§4. Asymptotic Stability of Canonically Polarized Curves

Objekttyp: **Chapter**

Zeitschrift: **L'Enseignement Mathématique**

Band (Jahr): **23 (1977)**

Heft 1-2: **L'ENSEIGNEMENT MATHÉMATIQUE**

PDF erstellt am: **26.09.2024**

§ 4. Asymptotic Stability of Canonically Polarized Curves

The chief difficulty of using the numerical criterion of Theorem 2.9 to prove the stability of a projective variety is that it is necessary to look inside $\mathcal{O}_{X \times \mathbf{A}^1}$ to compute the multiplicity $e_L(\mathcal{I})$. To circumvent this difficulty, we will construct an upper bound on $e_L(\mathcal{I})$ in terms of data on X alone. For curves, this bound involves only the multiplicities of ideals $\mathcal{I} \subset \mathcal{O}_X$, but for higher dimensional varieties—in particular, surfaces—it requires a theory of mixed multiplicities, i.e. multiplicities for several ideals simultaneously. To motivate the global theory, we will first describe what happens in the local case. Here the basic ideas were introduced by Teissier and Rissler [22]. Recall that if $\mathcal{O}$ is a local ring of dimension r with infinite residue field and I is an ideal of finite colength in it then whenever $f_1, \ldots f_r$ are sufficiently generic elements of I, $e(I) = e((f_1, \ldots, f_r))$. This suggests

Definition 4.1. *If $\mathcal{O}^r$ is a local ring and $I_1, \ldots, I_r$ are ideals of finite colength in $\mathcal{O}$, the mixed multiplicity of the I_i is defined by*

$$e(I_1, \ldots, I_r) = e((f_1, \ldots, f_r))$$

where $f_i \in I_i$ is a sufficiently generic element. (The set of integers $e((f_1, \ldots, f_r))$ has some minimal element and a choice $(f_1, \ldots, f_r)$ is sufficiently generic if the minimum is attained for these f_i.)

The basic property of these multiplicities is:

Proposition 4.2. *Let $I_1, \ldots, I_k$ be ideals of finite colength of a local ring $\mathcal{O}^r$ and let*

$$P_r(m_1, \ldots, m_k) = \sum_{\substack{\Sigma r_i = r \\ r_i \geq 0}} \frac{1}{\prod (r_i!)} \cdot e(I_1^{[r_1]}, \ldots, I_k^{[r_k]}) \cdot m_1^{r_1} \ldots m_k^{r_k}$$

where $I_i^{[r_i]}$ indicates that I_i appears r_i times. Then

i) $$\left| \dim (\mathcal{O} / \prod_{i=1}^{k} I_i^{m_i}) - P_r(m_1, \ldots, m_k) \right| = 0((\textstyle\sum m_i)^{r-1})$$

ii) *There exists a polynomial of total degree r*

$$P(m_1, \ldots, m_k) = P_r(m_1, \ldots, m_k) + \text{lower order terms}$$

and an N_0 such that if $m_i \geq N_0$ for all i, then

$$\dim (\mathcal{O} / \prod I_i^{m_i}) = P(m_1, \ldots, m_k).$$

Proof. See Teissier and Rissler [22].
Using this we obtain the estimate:

PROPOSITION 4.3. *Let* $I \subset \mathcal{O}[[t]]$ *be an ideal of finite codimension and let* $I_k = \{ a \in \mathcal{O} \mid at^k \in I \}$; *then* $I_0 \subseteq I_1 \subseteq \ldots \subseteq I_N = \mathcal{O}, N \gg 0$. *Then for all sequences* $0 = r_0 < r_1 < \ldots < r_l = N$,

$$e(I) \leq \sum_{k=0}^{l-1} (r_{k+1} - r_k) \sum_{j=0}^{r} e(I_{r_k}^{[j]}, I_{r_{k+1}}^{[r-j]}) .$$

Proof. Since $I \supset \oplus\, t^{r_i} I_{r_i}$

$$\begin{aligned} I^n \supset\ & I_{r_0}^n (\mathcal{O} + t\mathcal{O} + \ldots + t^{r_1 - 1}\mathcal{O}) + I_{r_0}^{n-1} I_{r_1} (t^{r_1}\mathcal{O} + t^{r_1+1}\mathcal{O} + \ldots + t^{2r_1 - 1}\mathcal{O}) \\ & + \ldots + I_{r_0} I_{r_1}^{n-1} (t^{(n-1)r_1}\mathcal{O} + \ldots + t^{nr_1 - 1}\mathcal{O}) \\ & + I_{r_1}^n (t^{nr_1}\mathcal{O} + \ldots + t^{(n-1)r_1 + r_2 - 1}\mathcal{O}) + I_{r_1}^{n-1} I_{r_2} (t^{(n-1)r_1 + r_2}\mathcal{O} + \ldots) \\ & + \ldots + I_{r_{l-1}}^n (t^{nr_{l-1}}\mathcal{O} + \ldots) + I_{r_{l-1}}^{n-1} (t^{(n-1)r_{l-1} + r_l}\mathcal{O} + \ldots) \\ & + \ldots + t^{nr_l}\mathcal{O}[[t]] . \end{aligned}$$

whence

$$\dim(\mathcal{O}[[t]]/I^n) \leq \sum_{k=0}^{l} (r_{k+1} - r_k) \sum_{i=0}^{n-1} \dim\left(\mathcal{O}/(I_{r_k}^{n-i} \,.\, I_{r_{k+1}}^{i})\right)$$

$$(4.4) \quad = \sum_{k=0}^{l} (r_{k+1} - r_k) \sum_{i=0}^{n-1} \left[\sum_{j=0}^{r} \frac{1}{j!\,(r-j)!} e\,(I_{r_k}^{[r-j]}, I_{r_{k+1}}^{[j]})\,(n-i)^{r-j} i^j + R_i \right]$$

By Proposition 4.2 i) each remainder terms R_i is $O(n^{r-1})$. Indeed, ii) of 4.2 says that except when i or $n - i < N_0$, the R_i are all represented by a polynomial of degree $r - 1$ so that we can obtain a uniform $O(n^{r-1})$ estimate for the R_i; hence $\sum_{i=0}^{n-1} R_i = O(n^r)$.

But the n.l.c. of the $(r+1)^{\text{st}}$ degree polynomial representing $\dim(\mathcal{O}[[t]]/I^n)$ is by definition $e(I)$; so evaluating the n.l.c. of the sum in (4.4) using the lemma below, gives the proposition.

LEMMA 4.5. $$\frac{j!\,(r-j)!}{(r+1)!} n^{r+1} = \sum_{i=0}^{n-1} (n-i)^{r-j} i^j + O(n^r)$$

Proof. We can reexpress the left hand side in terms of the β-function as

$$\frac{j!\,(r-j)!}{(r+1)!} n^{r+1} = \beta(j, r-j)\, n^{r+1} = \left(\int_0^1 t^j (1-t)^{r-j} dt \right) n^{r+1} ,$$

and the right hand side is just another expression for n^{r+1} times this integral as a Riemann sum plus error term.

To globalize these ideas we combine them with some results of Snapper [5, 21].

Definition 4.6. *Let* X^r *be a variety,* L *be a line bundle on* X *and* $\mathcal{I}_1, \ldots, \mathcal{I}_r$ *be ideals on* $\mathcal{O}_X$ *such that* $\operatorname{supp}(\mathcal{O}_X/\mathcal{I}_i)$ *is proper. Choose a compactification* $\bar{X}$ *of* X *on which* L *extends to a line bundle* $\bar{L}$ *and let* $\pi: \bar{B} \to \bar{X}$ *be the blowing up of* $\bar{X}$ *along* $\prod \mathcal{I}_i$ *so that* $\pi^{-1}(\mathcal{I}_i) = \mathcal{O}_{\bar{B}}(-E_i)$. *Let* $\pi^* L = \mathcal{O}_{\bar{B}}(D)$. *We define*

$$e_L(\mathcal{I}_1, \ldots, \mathcal{I}_r) = (D^r) - \big((D - E_1) \cdot \cdots\cdots \cdot (D - E_r)\big).$$

We omit the check that this definition is independent of the choice of $\bar{X}$ and $\bar{L}$.

4.7. Classical geometric interpretation. Suppose X is a projective variety, $L = \mathcal{O}_X(1)$ and $\mathcal{I}_i \cdot L$ is generated by a space of sections $W_i \subset \Gamma(\mathbf{P}^n, \mathcal{O}(1))$. If $H_1, \ldots, H_r$ are generic hyperplanes of $\mathbf{P}^n$, then $\#(H_1 \cap \ldots \cap H_r \cap X) = \deg X$. One sees by an argument like that of Proposition 2.5, that as the H_i specialize to hyperplanes defined by elements of W_i but otherwise generic, the number of points in $H_1 \cap \ldots \cap H_r \cap X$ which specialize to a point in one of the W_i's is just $e_L(\mathcal{I}_1, \ldots, \mathcal{I}_r)$.

We can globalize Proposition 4.2 to give an interpretation of the mixed multiplicity by Hilbert polynomials.

Proposition 4.8. i) *Let* X^r *be a variety,* $L_1, \ldots, L_n$ *be line bundles on* X *and* $\mathcal{I}_1, \ldots, \mathcal{I}_l$ *be ideals in* $\mathcal{O}_X$ *such that* $\operatorname{supp}(\mathcal{O}_X/\mathcal{I}_i)$ *is proper for all* i. *Then there is a polynomial* $P(n, m)$ *of total degree* r *and an* M_0 *such that if* $m_j \geqq M_0$ *for all* j *then*

$$\chi\Big(X, \bigotimes_{i=1}^{k} L_i^{n_i} \Big/ \prod_{j=1}^{l} \mathcal{I}_j^{m_j} \cdot \bigotimes_{i=1}^{k} L_i^{n_i}\Big) = P(n, m).$$

Now suppose all the line bundles are the same, say L *and let*

$$P_r(m_1, \ldots, m_l) = \sum_{\substack{\Sigma r_i = r \\ r_i \geqq 0}} \frac{1}{\prod (r_i!)} e_L(\mathcal{I}_1^{[r_1]}, \ldots, \mathcal{I}_l^{[r_l]})\, m_1^{r_1} \ldots m_l^{r_l}$$

Then

ii) $P(\sum m_i; m_1, \ldots, m_l) = P_r(m_1, \ldots, m_l)$ + lower order terms

iii) $| \chi(X, L^{\Sigma m_i} / \prod \mathscr{I}_j^{m_j} \otimes L^{\Sigma m_i}) - P_r(m_1, \ldots, m_l) | = O((\sum_{j=1}^{l} m_j)^{r-1})$
(i.e. *we retain an estimate assuming only* $\sum m_j$ *is large*).

Proof. Making a suitable compactification of X will not alter the Euler characteristics so we may assume X is compact.

Before proceeding we recall certain facts: If $R = \bigoplus_{n_i \geq 0} R_{n_1, \ldots, n_l}$ is a multigraded ring we can form a scheme Proj (R) in the obvious way from multi-homogeneous prime ideals. Quasi-coherent sheaves $\mathscr{F}$ on Proj (R) correspond to multigraded R-modules $M = \bigoplus M_{n_1, \ldots, n_l}$. Suppose $R_{0, \ldots, 0} = k$ a field and that R is generated by the homogeneous pieces $R_{0, \ldots, 0, 1, 0, \ldots, 0}$. Then we get invertible sheaves $L_1, \ldots, L_l$ on Proj (R) from the modules M_i, where M_i = (R with i^{th}-grading shifted by 1), and the multigraded variant of the F.A.C. vanishing theorem for higher cohomology says that if $\mathscr{F}$ is a coherent sheaf on Proj (R) then

$$H^i(\mathscr{F} \otimes (\otimes L_j^{n_j})) = \begin{cases} M_{n_1, \ldots, n_l}, & i = 0 \\ (0), & i > 0 \end{cases} \qquad \text{if } n_j \gg 0, \text{ all } j$$

Now if $\mathscr{I}_1, \ldots, \mathscr{I}_k$ are ideal sheaves on X such that supp $(\mathcal{O}_X / \mathscr{I}_j)$ is proper for all i, let $\mathscr{A} = \bigoplus_{m_j \geq 0} \mathscr{I}_1^{m_1} \ldots \mathscr{I}_l^{m_l}$. Then $\mathscr{A}$ is a multigraded sheaf of $\mathcal{O}_X$-algebras. Let B = Proj $(\mathscr{A})$; the blow up of X along $\prod \mathscr{I}_j$ is just $\pi: B \to X$. If E_j is the exceptional divisor corresponding to $\mathscr{I}_j$, then when $\mathcal{O}_B(-\sum m_j E_j)$ is coherent and when all the m_j are large the relative versions of the vanishing theorems say:

a) $R^i \pi_* (\mathcal{O}(-\sum m_j E_j)) = 0, \ i > 0$

b) $\pi_* \mathcal{O}(-\sum m_j E_j) = \prod_{j=1}^{l} \mathscr{I}_j^{m_j}$

In any case,

c) supp $R^i \pi_* (\mathcal{O}(-\sum m_j E_j))$ has dimension less than r, $i > 0$,

d) $\pi_* (\mathcal{O}(-\sum m_j E_j)) = \prod_i \mathscr{I}_i^{m_i}$ except on a set of dimension less than r.

From a) and b) we deduce that when all the m_j are large, $\chi(\prod \mathscr{I}_j^{m_j}) = \chi(\pi^* \mathcal{O}(-\sum m_j E_j))$. Thus, $\chi(X, \otimes L_i^{n_i} / \prod \mathscr{I}_j^{m_j} L_i^{n_i}) = \chi(X, \otimes L_i^{n_i}) - \chi(B, \otimes L_i^{n_i}(-\sum m_j E_j))$ and both of these last Euler characteristics polynomials of degree $\leq r$ by Snapper [5, 21]. Now if $\pi^* L = \mathcal{O}_B(D)$, his result also says,

$$r\,!\,.\,\text{n.l.c.}(\chi(X, L^{\Sigma m_j}/\prod \mathscr{I}_j^{m_j} \otimes L^{\Sigma m_j}) = (\sum m_j)^r (D^r) - ((\sum m_j (D - E_j))^r)$$

$$= \sum_{\substack{\Sigma r_j = r \\ r_j \geqq 0}} \frac{r\,!}{\prod (r_j\,!)} \prod (m_j (D - E_j))^{r_j}$$

$$= \sum_{\substack{\Sigma r_j = r \\ r_j \geqq 0}} \frac{r\,!}{\prod (r_j\,!)} e_L (\mathscr{I}_1^{[r_1]}, \ldots, \mathscr{I}_l^{[r_l]}) \,.\, m_1^{r_1} \ldots m_l^{r_l}$$

which is ii). Fix an N such that ii) holds when all $m_j \geqq N$.

Now suppose I is a proper subset of $\{1, \ldots, l\}$, J is its complement and that values $m_i < N$ are fixed for all $i \in I$. Let $\pi_J : B_J \to X$ be the blow up of X along $\prod_{j \in J} \mathscr{I}_j$. As above we deduce that $\exists N'$ depending on I and the m_i, $i \in I$ such that if $m_j > N'$, $\forall j \in J$, then

$$\chi (X, \mathscr{I}_1^{m_1} \ldots \mathscr{I}_k^{m_k}) = \chi (B_J, \prod_{i \in I} \mathscr{I}_i^{m_i} (- \sum_{j \in J} m_j E_j)) \,.$$

Then applying c) and d) we see that for some C, also depending on I and the m_i, $i \in I$,

$$|\, \chi (B, \mathcal{O} (- \sum m_i E_i)) - \chi (B_J, \prod_{i \in I} \mathscr{I}_i^{m_i} (- \sum_{j \in J} m_j E_j)) \,| \leqq C (\sum_{j \in J} m_j)^{r-1} \,.$$

Combining this with the argument used in the proof of i) and ii) shows that for some C' (depending on I and the m_i, $i \in I$)

$$|\, \chi (X, L^{\Sigma m_j}/\prod \mathscr{I}_j^{m_j} L^{\Sigma m_j}) - P_r (m_1 \ldots m_l) \,| \leqq C' (\sum_{j \in J} m_j)^{r-1} \,.$$

From ii), we get an estimate of this type with a uniform constant C', when all the $m_j \geqq N$. Since there are only finitely many sets I and for each of these only finitely many choices for the m_i, $i \in I$ with $m_i < N$ we can combine all these estimates to show: there exists M and C'' such that if any $m_i > M$, then

$$|\, \chi (X, L^{\Sigma m_j}/ \prod_j \mathscr{I}_j^{m_j} L^{\Sigma m_j}) - P_r (m_1, \ldots, m_l) \,| \leqq C'' ((\sum_j m_j)^{r-1})$$

which is iii).

The following analogue of Proposition 2.6 allows us to calculate mixed multiplicities in terms of the dimensions of spaces of sections.

PROPOSITION 4.9. *If L, $\mathscr{I}_1 L, \ldots, \mathscr{I}_l L$ are generated by their sections, then*

$$|\, \chi (X, L^{\Sigma m_j}/(\prod \mathscr{I}_j^{m_j}) L^{\Sigma m_j}) - \dim (\Gamma (X, L^{\Sigma m_j})/\Gamma (X, \prod \mathscr{I}^{m_j} L^{\Sigma m_j})) \,|$$
$$= O ((\sum m_j)^{r-1})$$

Proof. We give only a sketch of the proof which is very similar to that of Proposition 2.6. One first shows as in the proof of 2.6a), that for $i > 0$, $h^i(L^{\Sigma m_j}/\prod \mathscr{I}^{m_j} L^{\Sigma m_j}) = O((\sum m_j)^{r-1})$, hence that

$$| \chi(X, L^{\Sigma m_j}/\prod \mathscr{I}_j^{m_j} L^{\Sigma m_j}) - \dim \Gamma(X, L^{\Sigma m_j}/\prod \mathscr{I}_j^{m_j} L^{\Sigma m_j}) |$$
$$= O((\textstyle\sum m_j)^{r-1})$$

Using the long exact sequence

$$0 \to \Gamma(X, \prod \mathscr{I}_j^{m_j} L^{\Sigma m_j}) \to \Gamma(X, L^{\Sigma m_j}) \to \Gamma(X, L^{\Sigma m_j}/\prod \mathscr{I}_j^{m_j} L^{\Sigma m_j}) \to \ldots$$

this reduces the proposition to showing that

$$\dim(\operatorname{coker}(\Gamma(X, L^{\Sigma m_j}) \to \Gamma(X, L^{\Sigma m_j}/\prod \mathscr{I}_j^{m_j} L^{\Sigma m_j})) = O((\textstyle\sum m_j)^{r-1})$$

and this is done exactly as in the proof of 2.6b). (Note that the extra hypotheses of 2.6b) were not used in this part of the proof.)

The global form of Proposition 4.3 is:

PROPOSITION 4.10. *Given a variety X, a line bundle L on X and an ideal $\mathscr{I} \subset \mathscr{O}_{X \times \mathbf{A}^1}$ with $\operatorname{supp}(\mathscr{O}_{X\times \mathbf{A}^1}/\mathscr{I})$ proper in $X \times (0)$, let $\mathscr{I}_k = \{ a \in \mathscr{O}_X \mid t^k a \in \mathscr{I} \}$ so that $\mathscr{I}_0 \subseteq \mathscr{I}_1 \subseteq \ldots \subseteq \mathscr{I}_N = \mathscr{O}_X$ and let $L_1 = L \otimes \mathscr{O}_{\mathbf{A}^1}$. Suppose that L, $\mathscr{I}_k L$ and $\mathscr{I} L_1$ are generated by their sections. Then for all sequences $0 = r_0 < r_1 < \ldots < r_l = N$,*

$$e_{L_1}(\mathscr{I}) \leqq \sum_{k=0}^{l} (r_{k+1} - r_k) \sum_{j=0}^{r} e_L(\mathscr{I}_{r_k}^{[j]}, \mathscr{I}_{r_{k+1}}^{[r-j]}) .$$

Proof. By Proposition 4.9, $e_{L_1}(\mathscr{I})$ is calculated by the order of growth of

$$\dim [H^0(X \times \mathbf{A}^1, L_1^n)/H^0(X \times \mathbf{A}^1, \mathscr{I}^n . L_1^n)] .$$

Exactly as in Proposition 4.3, for each n, we introduce using the r_i's an approximating ideal sheaf $\mathscr{I}_n'$:

$$\mathscr{I}^n \supset \mathscr{I}_n' = \bigoplus_{k=0}^{\infty} t^k . \mathscr{I}_{n,k}$$

where $\mathscr{I}_{n,0} \subset \mathscr{I}_{n,1} \subset \ldots \subset \mathscr{I}_{n,N} = \mathscr{O}_X$ for $N \gg 0$. Since

$$H^0(X \times \mathbf{A}^1, \mathscr{I}^n . L_1^n) \supset H^0(X \times \mathbf{A}^1, \mathscr{I}_n' . L_1^n) = \bigoplus_{k=0}^{\infty} H^0(X, \mathscr{I}_{n,k} . L^n),$$

it follows that

$$\dim \left(H^0 (X \times \mathbf{A}^1, L_1^n)/H^0 (X \times \mathbf{A}^1, \mathcal{I}^n . L_1^n\right)$$

$$\leqq \sum_{k=0}^{\infty} \dim \left(H^0 (X, L^n)/H^0 (X, \mathcal{I}_{n,k} . L^n)\right)$$

The rest of the proof follows Proposition 4.3 exactly, using 4.9 again to get the estimate

$$\dim \left(H^0 (X, L^n)/H^0 (X, \mathcal{I}^i_{r_k} . \mathcal{I}^{n-i}_{r_{k+1}} . L^n)\right)$$

for $\chi (L^n/\mathcal{I}^i_{r_k} . \mathcal{I}^{n-i}_{r_{k+1}} . L^n)$.

COROLLARY 4.11. *If in Proposition* 4.10, X *is a curve*

$$e_{L_1} (\mathcal{I}) \leqq \min_{0 = r_0 < r_1 \ldots < r_l = N} \left[\sum_{k=0}^{e} (r_{k+1} - r_k) . (e_L (\mathcal{I}_{r_k}) + e_L (\mathcal{I}_{r_{k+1}}))\right]$$

If X *is a surface,*

$$e_{L_1} (\mathcal{I})$$

$$\leqq \min_{0 = r_0 < r_1 \ldots < r_l = N} \left[\sum_{k=0}^{l} (r_{k+1} - r_k) . (e_L (\mathcal{I}_{r_k}) + e_L (\mathcal{I}_{r_k}, \mathcal{I}_{r_{k+1}}) + e_L (\mathcal{I}_{r_k}))\right]$$

We now show how this upper bound proves the asymptotic stability of non-singular curves. It turns out that the estimate is, however, *not* sufficiently sharp to prove the asymptotic stability of curves with ordinary double points: more precisely, if $\mathcal{I}$ is the ideal associated to a 1-PS λ with normalized weights ρ_i then the estimate of the corollary may be greater than $\dfrac{2 \deg X}{n + 1} . \sum \rho_i$ (cf. Theorem 2.9)

THEOREM 4.12. *If* $C^1 \subset \mathbf{P}^N$ *is a linearly stable (resp.: semi-stable) curve, then* C *is Chow stable (resp.: semi-stable).*

Proof. We prove the stable case; the semi-stable case follows by replacing the strict inequalities in the proof by inequalities.

Fix coordinates $X_0, \ldots, X_N$ on $\mathbf{P}^N$ and a 1-PS

$$\lambda (t) = \begin{bmatrix} t^{\rho_0} & & & 0 \\ & \cdot & & \\ & & \cdot & \\ & & & \cdot \\ 0 & & & t^{\rho_N} \end{bmatrix}, \quad \rho_0 \geqq \rho_1 \geqq \ldots \geqq \rho_N = 0$$

Let $\mathscr{I}$ be the associated ideal on $\mathcal{O}_{C\times \mathbf{A}^1}$ and let $\mathscr{I}_k \subset \mathcal{O}_C$ be the ideal defined by $\mathscr{I}_k \,.\, L$ = [sheaf generated by $X_k, ..., X_N$]; thus $\mathscr{I} = \sum_{k=0}^{N} t^{\rho_k} \mathscr{I}_k$. The linear stability of X implies (cf. 2.16), $e(\mathscr{I}_k) < \frac{\deg C}{N} \,.\, \text{codim} < X_k, ..., X_N >$ $= \frac{\deg C \,.\, k}{N}$. So using Corollary 4.11,

$$e_L(\mathscr{I}) \leqq \min_{0=s_0<\ldots<s_k=N} \left[\sum (\rho_{s_k} - \rho_{s_{k+1}})(e_L(\mathscr{I}_{s_k}) + e_L(\mathscr{I}_{s_{k+1}}))\right]$$

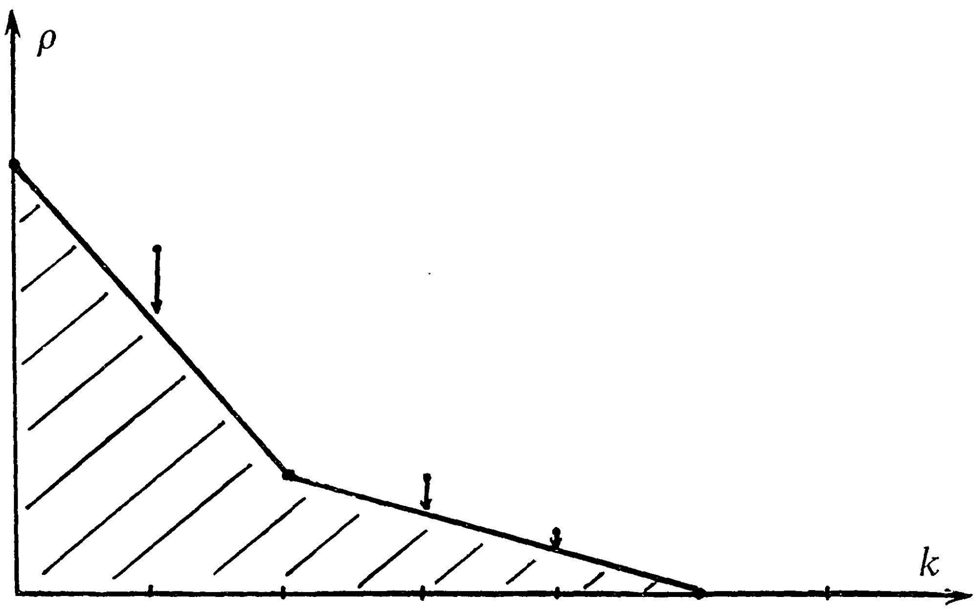
$$< \min_{0=s_0<\ldots<s_k=N} \left[\sum (\rho_{s_k} - \rho_{s_{k+1}})\,(s_k + s_{k+1})\,\frac{\deg C}{N}\right]$$

In view of the Lemma below this implies $e_L(\mathscr{I}) < \frac{2 \deg C}{N+1} \sum_{i=0}^{N} \rho_i$ which in turn implies C is stable by Theorem 2.9.

LEMMA 4.13. *If* $\rho_0 \geqq \ldots \geqq \rho_n = 0$, *then*

$$\min_{0=s_0<\ldots<s_l=n} \left[\sum (\rho_{s_k} - \rho_{s_{k+1}}) \cdot \left(\frac{s_k + s_{k+1}}{2}\right)\right] \leqq \frac{n}{n+1} \sum_{k=0}^{n} \rho_k$$

Proof. Draw the Newton polygon of the points (k, ρ_k) as shown below



The left hand side is just the area under this polygon so moving the points above the polygon down onto it as shown, does not affect this expression. Since this can only decrease the right hand side we may assume all the ρ_i are on this polygon. Then the left hand expression can be calculated with $s_k = k$ and it becomes

$$\frac{1}{2}\rho_0 + \rho_1 + \dots + \rho_{n-1} + \frac{1}{2}\rho_n = \rho_0 + \dots + \rho_n - \frac{1}{2}(\rho_0 + \rho_n)$$

$$\leqq \rho_0 + \dots + \rho_n - \frac{1}{n+1}(\rho_0 + \dots + \rho_n)$$

since the Newton polygon is convex. But the last expression is just $\frac{n}{n+1}(\rho_0 + \dots + \rho_n)$, hence the lemma.

THEOREM 4.14. *If* $C \subset \mathbf{P}^N$ *is a smooth curve embedded by* $\Gamma(C, L)$ *where* L *is a line bundle of degree* d, *then*

i) $d > 2g > 0 \Rightarrow C$ *linearly stable,*

ii) $d \geqq 2g \geqq 0 \Rightarrow C$ *linearly semi-stable.*

Combining this result with Theorem 4.13 gives the main theorem of this section:

THEOREM 4.15. *If* C *is a smooth curve of genus* $g \geqq 1$ *embedded by a complete linear system of degree* $d > 2g$ *then* C *is Chow-stable.*

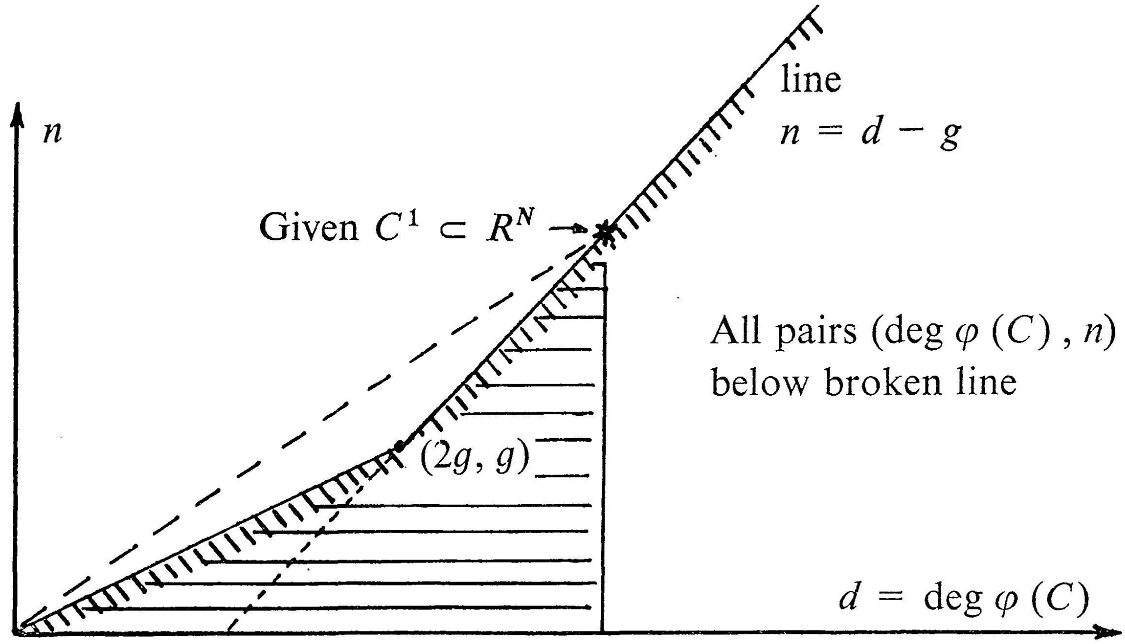
Proof of 4.14. Consider all morphisms $\varphi: C \to \mathbf{P}^n$ for all n, where $\varphi(C) \not\subset$ hyperplane. Let us plot the locus of pairs $(\deg \varphi(C), n)$, where $\varphi(C)$ is counted with multiplicity if φ is not birational. Note that, if $\varphi^*\mathcal{O}(1)$ is non-special, then by Riemann-Roch on C:

$$\begin{aligned} n &= \dim H^0(\mathcal{O}_{\mathbf{P}^n}(1)) - 1 \leqq \dim H^0(\varphi^*\mathcal{O}(1)) - 1 \\ &= \deg \varphi^*\mathcal{O}(1) - g = \deg \varphi(C) - g \end{aligned}$$

while if $\varphi^*\mathcal{O}(1)$ is special, then by Clifford's Theorem on C:

$$\begin{aligned} n &\leqq \dim H^0(\varphi^*\mathcal{O}(1)) - 1 \\ &\leqq \frac{\deg \varphi^*(\mathcal{O}(1))}{2} = \frac{\deg \varphi(C)}{2} \end{aligned}$$

This gives us the diagram



The reduced degree of $\varphi(C)$ is just d/n, the inverse of the slope of the joining (0, 0) to the plotted point (n, d). In case (i), by assumption, the given curve $C^1 \subset \mathbf{P}^N$ corresponds to a point on the upper bounding segment, such as * in our picture. Any projection of C corresponds to a point (n', d') in the shaded area with $d' \leq d$, $n' < n$. From the diagram it is clear that the slope decreases, or the reduced degree increases: this is exactly what linear stability means. In case (ii), we allow the given curve C to correspond to the vertex $(2g, g)$ of the boundary, or allow $g = 0$, when the boundary line is just $n = d$. In these cases, the slope at least cannot increase, or the reduced degree cannot decrease under projection.

Remark. Curves with ordinary double points are *not*, in general, linearly stable since projecting from a double point lowers the degree by 2, but decreases the dimension of the ambient space by only 1. In fact, linear stability is somewhat too strong a condition for most moduli problems: Chow stability for varieties of dimension r apparently allows points of multiplicity up to $(r+1)$! while linear stability allows only points of multiplicity up to r !