

§5. The set of linear superpositions in the space of continuous functions is closed

Objekttyp: **Chapter**

Zeitschrift: **L'Enseignement Mathématique**

Band (Jahr): **23 (1977)**

Heft 1-2: **L'ENSEIGNEMENT MATHÉMATIQUE**

PDF erstellt am: **19.04.2024**

Nutzungsbedingungen

Die ETH-Bibliothek ist Anbieterin der digitalisierten Zeitschriften. Sie besitzt keine Urheberrechte an den Inhalten der Zeitschriften. Die Rechte liegen in der Regel bei den Herausgebern.

Die auf der Plattform e-periodica veröffentlichten Dokumente stehen für nicht-kommerzielle Zwecke in Lehre und Forschung sowie für die private Nutzung frei zur Verfügung. Einzelne Dateien oder Ausdrucke aus diesem Angebot können zusammen mit diesen Nutzungsbedingungen und den korrekten Herkunftsbezeichnungen weitergegeben werden.

Das Veröffentlichen von Bildern in Print- und Online-Publikationen ist nur mit vorheriger Genehmigung der Rechteinhaber erlaubt. Die systematische Speicherung von Teilen des elektronischen Angebots auf anderen Servern bedarf ebenfalls des schriftlichen Einverständnisses der Rechteinhaber.

Haftungsausschluss

Alle Angaben erfolgen ohne Gewähr für Vollständigkeit oder Richtigkeit. Es wird keine Haftung übernommen für Schäden durch die Verwendung von Informationen aus diesem Online-Angebot oder durch das Fehlen von Informationen. Dies gilt auch für Inhalte Dritter, die über dieses Angebot zugänglich sind.

$$\lambda_i = \min_{j \leq i} \inf_{t \in q_j(G_i)} \lambda(t, G_i, q_j, \{p_i^k\}) > \frac{1}{2} \min \left\{ \frac{\delta}{2}, \min_{j < i} \lambda_j \right\}.$$

Thus, the regular regions $G_1, G_2, \dots, G_n$ can be constructed. The regular region $G = G_n$ satisfies all the requirements of our lemma ($\lambda = \lambda_n$), which is now proved.

§ 5. *The set of linear superpositions in the space of continuous functions is closed*

THEOREM 4.5.1. *Suppose that continuous functions $p_m(x, y)$ and continuously differentiable functions $q_m(x, y)$ ($m=1, 2, \dots, N$) are fixed. Then in any region D of the plane of the variables x, y , there exists a closed subregion $G \subset D$ such that the set of superpositions of the form*

$$\sum_{m=1}^N p_m(x, y) f_m(q_m(x, y)),$$

where $\{f_m(t)\}$ are arbitrary continuous functions, is closed (in the uniform metric) in the set of all functions continuous on the set G .

By Lemma 4.2.2 and 4.4.3 we can find a subset $G \subset D$, determine constants $\gamma > 0$ and $\lambda > 0$, and renumber the functions $\{p_m(x, y)\}$ and $\{q_m(x, y)\}$ with two indices so that the functions obtained after the renumbering, $\{p_i^k(x, y)\}$ and $\{q_i^k(x, y)\}$ ($i=0, 1, 2, \dots, n; k=1, 2, \dots, m_i; \sum_{i=0}^n m_i \leq N$) that is, some functions may be omitted in the renumbering) satisfy conditions (1), (2), (3) of Lemma 4.2.2, and also the conditions:

(4') for any continuous functions $\{f_m(t)\}$ there exists continuous functions $\{f_i^k(t)\}$ such that on G

$$\sum_{m=1}^N p_m(x, y) f_m(q_m(x, y)) = \sum_{i=0}^n \sum_{k=1}^{m_i} p_i^k(x, y) f_i^k(q_i^k(x, y));$$

(5') for every i and $t \in q_i^1(G)$ and for any functions $\{f_i^k(t)\}$

$$\max_{(x, y) \in e(q_i^1, t) \cap G} \left| \sum_{k=1}^{m_i} p_i^k(x, y) f_i^k(q_i^1(x, y)) \right| \leq \lambda \max_k |f_i^k(t)|;$$

(6') G is a regular region with respect to the functions $\{q_i^k(x, y)\}$.

LEMMA 4.5.1. In the sets $\{q_i^1(G)\}$ we can select subsets consisting of a finite number of points $t_{i,j} \in q_i^1(G)$ ($i=0, 1, 2, \dots, n$; $j=1, 2, \dots, s_i$) such that for any continuous functions $\{f_i^k(t)\}$

$$\max_{i,k} \max_{t \in q_i^1(G)} |f_i^k(t)| \leq C \left(\max_{(x,y) \in G} \left| \sum_{i=0}^n \sum_{k=1}^{m_i} p_i^k(x,y) f_i^k(q_i^1(x,y)) \right| + \max_k |f_i^k(t_{i,j})| \right),$$

where C is a constant not depending on the functions $\{f_i^k(t)\}$.

Proof. Since G is polyhedral, for each i we can choose in $q_i(G)$ a finite set of points $\{t_{i,j}\}$ so dense that the components of the level curves $e(q_i^1, t_{i,j}) \cap G$ form a δ -net in the set of all components of the level curves $e(q_i^1, t) \cap G$, $t \in q_i^1(G)$. A sufficiently small δ , not depending on the functions $\{f_i^k(t)\}$, will be chosen below. We put

$$\mu = \max_{i,k} \max_{(x,y) \in G} |f_i^k(q_i^1(x,y))|;$$

$$\varepsilon_1 = \max_{(x,y) \in G} \left| \sum_{i=0}^n \sum_{k=1}^{m_i} p_i^k(x,y) f_i^k(q_i^1(x,y)) \right|; \quad \varepsilon_2 = \max_{k,i,j} |f_i^k(t_{i,j})|.$$

For definiteness, let $f_1^1(q_1^1(a)) = \mu$ at the point $a \in G$. By (5') there exists a point $a' \in G$ such that $\left| \sum_{k=1}^{m_1} p_1^k(a') f_1^k(q_1^1(a')) \right| \geq \lambda \mu$. Let $[a', a^*]$ be a segment of the level curve of the function $q_1^1(x,y)$ with end-points at a' and a^* such that $h_1([a', a^*]) \geq \gamma G/2$ (see the definition of a regular region in § 4). On the arc $[a', a^*]$ we fix a point a'' such that $\omega(\alpha) \leq \frac{\lambda}{2m_1}$, where $\alpha = h_1([a', a''])$. Then on the segment $[a', a'']$ the function $\varphi_1(x,y) = \sum_{k=1}^{m_1} p_1^k(x,y) f_1^k(q_1^1(x,y))$ keeps a constant sign and satisfies the inequality $|\varphi_1(x,y)| \geq \lambda \mu/2$. In fact, $|\varphi_1(a')| \geq \lambda \mu$ at the point a' , and for any point $s \in [a', a'']$

$$|\varphi_1(s) - \varphi_1(a')| = \left| \sum_{k=1}^{m_1} (p_1^k(s) - p_1^k(a')) f_1^k(a') \right| \leq m_1 \mu \omega(\alpha) \leq \frac{\lambda \mu}{2}.$$

Consequently,

$$\left| \int_{s \in [a', a'']} \varphi_1(s) ds \right| \geq \frac{1}{2} \lambda \mu \alpha.$$

By construction there is an index j and a segment $[b', b'']$ of the level curve $e(q_1^1, t_{1,j}) \cap G$ such that $\rho([a', a''], [b', b'']) < \delta$. We have

$$\left| \int_{s \in [b', b'']} \varphi_1(s) ds \right| \leq c_1 \varepsilon_2 \beta,$$

where $\beta = h_1([b', b''])$, $C_1 = m_1 \max_k \max_{(x, y) \in G} |p_1^k(x, y)|$. And since α and β are commensurable (δ will be chosen small in comparison with α),

$$\left| \int_{s \in [a', a'']} \varphi_1(s) ds - \int_{s \in [b', b'']} \varphi_1(s) ds \right| \geq \frac{1}{2} \lambda \mu \alpha - c'_1 \varepsilon_2 \alpha.$$

By Lemma 4.2.3

$$\left| \int_{s \in [a', a'']} \varphi_1(s) ds - \int_{s \in [b', b'']} \varphi_1(s) ds \right| \leq c_3 (\alpha \varepsilon_1 + \mu \alpha \omega(\delta) + \mu \delta).$$

Thus, $c_3 (\alpha \varepsilon_1 + \mu \alpha \omega(\delta) + \mu \delta) \geq \lambda \mu \alpha / 2 - c'_1 \alpha \cdot \varepsilon_2$. If δ is taken sufficiently small in comparison with α (in order that $c_3 (\alpha \omega(\delta) + \delta) < \lambda \alpha / 2$), then we have $\mu \leq C (\varepsilon_1 + \varepsilon_2)$. This proves the lemma.

Let B be the Banach space consisting of all systems of functions $\{f_i^k(t)\}$, defined and continuous on the sets $\{q_i^1(G)\}$, with the norm

$$\|\{f_i^k(t)\}\|_B = \max_{i, k} \max_{t \in q_i^1(G)} |f_i^k(t)| \quad (i=0, 1, 2, \dots, n; k=1, 2, \dots, m_i).$$

We denote by $C(G)$ the space of all functions $f(x, y)$ continuous on G with the uniform metric:

$$\|f(x, y)\|_{C(G)} = \max_{(x, y) \in G} |f(x, y)|.$$

LEMMA 4.5.2. *The linear operator $T: B \rightarrow C(G)$ acting by the formula*

$$T(\{f_i^k(t)\}) = f(x, y) = \sum_{i=0}^n \sum_{k=1}^{m_i} p_i^k(x, y) f_i^k(q_i^1(x, y)),$$

maps bounded closed sets of B onto closed sets of $C(G)$.

Proof. Let $F \subset B$ be a closed and bounded set of elements of B . Suppose that $f_n(x, y)$ is a sequence of functions in $T(F) \subset C(G)$, and that $f(x, y) \in C(G)$, where $\|f(x, y) - f_n(x, y)\|_{C(G)} \rightarrow 0$ as $n \rightarrow \infty$. We show that then $f(x, y) \in T(F)$. Since $f_n(x, y) \in T(F)$, there exists a sequence of elements $\{f_{i,n}^k(t)\} \in F$ such that $T(\{f_{i,n}^k(t)\}) = f_n(x, y)$. By Lemma 4.5.1 we can select in the sets $\{q_i^1(G)\}$ subsets consisting of a finite number of points $t_{i,j} \in q_i^1(G)$ ($i=0, 1, \dots, n; j=1, 2, \dots, s_i$) such that for each element $\{f_i^k(t)\} \in B$ the inequality

$$\|\{f_i^k(t)\}\|_B \leq c (\|f(x, y)\|_{C(G)} + \max_{k, j, i} |f_i^k(t_{i,j})|),$$

is satisfied, where the constant C does not depend on the functions $\{f_i^k(t)\}$. Since F is a bounded set, there exists a subsequence of suffixes $n_1, n_2, \dots$ such that for any $i = 0, 1, \dots, n$; $k = 1, 2, \dots, m_i$; $j = 1, 2, \dots, s_i$ the numerical sequence $f_{i,n_v}^k \rightarrow C_{k,i,j}$ as $v \rightarrow \infty$. From this and the previous inequality it follows that $\{f_{i,n_v}^k(t)\} \in F (v=1, 2, \dots)$ is a Cauchy sequence, because it is known that the sequence $f_n(x, y) \in T(F)$ is Cauchy sequence. Consequently there exists an element $\{f_i^k(t)\} \in B$ such that $\|\{f_i^k(t) - f_{i,n_v}^k(t)\}\|_B \rightarrow 0$. Since F is a closed set, $\{f_i^k(t)\} \in F$. The operator $T: B \rightarrow C(G)$ is bounded. Therefore $T(\{f_i^k(t)\}) = f(x, y)$. Consequently $f(x, y) \in T(F)$. This proves the lemma.

The following lemma from the theory of linear operators [28] turns out to be useful.

LEMMA 4.5.3. *Let B_1 and B_2 be Banach spaces. If a linear operator $T: B_1 \rightarrow B_2$ maps bounded closed sets of B_1 onto closed sets of B_2 , then its domain of values is closed.*

Proof of Theorem 4.5.1. The set of superpositions of the form $\sum_{m=1}^N p_m(x, y) f_m(g_m(x, y))$ coincides on G with the set of superpositions of the form $\sum_{i=0}^n \sum_{k=1}^{m_i} p_i^k(x, y) f_i^k(q_i^1(x, y))$. By Lemma 4.5.2 and 4.5.3 the set of the latter superpositions is closed in the space $C(G)$. This proves the theorem.

§ 6. *The set of linear superpositions in the space of continuous functions is nowhere dense*

THEOREM 4.6.1. *For any continuous functions $p_m(x, y)$ and continuously differentiable functions $q_m(x, y)$ ($m=1, 2, \dots, N$) and any region D of the plane of the variables x, y the set of superpositions of the form*

$$\sum_{m=1}^N p_m(x, y) f_m(q_m(x, y)),$$

where $\{f_m(t)\}$ are arbitrary continuous functions, is nowhere dense in the space of all functions continuous in D with uniform convergence.

By Lemma 4.2.2 we can find a subregion $G^* \subset D$, determine a constant $\gamma^* > 0$, and renumber the functions $\{q_m(x, y)\}$, with two indices so that