

# §1. Certain improvements of Kolmogorov 's theorem

Objekttyp: **Chapter**

Zeitschrift: **L'Enseignement Mathématique**

Band (Jahr): **23 (1977)**

Heft 1-2: **L'ENSEIGNEMENT MATHÉMATIQUE**

PDF erstellt am: **22.09.2024**

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LEMMA 2.3.3. *If  $\frac{n}{s} > \frac{n'}{s'}$  then for any natural  $k$  the set  $\Omega_k \cap C_s(\mathcal{I}^n)$  is nowhere dense in  $C_s(\mathcal{I}^n)$ .*

By lemma 2.3.1 and the theorem 2.2.1 for any natural  $k$   $H_\varepsilon(\Omega_k) \leq C \left(\frac{1}{\varepsilon}\right)^{n'/s'}$ , where  $C$  does not depend on  $\varepsilon$ . Hence, it follows from the inequality  $\frac{n}{s} > \frac{n'}{s'}$  and lemma 2.3.2 that the set  $\Omega_k \cap C_s(\mathcal{I}^n)$  is nowhere dense in  $C_s(\mathcal{I}^n)$ .

Now to prove the theorem we have to notice only that the set of functions from  $C_s(\mathcal{I}^n)$  representable by superpositions coincides with  $\bigcup_{k=1}^{\infty} (\Omega_k \cap C_s(\mathcal{I}^n))$ . By lemma 2.3.3 the sets  $\{\Omega_k \cap C_s(\mathcal{I}^n)\}$  are nowhere dense and consequently the set of not representable functions is a set of second category.

### CHAPTER 3. — SUPERPOSITIONS OF CONTINUOUS FUNCTIONS

In this chapter we present the proof of the theorem of Kolmogorov given by Kahane [36]. This proof which is based on Baire's theory contains a minimum of concrete constructions and shows that there exists a wide choice of inner functions for Kolmogorov's formula.

#### § 1. *Certain improvements of Kolmogorov's theorem*

By the theorem of Kolmogorov any function defined and continuous on the cube  $\mathcal{I}^n$  can be represented as

$$f(x_1, \dots, x_n) = \sum_{q=1}^{2n+1} g_q \left( \sum_{p=1}^n \varphi_{p,q}(x_p) \right),$$

where  $\{\varphi_{p,q}\}$  are specially chosen continuous and monotonic functions which do not depend on  $f$ , and where  $\{g_q\}$  are continuous functions.

Lorentz [12] has noticed that in the theorem of Kolmogorov the functions  $\{g_q\}$  can be chosen independently of  $q$ . In fact, by adding constants to the functions  $t_q = \sum_{p=1}^n \varphi_{p,q}(x_p)$  ( $q = 1, \dots, 2n+1$ ) one can make the ranges

of the functions pairwise disjoint and consequently the functions  $\{t_q\}$  can be considered as the restrictions of a single function  $\{g_q\}$ .

Sprecher [40] has shown that the functions  $\{\varphi_{p,q}\}$  can be chosen in the form  $\varphi_{p,q}(x_p) = \lambda_p \varphi_q(x_p)$  where  $\{\lambda_p\}$  are constants and  $\{\varphi_q\}$  are continuous monotonic functions.

Thus any continuous function can be represented as

$$f(x_1, \dots, x_n) = \sum_{q=1}^{2n+1} g \left( \sum_{p=1}^n \lambda_p \varphi_q(x_p) \right),$$

where the constants  $\{\lambda_p\}$  and the continuous monotone functions  $\{\varphi_q\}$  do not depend on  $f$ , and where  $g$  is a continuous function.

Kahane [36] has shown that such a representation is possible with almost every collection of constants  $\{\lambda_p\}$  and “quasi every” collection of continuous functions  $\{\varphi_q\}$ . The precise statement of this theorem will be given below. Here we consider some further results concerning the formula of Kolmogorov.

Doss [38] has shown that for any continuous monotonic functions  $\varphi_{p,q}$  ( $p=1, 2; q=1, 2, 3, 4$ ) there exists a continuous function  $f(x_1, x_2)$  of two variables not representable as a superposition of the form  $\sum_{q=1}^4 g_q \left( \sum_{p=1}^2 \varphi_{p,q}(x_p) \right)$ , where  $\{g_q\}$  are continuous functions.

Bassalygo [39] succeeded in showing that for any continuous functions  $\varphi_i(x_1, x_2)$  ( $i=1, 2, 3$ ) there exists a continuous function  $f(x_1, x_2)$  that is not equal to any superposition of the form  $\sum_{i=1}^3 g_i(\varphi_i(x_1, x_2))$ , where  $\{g_i\}$  are continuous functions.

Tihomirov showed that Kolmogorov's theorem can be generalized as follows: for any compact  $K$  of dimension  $n$  there exists a homeomorphic embedding  $\Psi(x) = \{\Psi_1(x), \dots, \Psi_{2n+1}(x)\}$ ,  $x \in K$  into  $(2n+1)$ -dimensional euclidean space such that any continuous function  $f(x)$  on  $K$  can be represented in the form  $f(x) = \sum_{i=1}^{2n+1} g_i(\Psi_i(x))$ , where  $\{g_i\}$  are continuous functions of one variable.

In the same paper [36] Kahane has shown that there exist complex numbers  $\lambda_p$  ( $p=1, \dots, n$ ) and complex valued functions  $\varphi_q$  ( $q=1, \dots, 2n+1$ ) possessing the following properties.

1. The function  $\varphi_q$  is a monotonic continuous transformation of the real axis onto the circle  $|t| = 1$  ( $q=1, \dots, 2n+1$ ).

2. The function  $t_q = \sum_{p=1}^n \lambda_p \varphi_q(x_p)$  maps the cube  $\mathcal{I}^n$  into the circle  $|t| = 1$ .

3. The transformation  $\Psi$  given by the equalities  $t_q = \sum_{p=1}^n \lambda_p \varphi_q(x_p)$  ( $q=1, \dots, 2n+1$ ) is one-to-one on  $\mathcal{I}^n$ .

4. For any function  $f$  continuous on  $\mathcal{I}^n$  there exists a function  $g(z)$  continuous on the disk  $|z| \leq 1$ , holomorphic inside that disk, and such that  $f = \sum_{q=1}^{2n+1} g\left(\sum_{p=1}^n \lambda_p \varphi_q(x_p)\right)$ .

The transformation  $\Psi$  gives an embedding of the cube  $\mathcal{I}^n$  into the torus  $|t| = 1$  ( $q=1, \dots, 2n+1$ ) such that any function continuous on the cube  $\tilde{\mathcal{I}}^n = \Psi(\mathcal{I}^n)$  is represented in the form  $f(t_1, \dots, t_{2n+1}) = \sum_{q=1}^{2n+1} g(t_q)$ , where  $g$  is a function holomorphic in the unit disk. This means in particular that any function continuous on  $\tilde{\mathcal{I}}^n$  has an analytic extension to the polydisk  $|t_q| \leq 1$  ( $q=1, \dots, 2n+1$ ).

## § 2. The theorem of Kahane

Let  $M$  be a complete metric space. We recall that a set is called a set of second category if it is the intersection of a countable family of open sets which are everywhere dense in  $M$ . By the theorem of Baire in a complete metric space no set of second category is empty. The massivity of such sets is characterized by the fact that the intersection of a countable family of sets of second category is again a set of second category and consequently is not empty.

We will say that a statement is true for quasi every element of  $M$  if it is true for a set of elements of second category.

Let us consider an example. Let  $\Phi$  be the space with uniform norm consisting of all functions continuous and non-decreasing on the segment  $\mathcal{I}^1$  ( $0 \leq t \leq 1$ ). It can be shown easily that quasi every element of  $\Phi$  is a strictly increasing function.

In fact, any strictly increasing function belongs to any set defined as  $\varphi(r') < \varphi(r'')$ , where  $r' < r''$  are fixed rational numbers. Any set defined by an inequality of that type is open and everywhere dense in  $\Phi$ , and the set of all such sets is countable.