and  $\pi_*(c_1(\omega_{C/H})^2) = 12\lambda - [\Delta]$ . Plugging these values back in gives us the theorem up to torsion. But in fact:

LEMMA 5.14. *Over  $\mathbf{C}$ ,  $\text{Pic}(H, \text{PGL}(v))$  is torsion free.*

Note that this will prove what we want because the invertible sheaves that we are trying to show are isomorphic all “live” on the full scheme  $H_{\mathbf{Z}}$  over  $\text{Spec } \mathbf{Z}$  of stable  $\mathcal{C}$ -canonical curves. If they are isomorphic on  $H_{\mathbf{Z}}$ , they are isomorphic after any base change. But on the other hand, I claim that  $\text{Pic}(H, \text{PGL}(v))$  injects into  $\text{Pic}(H_{\mathbf{C}}, \text{PGL}_{\mathbf{C}}(v))$ :

<sup>1)</sup> For  $n = 1$ ,  $R^1 \pi_*(\omega_{C/H})$  is not zero, but it is the trivial line bundle, hence doesn't affect  $\pi_!$ .

If  $L$  is a line bundle on  $H$  with  $PGL(v)$  action such that  $L \otimes \mathbf{C}$  is trivial over  $H_{\mathbf{C}}$ , then

$$\begin{array}{c} H^0(H, L)^{PGL(v)} \otimes \mathbf{C} = H^0(H_{\mathbf{C}}, L \otimes \mathbf{C})^{PGL(v)} \\ \Downarrow \alpha \\ H^0(H_{\mathbf{C}}, \mathcal{O}_{H_{\mathbf{C}}})^{PGL(v)} = \mathbf{C} \end{array}$$

since  $H_{\mathbf{C}}/PGL(v)$  is compact. Thus we can find a non-zero section  $s \in H^0(H, L)^{PGL(v)}$ , which over  $\mathbf{C}$  can be used to give the trivialization  $\alpha$ . Over  $\mathbf{C}$ ,  $s$  has no zeros so the divisor  $(s)_0$  of the zeros of  $s$  on  $H$ , has support only over the closed fibres of  $\text{Spec}(\mathbf{Z})$ . Mumford and Deligne [6] have shown that  $H \rightarrow \text{Spec} \mathbf{Z}$  is smooth with irreducible fibres, hence  $(s)_0 = \sum r_i \pi^{-1}(p)$ ,  $r_i \geq 0$  i.e.  $(s)_0 = (n)$  for some integer  $n$ . Then  $\left(\frac{s}{n}\right)$  is a global section of  $L$  with no zeros so  $L$  is trivial.

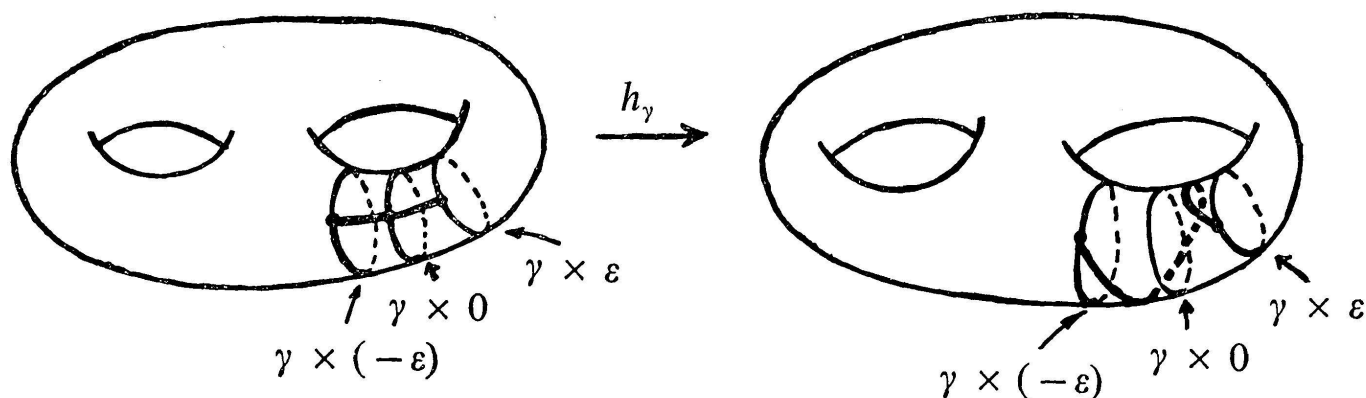
*Proof of Lemma.* Over  $\mathbf{C}$ , we have Teichmüller theory at our disposal. Let  $\Pi$  be a standard model of a group with generators  $\{a_i, b_i \mid 1 \leq i \leq g\}$  mod the relation  $\prod_{i=1}^g (a_i b_i a_i^{-1} b_i^{-1}) = 1$ . Then the Teichmüller modular group  $\Gamma$  is

$$\Gamma = \{ \alpha \mid \alpha : \Pi \rightarrow \Pi \text{ is an orientation preserving } \} / \begin{array}{l} \text{inner} \\ \text{isomorphism} \quad \text{automorphisms} \end{array}$$

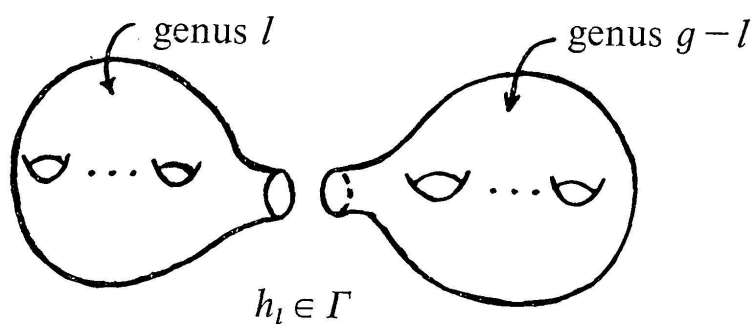
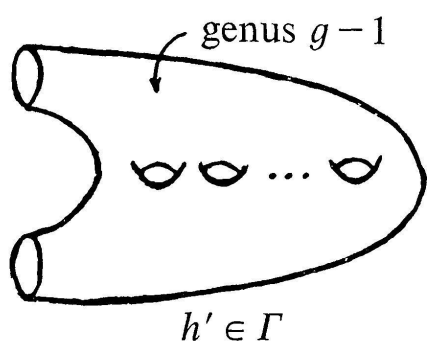
The Teichmüller space  $\mathcal{T}_g$  is given by

$$\mathcal{T}_g = \left\{ (C, \alpha) \left| \begin{array}{l} C \text{ a smooth curve of genus } g \text{ and } \alpha : \pi_1(C) \rightarrow \Pi \text{ an} \\ \text{orientation preserving isomorphism given up to inner} \\ \text{automorphism} \end{array} \right. \right\}$$

Fix a model  $M_g$  of the real surface of genus  $g$ , and identify  $\pi_1(M_g)$  and  $\Pi$ . Then  $\Gamma$  is generated by the maps which are induced by certain automorphisms of  $M_g$ , called Dehn twists. The Dehn twist  $h_\gamma$  corresponding to a loop  $\gamma : [0, 1] \rightarrow M_g$  on  $M_g$  is given by taking an  $\varepsilon$ -collar  $\gamma \times [-\varepsilon, \varepsilon]$  about  $\gamma$ , letting  $h$  = identify off the collar and letting  $h(\gamma(t), \eta - \varepsilon) = \left( \gamma\left(t + \frac{\eta}{2\varepsilon}\right), \eta - \varepsilon \right)$  as shown below.

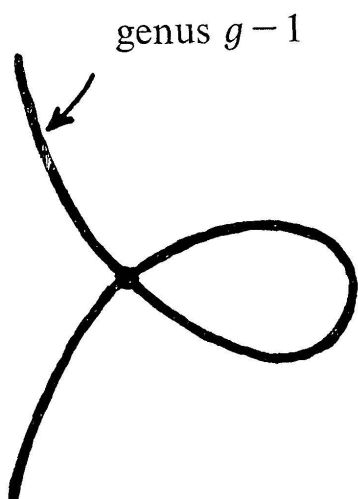


Up to inner automorphism  $h_\gamma$  is determined by which of the pictures below results from cutting open  $M_g$  along  $\gamma$ . We have name these elements of  $\Gamma$  in the diagrams:

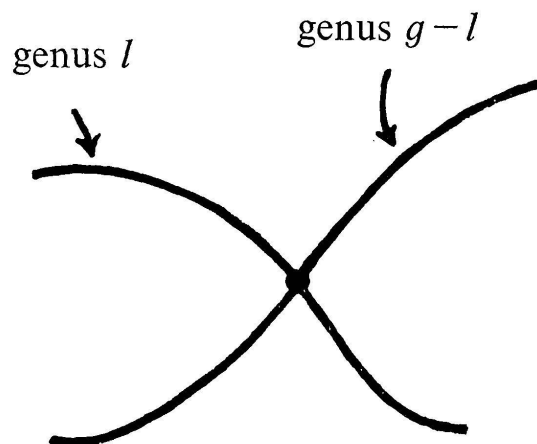


The Dehn twist  $h_\gamma$  can also be described as the monodromy map obtained by going around a curve  $C_0$  with one double point for which  $\gamma$  is the vanishing cycle.

The components of  $\Delta \subset H$  correspond to the different ways of putting a stable double point on a smooth moduli stable curve  $C$ . They are the closures of the sets of curves of the forms shown below: again, we name these components in the diagram:



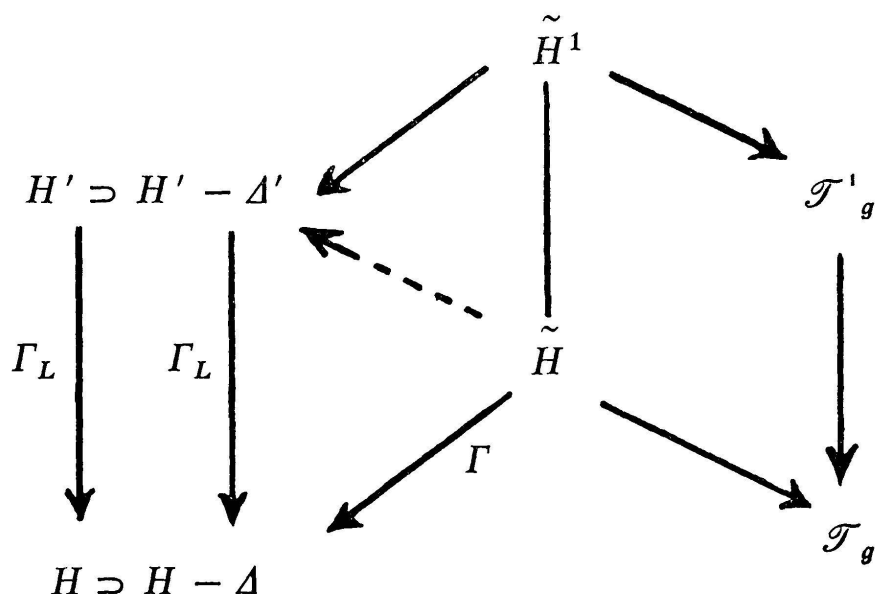
$$\Delta' = \left( \begin{array}{l} \text{closure of set} \\ \text{formed by curves} \\ \text{like this} \end{array} \right)$$



$$\Delta_l = \left( \begin{array}{l} \text{closure of set} \\ \text{formed by curves} \\ \text{like this} \end{array} \right)$$

Let  $\tilde{H} = \left\{ (C, \alpha, B) \mid \begin{array}{l} (C, \alpha) \in \mathcal{T}_g, B \text{ a basis of the } e\text{-tuple dif-} \\ \text{ferentials on } C \text{ given up to a scalar} \end{array} \right\}$

Suppose we are given a line bundle  $L$  on  $H$  with  $PGL(v)$ -action such that  $L^n \cong \mathcal{O}_H$ .  $L$  induces a cyclic covering  $H'$  of  $H$  plus a lifting of the  $PGL(v)$ -action to  $H'$ . If we choose  $n$  minimal this covering is not split: we denote its structure group by  $\Gamma_L$ . Let  $\tilde{H}'$  be the pullback of covering over  $\tilde{H}$ , and let  $\mathcal{T}'_g$  denote the quotient of  $\tilde{H}'$  by  $PGL(v)$ —this is a covering of  $\mathcal{T}_g$ . These coverings are related by



$\mathcal{T}_g$  is simply connected so the cover  $\mathcal{T}'_g \rightarrow \mathcal{T}_g$  splits, hence so does  $\tilde{H}' \rightarrow \tilde{H}$ . A section of this last cover gives a map from  $\tilde{H}$  to  $H' - \Delta'$  (shown dashed in the diagram), so  $\Gamma_L$  is a quotient of  $\Gamma$ , of finite order.

Let  $\gamma'$  [resp.  $\gamma_e$ ] be a loop at a fixed base point  $P_0 \in H - \Delta$  going around  $\Delta'$  [resp.:  $\Delta_e$ ] but homotopic to 0 in  $H$ . Fix a point  $\tilde{P}_0 \in \tilde{H}$  over  $P_0$ . The monodromy characterization of the Dehn twists implies that  $\gamma'$  [resp.:  $\gamma_e$ ] lifted to  $\tilde{H}$  goes from  $\tilde{P}_0$  to  $h'(\tilde{P}_0)$  [resp.: to  $h_e(\tilde{P}_0)$ ]. Since  $\gamma'$  [resp.:  $\gamma_e$ ] are homotopic to 0 in  $H$ , and the covering  $H' - \Delta'$  extends over  $H$ , this implies that the image of  $h'$  [resp.:  $h_e$ ] in  $\Gamma_L$  is 0. But these elements and their conjugates generate  $\Gamma_L$ , so  $\Gamma_L = \{1\}$ , hence  $L \cong \mathcal{O}_H$ , proving the lemma and the theorem.

In order to describe the ample cone on  $\text{Pic}(\overline{\mathcal{M}}_g)$  we prove:

THEOREM 5.15.  $\text{Ch}^*(\mathcal{O}_{\text{Div}}(v)) = (\mu^e \otimes \lambda^{-4})^{e(g-1)}$

*Proof.* The proof depends on a result which we simply quote from Fogarty [8] or Knudsen [12]:

PROPOSITION 5.16. *Let  $S$  be a locally closed subscheme of a Hilbert scheme  $\text{Hilb}_{\mathbf{P}^v-1}^P$ ,  $\text{Ch}$  be the associated Chow map  $\text{Ch}: S \rightarrow \text{Div}$  and  $Z \subset \mathbf{P}^v \times S$  have relative dimension  $r$  over  $S$ . Then if  $n \geq 0$ ,  $\Lambda^{\max} p_{2,*}(\mathcal{O}_Z(n)) = \bigotimes_{i=0}^{r+1} \mu_i^{\binom{n}{i}}$  and  $\text{Ch}^*(\mathcal{O}_{\text{Div}}(1)) = \mu_{r+1}$ , where  $\mu_i$  are suitable invertible sheaves on  $S$ .*

In the situation of our theorem, with  $S = H$  and  $Z = C$ ,  $\mathcal{O}_C(1) = \omega_{C/H}^{\otimes e} \otimes \pi^*Q$  where  $Q$  is the invertible sheaf determined by  $(\pi_*\omega_{C/H}^{\otimes e}) \otimes Q = \pi_*\mathcal{O}_C(1) = \pi_*\mathcal{O}_{\mathbf{P}^v-1}(1) = \mathcal{O}_H^v$ , hence

$$(5.17) \quad \mathcal{O}_H = [\Lambda^{\max} \pi_*(\omega_{C/H}^{\otimes e})] \otimes Q^v = \mu^{\binom{e}{2}} \otimes \lambda \otimes Q^v.$$

On the other hand,

$$\Lambda^{\max}(\pi_*\mathcal{O}_C(n)) = \Lambda^{\max}[\pi_*(\omega_{C/H}^{\otimes ne} \otimes Q^n)] = \mu^{\binom{ne}{2}} \otimes \lambda \otimes Q^{P(n).n}.$$

This has leading term in  $n$  of  $\mu^{n^2e^2/2} \otimes Q^{2e(g-1)n^2}$  so

$$\begin{aligned} \text{Ch}^*(\mathcal{O}_{\text{Div}}(v)) &= \mu^{ve^2} \otimes Q^{4e(g-1)v} \\ &= \mu^{ve^2 - \binom{e}{2}.4e(g-1)} \otimes \lambda^{-4e(g-1)} \quad \text{using (5.17)}. \end{aligned}$$

Finally, therefore,  $\text{Ch}^*(\mathcal{O}_{\text{Div}}(v)) = \mu^{e^2(g-1)} \otimes \lambda^{-4e(g-1)}$  as required.

COROLLARY 5.18. *If  $e \geq 5$ ,  $\mu^e \otimes \lambda^{-4} (= \lambda^{12e-4} \otimes \delta^{-e})$  is “ample on  $\overline{\mathcal{M}}_g$ ”, i.e. those positive powers of this bundle which are pull-backs of bundles on  $\overline{\mathcal{M}}_g$  are ample on  $\mathcal{M}_g$ .*

*Proof.* This is an immediate consequence of the Theorem and our main result: that  $PGL(v)$ -invariant sections of  $\text{Ch}^*(\mathcal{O}_{\text{Div}}(1))$  define a projective embedding of  $\overline{\mathcal{M}}_g$ .

REMARK 5.19. A similar argument using the facts that

- (1)  $\omega^{\otimes e}$  is base point free for all canonical curves when  $e \geq 2$ ,
- (2) smooth curves are stable if  $d > 2g$ ,

shows that if  $e \geq 2$ , the sections of  $\lambda^{12e-4} \otimes \delta^{-e}$  on  $\overline{\mathcal{M}}_g$  separate points on  $\mathcal{M}_g$ .

To get a good picture of the ample cone on  $\overline{\mathcal{M}}_g$  we need to use the realization via  $\Theta$  functions  $\mathcal{A}_{g,1} \xrightarrow{\Theta} \mathbf{P}^N$  of the moduli scheme  $\mathcal{A}_{g,1}$  of

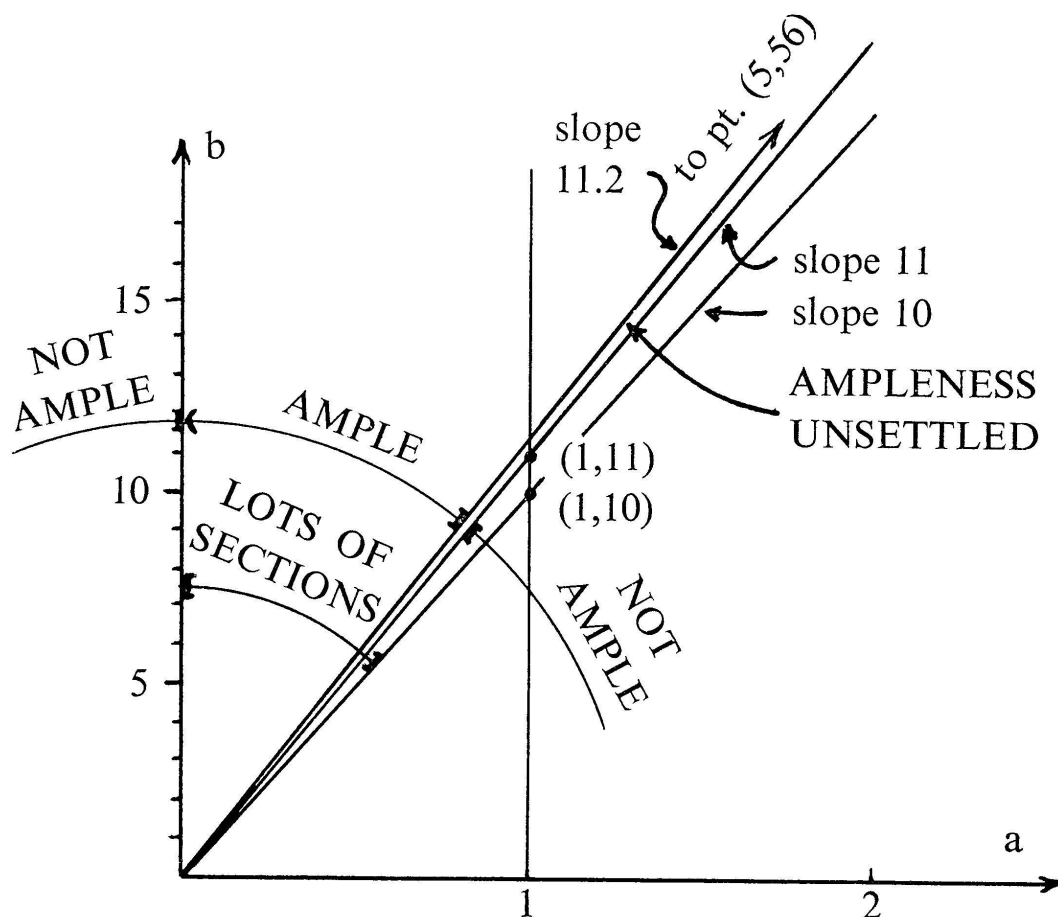
principally polarized abelian varieties. More precisely, let  $J : \mathcal{M}_g \rightarrow \mathcal{A}_{g,1}$  be the map taking a curve  $C$  to its Jacobian. Then we have:

**THEOREM 5.20.** *In characteristic 0, the morphism  $\mathcal{M}_g \xrightarrow{J} \mathcal{A}_{g,1} \xrightarrow{\theta} \mathbf{P}^N$  extends to a morphism  $\overline{\mathcal{M}}_g \xrightarrow{\theta} \mathbf{P}^N$  so that for some  $m$ ,  $\theta^*(\mathcal{O}_{\mathbf{P}^N}(1)) = \lambda^m$ .*

*Proof.* See Arakelov [1] or Knudsen [12].

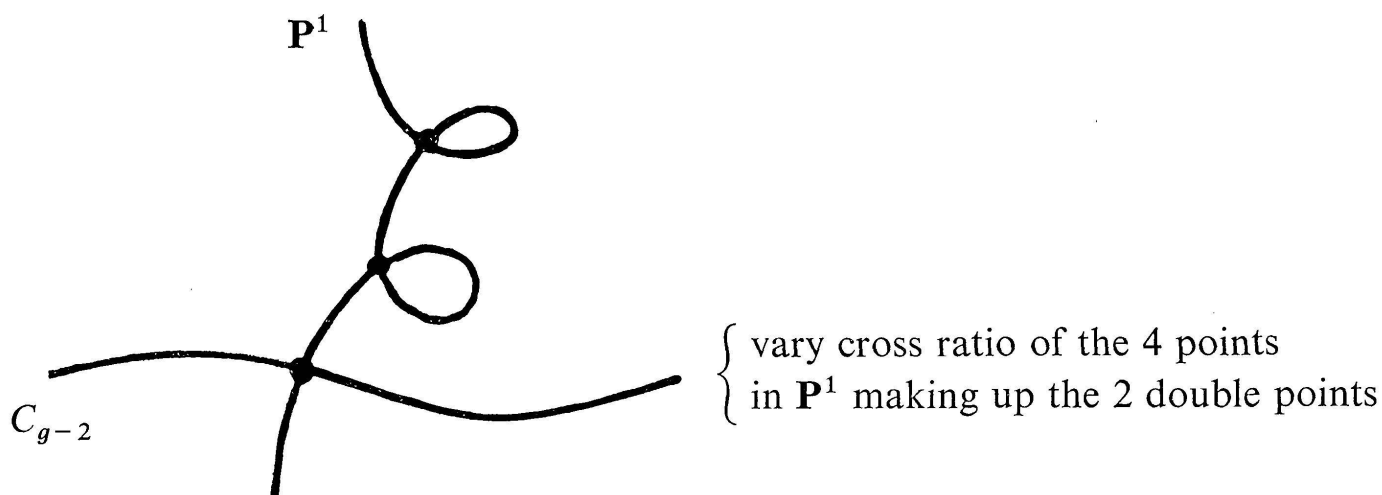
**REMARK.** This should also hold in characteristic  $p$ , but it seems to be a rather messy problem there.

Putting together 5.18 and 5.20, we get a whole sector in the  $(a, b)$ -plane such that  $\lambda^b \otimes \delta^{-a}$  is ample for  $(a, b)$  in this sector. This is depicted in the diagram below:



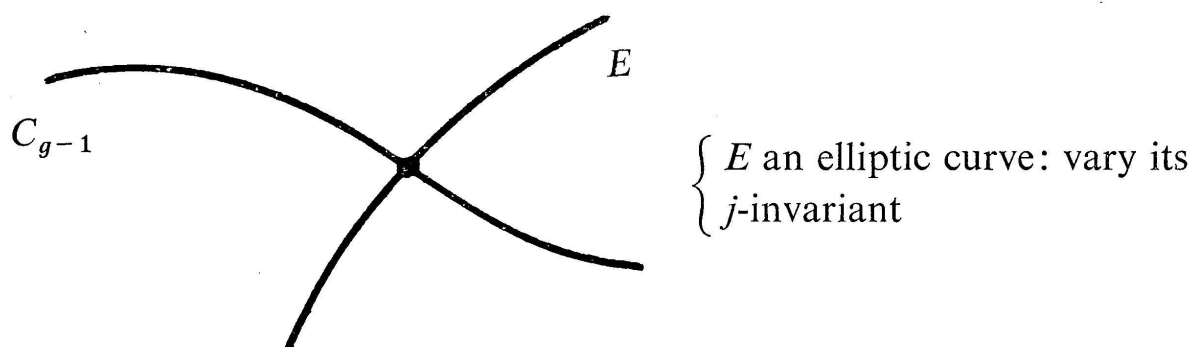
The fact that  $\lambda$  and  $\lambda^{11} \otimes \delta^{-1}$  are not ample can be seen by examining the following 2 curves in  $\overline{\mathcal{M}}_g$ :

- (1) If  $S_1$  is a curve in  $\overline{\mathcal{M}}_g$  composed of curves of the form:



where  $C_{g-2}$  is a fixed genus  $(g-2)$  component, then  $\lambda|_{S_1} = \mathcal{O}_{S_1}$ , hence sections of  $\lambda$  always collapse such families.

(2) If  $S_2$  is a curve in  $\overline{\mathcal{M}}_g$  composed of curves of the form:



where  $C_{g-1}$  is a fixed genus  $(g-1)$  component, then  $\lambda^{11} \otimes \delta^{-1}|_{S_2} = \mathcal{O}_{S_2}$  i.e.  $\lambda^{11} \otimes \delta^{-1}$  collapses these families.

We omit the details.

## APPENDIX

We wish to fill in the gap in the proof of Proposition 5.5 on page 95. The difficulty occurs if the support of  $\mathcal{I}$ , i.e.  $(0) \times L_1$ , contains some of the components of  $C_2$  meeting  $C_1$ . In this case, the inequality

$$e_L(\mathcal{I}_2) \geq w$$

is not clear. Indeed, if  $D_1, \dots, D_k$  are the components of  $C_2$  meeting  $C_1$ ,  $w_i = \#(D_i \cap C_1)$ , and  $\mathcal{K}_i$  is the pull-back of  $\mathcal{I}_2$  to  $D_i$ , then