

3. An estimation by interpolation

Objektyp: **Chapter**

Zeitschrift: **L'Enseignement Mathématique**

Band (Jahr): **23 (1977)**

Heft 1-2: **L'ENSEIGNEMENT MATHÉMATIQUE**

PDF erstellt am: **20.09.2024**

Nutzungsbedingungen

Die ETH-Bibliothek ist Anbieterin der digitalisierten Zeitschriften. Sie besitzt keine Urheberrechte an den Inhalten der Zeitschriften. Die Rechte liegen in der Regel bei den Herausgebern.

Die auf der Plattform e-periodica veröffentlichten Dokumente stehen für nicht-kommerzielle Zwecke in Lehre und Forschung sowie für die private Nutzung frei zur Verfügung. Einzelne Dateien oder Ausdrucke aus diesem Angebot können zusammen mit diesen Nutzungsbedingungen und den korrekten Herkunftsbezeichnungen weitergegeben werden.

Das Veröffentlichen von Bildern in Print- und Online-Publikationen ist nur mit vorheriger Genehmigung der Rechteinhaber erlaubt. Die systematische Speicherung von Teilen des elektronischen Angebots auf anderen Servern bedarf ebenfalls des schriftlichen Einverständnisses der Rechteinhaber.

Haftungsausschluss

Alle Angaben erfolgen ohne Gewähr für Vollständigkeit oder Richtigkeit. Es wird keine Haftung übernommen für Schäden durch die Verwendung von Informationen aus diesem Online-Angebot oder durch das Fehlen von Informationen. Dies gilt auch für Inhalte Dritter, die über dieses Angebot zugänglich sind.

3. AN ESTIMATION BY INTERPOLATION

Lemma 2 reduces the problem of estimating the number of zeros to one of finding an upper bound for determinantal combinations of the shape

$$\left| \sum_{k=1}^{\sigma} \frac{\Delta_{\lambda,k}}{\Delta} g_k(w) \right|.$$

As we propose to discuss only some very special cases, we alert the reader on the one hand to the encyclopaedic Muir [14], and, for some determinants relevant in transcendence work, to van der Poorten [21].

LEMMA 3. Let $\omega_1, \dots, \omega_{\sigma}$ be complex numbers and denote by $D_{j,i}$ the cofactor of the typical element in the $\sigma \times \sigma$ determinant

$$D = |\omega_i^{j-1}|_{1 \leq i, j \leq \sigma}.$$

Let n be a positive integer, and write $\max_k |\omega_k| \leq \Omega$. Then for each $\lambda = 1, 2, \dots, \sigma$

$$(7) \quad \left| \sum_{k=1}^{\sigma} \frac{D_{\lambda,k}}{D} \frac{(\omega_k w)^{n-1}}{(n-1)!} \right| \leq \frac{1}{\Omega^{\lambda-1}} \sum_{h=1}^{\sigma} \frac{(\Omega|w|)^{h-1}}{(h-1)!} \frac{(\Omega|w|)^{n-h}}{(n-h)!} \binom{h-1}{\lambda-1}$$

Note. The quantity on the left of (7) remains well-defined by continuity even though the ω_k be not distinct. However, we treat the ω_k as formally distinct.

Proof. We commence by asserting that $\sum_{k=1}^{\sigma} \frac{D_{\lambda,k}}{D} \omega_k^{n-1}$ is the coefficient of $z^{\lambda-1}$ in the polynomial

$$(8) \quad P(z) = \sum_{k=1}^{\sigma} \omega_k^{n-1} \prod_{\substack{h=1 \\ h \neq k}}^{\sigma} \left(\frac{z - \omega_h}{\omega_k - \omega_h} \right)$$

To see this, observe that $P(z)$ is the unique polynomial of degree at most $\sigma - 1$ determined by the σ conditions (this is just Lagrange interpolation)

$$(9) \quad P(\omega_h) = \omega_h^{n-1}, \quad (h = 1, \dots, \sigma).$$

On the other hand, if

$$Q(z) = \sum_{\lambda=1}^{\sigma} \left(\sum_{k=1}^{\sigma} \frac{D_{\lambda,k}}{D} \omega_k^{n-1} \right) z^{\lambda-1},$$

then

$$Q(\omega_h) = \sum_{k=1}^{\sigma} \omega_k^{n-1} \left(\sum_{\lambda=1}^{\sigma} \omega_h^{\lambda-1} \frac{D_{\lambda,k}}{D} \right) = \sum_{k=1}^{\sigma} \omega_k^{n-1} \delta_{kh} = \omega_h^{n-1},$$

and it follows that $Q(z) \equiv P(z)$ as asserted.

To now evaluate the coefficients of $P(z)$ we expand P in a Newton interpolation series

$$(10) \quad P(z) = \sum_{h=1}^{\sigma} b_h (z - \omega_1) \dots (z - \omega_{h-1}),$$

and observe that by virtue of the residue formula we actually have

$$b_h = \frac{1}{2\pi i} \int_C \frac{P(\gamma)}{(\gamma - \omega_1) \dots (\gamma - \omega_h)} d\gamma = \frac{1}{2\pi i} \int_C \frac{\gamma^{n-1}}{(\gamma - \omega_1) \dots (\gamma - \omega_h)} d\gamma,$$

($h = 1, \dots, \sigma$),

where the contour C is, say, any circle about the origin of sufficiently large radius in order that C contain the points $\omega_1, \dots, \omega_{\sigma}$. The second, rather remarkable, equality is of course a consequence of the fact that the residue formula only “notices” P at the poles $\omega_1, \dots, \omega_h$, and at these points, (8) implies (9), so $P(\gamma)$ coincides with γ^{n-1} .

It is convenient to evaluate the second integral at its pole (if there is indeed such a pole) at ∞ . Accordingly we obtain

$$(11) \quad b_h = \frac{1}{2\pi i} \int_C \frac{\gamma^{n-1}}{(\gamma - \omega_1) \dots (\gamma - \omega_h)} d\gamma$$

$$= \frac{1}{2\pi i} \int_{C'} \frac{d\gamma}{\gamma^{n-h+1} (1 - \omega_1 \gamma) \dots (1 - \omega_h \gamma)}$$

where C' is now a circle about the origin of sufficiently small radius in order that C' not contain the points $\omega_1^{-1}, \dots, \omega_{\sigma}^{-1}$ (if some ω_k should vanish treat it as formally nonzero albeit arbitrarily small). It follows that b_h is exactly the coefficient of γ^{n-h} in the power series expansion about the origin of $\{(1 - \omega_1 \gamma) \dots (1 - \omega_h \gamma)\}^{-1}$, that is

$$(12) \quad |b_h| = \left| \sum_{|\mu|=n-h} \omega_1^{\mu(1)} \dots \omega_h^{\mu(h)} \right| \leq \binom{n-1}{h-1} \Omega^{n-h}.$$

It is now no longer of any matter that the ω_k not be distinct or that any should vanish. Inserting the estimate (12) in (10) we easily see that

$$(13) \quad \sum_{h=1}^{\sigma} \binom{h-1}{\lambda-1} \Omega^{h-\lambda} \binom{n-1}{h-1} \Omega^{n-h}$$

is an upper bound for the coefficient of $z^{\lambda-1}$ in the polynomial $P(z)$ of (8). Accordingly we have that

$$\begin{aligned} \left| \sum_{k=1}^{\sigma} \frac{D_{\lambda,k}}{D} \frac{(\omega_k w)^{h-1}}{(n-1)!} \right| &\leq \sum_{h=1}^{\sigma} \frac{|w|^{n-1}}{(n-1)!} \binom{n-1}{h-1} \binom{h-1}{\lambda-1} \Omega^{n-\lambda} \\ &= \frac{1}{(\lambda-1)!} \sum_{h=1}^{\sigma} \frac{\Omega^{h-\lambda}}{(h-\lambda)!} \frac{(\Omega|w|)^{n-h}}{(n-h)!} |w|^{h-1}, \end{aligned}$$

which is the assertion.

The following is essentially an immediate corollary of the previous lemma.

LEMMA 4. Let g be a function analytic in a sufficiently large disc about the origin and suppose that in that disc

$$(14) \quad g(z) = \sum_{n=1}^{\infty} \frac{c_{n-1}}{(n-1)!} z^{n-1}.$$

Let $g_k(z) = g(\omega_k z)$, ($k=1, \dots, \sigma$) and otherwise let the notation be as in lemma 3. Then if $|g|$ is the function

$$(15) \quad |g|(z) = \sum_{n=1}^{\infty} \frac{|c_{n-1}|}{(n-1)!} z^{n-1},$$

we have for each $\lambda = 1, \dots, \sigma$

$$(16) \quad \left| \sum_{k=1}^{\sigma} \frac{D_{\lambda,k}}{D} g(\omega_k w) \right| \leq \frac{1}{\Omega^{\lambda-1}} \sum_{h=1}^{\sigma} \frac{(\Omega|w|)^{h-1}}{(h-1)!} |g|^{(h-1)}(\Omega|w|) \binom{h-1}{\lambda-1}$$

Proof. By lemma 3 we have

$$\begin{aligned} \left| \sum_{k=1}^{\sigma} \frac{D_{\lambda,k}}{D} g(\omega_k w) \right| &\leq \left| \sum_{k=1}^{\sigma} \sum_{n=1}^{\infty} \frac{D_{\lambda,k}}{D} c_{n-1} \frac{(\omega_k w)^{n-1}}{(n-1)!} \right| \\ &\leq \frac{1}{\Omega^{\lambda-1}} \sum_{h=1}^{\sigma} \frac{(\Omega|w|)^{h-1}}{(h-1)!} \binom{h-1}{\lambda-1} \sum_{n=1}^{\infty} |c_{n-1}| \frac{(\Omega|w|)^{n-h}}{(n-h)!}, \end{aligned}$$

which is the assertion.

The critical aspect of the above estimates is that they are independent of $\min_{h \neq k} |\omega_k - \omega_h| = d$. The interpolation method of lemma 3 is not at all new nor is the idea of obtaining results independent of d . The latter seems appropriately attributable to Turán [30], whilst the former occurs in Makai [11], [12] in the context of our problem. The interpolation method appears in a more general way in the thesis of van der Poorten [16], and thence in the papers [17], [18] [19]. However the recognition of the general

pattern is due to Tijdeman [26], whence see Balkema and Tijdeman [1]. For further details see the references cited in the papers mentioned above.

4. EXPONENTIAL POLYNOMIALS

We commence by making explicit some folklore the principles of which can be found in [16] and Tijdeman [26], and which is made explicit in another context in van der Poorten [20].

LEMMA 5. *For some fixed positive integer σ , and some given function g , supposed holomorphic in the domain under consideration, denote by J the set of functions G of the shape*

$$G(z) = \sum_{k=1}^{\sigma} b_k g(\alpha_k z),$$

where $b_1, \dots, b_{\sigma}; \alpha_1, \dots, \alpha_{\sigma}$ are complex numbers. Then, for all sets of non-negative integers $\rho(1), \dots, \rho(m)$ with sum $\sum_{h=1}^m \rho(h) = \sigma$ (and all positive integers m such that $1 \leq m \leq \sigma$), for each function F of the shape

$$F(z) = \sum_{h=1}^m \sum_{t=1}^{\rho(h)} a_{ht} z^{t-1} g^{(t-1)}(\omega_h z),$$

the a_{ht} complex constants, there is a sequence of functions in J converging uniformly to F in compact sets.

Proof. The lemma depends upon noticing that functions of the shape F are actually, in a sense, particular cases of, rather than generalisations of functions of the shape G . Indeed, reindex so that G appears as

$$(17) \quad G(z) = \sum_{h=1}^m \sum_{t=1}^{\rho(h)} b_{ht} g(\omega_{ht} z),$$

and choose the coefficients b_{ht} as functions of $\omega_{11}, \dots, \omega_{m\rho(m)}$ (so of $\alpha_1, \dots, \alpha_{\sigma}$) so that for each $h = 1, \dots, m$

$$(18) \quad \sum_{t=1}^{\rho(h)} b_{ht} g(\omega_{ht} z) = \sum_{t=1}^{\rho(h)} a_{ht} \frac{(t-1)!}{2\pi i} \int_C g(\gamma z) \prod_{s=1}^t (\gamma - \omega_h s)^{-1} d\gamma,$$

where the closed contour C contains all the ω_{ht} but excludes any singularities of g . Clearly there exists a sequence of σ -tuples $(\omega_{11}, \dots, \omega_{m\rho(m)})$ which converges to $(\omega_1, \dots, \omega_1; \omega_2, \dots, \omega_m)$ componentwise, and in the limit, (18) shows that (17) becomes $F(z)$.

I am indebted to D. W. Masser for any felicities in the terminology used in the lemma.