

§2. The theorem of Kahane

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2. The function $t_q = \sum_{p=1}^n \lambda_p \varphi_q(x_p)$ maps the cube $\mathcal{J}^n$ into the circle $|t| = 1$.

3. The transformation Ψ given by the equalities $t_q = \sum_{p=1}^n \lambda_p \varphi_q(x_p)$ ($q=1, \dots, 2n+1$) is one-to-one on $\mathcal{J}^n$.

4. For any function f continuous on $\mathcal{J}^n$ there exists a function $g(z)$ continuous on the disk $|z| \leq 1$, holomorphic inside that disk, and such that $f = \sum_{q=1}^{2n+1} g\left(\sum_{p=1}^n \lambda_p \varphi_q(x_p)\right)$.

The transformation Ψ gives an embedding of the cube $\mathcal{J}^n$ into the torus $|t| = 1$ ($q=1, \dots, 2n+1$) such that any function continuous on the cube $\tilde{\mathcal{J}}^n = \Psi(\mathcal{J}^n)$ is represented in the form $f(t_1, \dots, t_{2n+1}) = \sum_{q=1}^{2n+1} g(t_q)$, where g is a function holomorphic in the unit disk. This means in particular that any function continuous on $\tilde{\mathcal{J}}^n$ has an analytic extension to the polydisk $|t_q| \leq 1$ ($q=1, \dots, 2n+1$).

§ 2. The theorem of Kahane

Let M be a complete metric space. We recall that a set is called a set of second category if it is the intersection of a countable family of open sets which are everywhere dense in M . By the theorem of Baire in a complete metric space no set of second category is empty. The massivity of such sets is characterized by the fact that the intersection of a countable family of sets of second category is again a set of second category and consequently is not empty.

We will say that a statement is true for quasi every element of M if it is true for a set of elements of second category.

Let us consider an example. Let Φ be the space with uniform norm consisting of all functions continuous and non-decreasing on the segment $\mathcal{J}^1$ ($0 \leq t \leq 1$). It can be shown easily that quasi every element of Φ is a strictly increasing function.

In fact, any strictly increasing function belongs to any set defined as $\varphi(r') < \varphi(r'')$, where $r' < r''$ are fixed rational numbers. Any set defined by an inequality of that type is open and everywhere dense in Φ , and the set of all such sets is countable.

Let $\mathcal{J}^n$ be the cube $\{0 \leq x_i \leq 1, i = 1, \dots, n\}$; $C(\mathcal{J}^n)$ -the space of all functions continuous on $\mathcal{J}^n$ with the uniform norm; Φ -the space of functions continuous and non-decreasing on the segment $\mathcal{J}^1$ (with the uniform norm); $\Phi^k = \Phi \times \dots \times \Phi$ the k -th power of the space Φ .

THEOREM 3.2.1. *Let λ_p ($p=1, \dots, n$) be a collection of rationally independent constants. Then for quasi every collection $\{\varphi_1, \dots, \varphi_{2n+1}\} \in \Phi^{2n+1}$ it is true that any function $f \in C(\mathcal{J}^n)$ can be represented on $\mathcal{J}^n$ in the form*

$$f(x) = \sum_{q=1}^{2n+1} g\left(\sum_{p=1}^n \lambda_p \varphi_q(x_p)\right),$$

where g is a continuous function.

§ 3. The main lemma

We fix a function $f \in C(\mathcal{J}^n)$, positive numbers λ_p ($p=1, \dots, n$) and a positive ε . We will denote by Ω_f the set of all collections $\{\varphi_1, \dots, \varphi_{2n+1}\} \in \Phi^{2n+1}$ for each of which there exists a continuous function h such that $\|h\| \leq \|f\|$ and $\|f(x) - \sum_{q=1}^{2n+1} h\left(\sum_{p=1}^n \lambda_p \varphi_q(x_p)\right)\| < (1-\varepsilon)\|f\|$. The latter inequality is strict and consequently the set Ω_f is open.

The idea of the construction is contained in the following statement.

LEMMA 3.3.1. *If $\|f\| \neq 0$, the numbers $\{\lambda_p\}$ are rationally independent, and $0 < \varepsilon < \frac{1}{2n+2}$, then the corresponding set Ω_f is everywhere dense in Φ^{2n+1} .*

Proof. Let us fix an open set $\Omega \subset \Phi^{2n+1}$ and prove that $\Omega \cap \Omega_f$ is not empty. This will imply that Ω_f is everywhere dense in Φ^{2n+1} .

We choose a number $\delta > 0$ and denote by $\mathcal{J}_q(j)$ the segment defined by the inequality

$$q \cdot \delta + (2n+1)j \cdot \delta \leq t \leq q \cdot \delta + (2n+1)j\delta + 2n\delta$$

$$(q=1, \dots, 2n+1, j \text{ is an integer}).$$

The value δ will be determined below. Now we notice, firstly, that for any q the segments $\mathcal{J}_q(j)$ ($j=0, \pm 1, \pm 2$) are pairwise disjoint and every two consecutive segments are separated by an interval of length δ and, secondly, that, every point of the real axis belongs to at least $2n$ of the sets $\sum_j \mathcal{J}_q(j)$, ($q=1, \dots, 2n+1$).