3. An estimation by interpolation

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Lemma 2 reduces the problem of estimating the number of zeros to one of finding an upper bound for determinantal combinations of the shape

$$\left| \sum_{k=1}^{\sigma} \frac{\Delta_{\lambda,k}}{\Delta} g_k(w) \right| .$$

As we propose to discuss only some very special cases, we alert the reader on the one hand to the encyclopaedic Muir [14], and, for some determinants relevant in transcendence work, to van der Poorten [21].

Lemma 3. *Let* $\omega_1, ..., \omega_\sigma$ *be complex numbers and denote by* $D_{j,i}$ *the cofactor of the typical element in the* $\sigma \times \sigma$ *determinant*

$$D = | \omega_i^{j-1} |_{1 \leq i,j \leq \sigma} .$$

Let n *be a positive integer, and write* $\max_k | \omega_k | \leqslant \Omega$. *Then for each* $\lambda = 1, 2,..., \sigma$

$$(7) \qquad \left| \sum_{k=1}^{\sigma} \frac{D_{\lambda,k}}{D} \frac{(\omega_k w)^{n-1}}{(n-1)!} \right| \leqslant \frac{1}{\Omega^{\lambda-1}} \sum_{h=1}^{\sigma} \frac{(\Omega |w|)^{h-1}}{(h-1)!} \frac{(\Omega |w|)^{n-h}}{(n-h)!} \binom{h-1}{\lambda-1}$$

Note. The quantity on the left of (7) remains well-defined by continuity even though the ω_k be not distinct. However, we treat the ω_k as formally distinct.

Proof. We commence by asserting that $\sum_{k=1}^{\sigma} \frac{D_{\lambda,k}}{D} \omega_k^{n-1}$ is the coefficient of $z^{\lambda-1}$ in the polynomial

$$(8) \qquad P(z) = \sum_{k=1}^{\sigma} \omega_k^{n-1} \prod_{\substack{h=1 \\ h \neq k}}^{\sigma} \left(\frac{z - \omega_h}{\omega_k - \omega_h} \right)$$

To see this, observe that $P(z)$ is the unique polynomial of degree at most $\sigma - 1$ determined by the σ conditions (this is just Lagrange interpolation)

$$(9) \qquad P(\omega_h) = \omega_h^{n-1} , \; (h = 1, ..., \sigma) .$$

On the other hand, if

$$Q(z) = \sum_{\lambda=1}^{\sigma} \left(\sum_{k=1}^{\sigma} \frac{D_{\lambda,k}}{D} \omega_k^{n-1} \right) z^{\lambda-1} ,$$

then

$$Q(\omega_h) = \sum_{k=1}^{6} \omega_k^{n-1} \left(\sum_{\lambda=1}^{\sigma} \omega_h^{\lambda-1} \frac{D_{\lambda,k}}{D} \right) = \sum_{k=1}^{\sigma} \omega_k^{n-1} \delta_{kh} = \omega_h^{n-1},$$

and it follows that $Q(z) \equiv P(z)$ as asserted.

To now evaluate the coefficients of $P(z)$ we expand P in a Newton interpolation series

(10) $$P(z) = \sum_{h=1}^{\sigma} b_h (z-\omega_1)\dots(z-\omega_{h-1}),$$

and observe that by virtue of the residue formula we actually have

$$b_h = \frac{1}{2\pi i} \int_C \frac{P(\gamma)}{(\gamma-\omega_1)\dots(\gamma-\omega_h)} d\gamma = \frac{1}{2\pi i} \int_C \frac{\gamma^{n-1}}{(\gamma-\omega_1)\dots(\gamma-\omega_h)} d\gamma,$$
$$(h = 1, \dots, \sigma),$$

where the contour C is, say, any circle about the origin of sufficiently large radius in order that C contain the points $\omega_1, \dots, \omega_\sigma$. The second, rather remarkable, equality is of course a consequence of the fact that the residue formula only " notices " P at the poles $\omega_1, \dots, \omega_h$, and at these points, (8) implies (9), so $P(\gamma)$ coincides with γ^{n-1}.

It is convenient to evaluate the second integral at its pole (if there is indeed such a pole) at ∞. Accordingly we obtain

(11) $$b_h = \frac{1}{2\pi i} \int_C \frac{\gamma^{n-1}}{(\gamma-\omega_1)\dots(\gamma-\delta_h)} d\gamma$$
$$= \frac{1}{2\pi i} \int_{C'} \frac{d\gamma}{\gamma^{n-h+1}(1-\omega_1\gamma)\dots(1-\omega_h\gamma)}$$

where C' is now a circle about the origin of sufficiently small radius in order that C' not contain the points $\omega_1^{-1}, \dots, \omega_\sigma^{-1}$ (if some ω_k should vanish treat it as formally nonzero albeit arbitrarily small). It follows that b_h is exactly the coefficient of γ^{n-h} in the power series expansion about the origin of $\{(1 - \omega_1 \gamma) \dots (1 - \omega_h \gamma)\}^{-1}$, that is

(12) $$|b_h| = |\sum_{|\mu|=n-h} \omega_1^{\mu(1)} \dots \omega_h^{\mu(h)}| \leqslant \binom{n-1}{h-1} \Omega^{n-h}.$$

It is now no longer of any matter that the ω_k not be distinct or that any should vanish. Inserting the estimate (12) in (10) we easily see that

(13) $$\sum_{h=1}^{\sigma} \binom{h-1}{\lambda-1} \Omega^{h-\lambda} \binom{n-1}{h-1} \Omega^{n-h}$$

is an upper bound for the coefficient of $z^{\lambda-1}$ in the polynomial $P(z)$ of (8). Accordingly we have that

$$\left| \sum_{k=1}^{\sigma} \frac{D_{\lambda,k}}{D} \frac{(\omega_k w)^{h-1}}{(n-1)!} \right| \leqslant \sum_{h=1}^{\sigma} \frac{|w|^{n-1}}{(n-1)!} \binom{n-1}{h-1} \binom{h-1}{\lambda-1} \Omega^{n-\lambda}$$
$$= \frac{1}{(\lambda-1)!} \sum_{h=1}^{\sigma} \frac{\Omega^{h-\lambda}}{(h-\lambda)!} \frac{(\Omega|w|)^{n-h}}{(n-h)!} |w|^{h-1},$$

which is the assertion.

The following is essentially an immediate corollary of the previous lemma.

LEMMA 4. *Let* g *be a function analytic in a sufficiently large disc about the origin and suppose that in that disc*

$$g(z) = \sum_{n=1}^{\infty} \frac{c_{n-1}}{(n-1)!} z^{n-1} . \tag{14}$$

Let $g_k(z) = g(\omega_k z)$, $(k=1, \ldots, \sigma)$ and *otherwise let the notation be as in lemma 3. Then if* $|g|$ *is the function*

$$|g|(z) = \sum_{n=1}^{\infty} \frac{|c_{n-1}|}{(n-1)!} z^{n-1} , \tag{15}$$

we have for each $\lambda = 1, \ldots, \sigma$

$$\left| \sum_{k=1}^{\sigma} \frac{D_{\lambda,k}}{D} g(\omega_k w) \right| \leqslant \frac{1}{\Omega^{\lambda-1}} \sum_{h=1}^{\sigma} \frac{(\Omega|w|)^{h-1}}{(h-1)!} |g|^{(h-1)}(\Omega|w|) \binom{h-1}{\lambda-1} \tag{16}$$

Proof. By lemma 3 we have

$$\left| \sum_{k=1}^{\sigma} \frac{D_{\lambda,k}}{D} g(\omega_k w) \right| \leqslant \left| \sum_{k=1}^{\sigma} \sum_{n=1}^{\infty} \frac{D_{\lambda,k}}{D} c_{n-1} \frac{(\omega_k w)^{n-1}}{(n-1)!} \right|$$
$$\leqslant \frac{1}{\Omega^{\lambda-1}} \sum_{h=1}^{\sigma} \frac{(\Omega|w|)^{h-1}}{(h-1)!} \binom{h-1}{\lambda-1} \sum_{n=1}^{\infty} |c_{n-1}| \frac{(\Omega|w|)^{n-h}}{(n-h)!},$$

which is the assertion.

The critical aspect of the above estimates is that they are independent of $\min_{h \neq k} |\omega_k - \omega_h| = d$. The interpolation method of lemma 3 is not at all new nor is the idea of obtaining results independent of d. The latter seems appropriately attributable to Turán [30], whilst the former occurs in Makai [11], [12] in the context of our problem. The interpolation method appears in a more general way in the thesis of van der Poorten [16], and thence in the papers [17], [18] [19]. However the recognition of the general pattern is due to Tijdeman [26], whence see Balkema and Tijdeman [1]. For further details see the references cited in the papers mentioned above.

4. Exponential polynomials

We commence by making explicit some folklore the principles of which can be found in [16] and Tijdeman [26], and which is made explicit in another context in van der Poorten [20].

Lemma 5. *For some fixed positive integer* σ, *and some given function* g, *supposed holomorphic in the domain under consideration, denote by* J *the set of functions* G *of the shape*

$$G(z) = \sum_{k=1}^{\sigma} b_k g(\alpha_k z),$$

where $b_1, \ldots, b_\sigma; \alpha_1, \ldots, \alpha_\sigma$ *are complex numbers. Then, for all sets of non-negative integers* $\rho(1), \ldots, \rho(m)$ with *sum* $\sum_{h=1}^{m} \rho(h) = \sigma$ *(and all positive integers* m *such that* $1 \leqslant m \leqslant \sigma$*), for each function* F *of the shape*

$$F(z) = \sum_{h=1}^{m} \sum_{t=1}^{\rho(h)} a_{ht} z^{t-1} g^{(t-1)}(\omega_h z),$$

the a_{ht} *complex constants, there is a sequence of functions in* J *converging uniformly to* F *in compact sets.*

Proof. The lemma depends upon noticing that functions of the shape F are actually, in a sense, particular cases of, rather than generalisations of functions of the shape G. Indeed, reindex so that G appears as

$$G(z) = \sum_{h=1}^{m} \sum_{t=1}^{\rho(h)} b_{ht} g(\omega_{ht} z), \tag{17}$$

and choose the coefficients b_{ht} as functions of $\omega_{11}, \ldots, \omega_{m\rho(m)}$ (so of $\alpha_1, \ldots, \alpha_\sigma$) so that for each $h = 1, \ldots, m$

$$\sum_{t=1}^{\rho(h)} b_{ht} g(\omega_{ht} z) = \sum_{t=1}^{\rho(h)} a_{ht} \frac{(t-1)!}{2\pi i} \int_C g(\gamma z) \prod_{s=1}^{t} (\gamma - \omega_{hs})^{-1} d\gamma, \tag{18}$$

where the closed contour C contains all the ω_{ht} but excludes any singularities of g. Clearly there exists a sequence of σ-tuples $(\omega_{11}, \ldots, \omega_{m\rho(m)})$ which converges to $(\omega_1, \ldots, \omega_1; \omega_2, \ldots, \omega_m)$ componentwise, and in the limit, (18) shows that (17) becomes $F(z)$.

I am indebted to D. W. Masser for any felicities in the terminology used in the lemma.