# §3. Superpositions of smooth functions and the theory of approximation

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## § 3. *Superpositions of smooth functions and the theory of approximation*

In [4] it was proved that in the class of all $S$ times continuously differentiable functions of $n$ variables there exist some that cannot be represented as a finite superposition of functions for which the ratio of the number of arguments to the number of derivatives they have is strictly less than $n/S$.

This theorem shows that the ratio $n/S$ can serve as a measure of the complexity of $S$ times differentiable functions of $n$ variables. The original proof of this theorem made use of the theory of multi-dimensional variations of sets and estimates of the number of $\varepsilon$-distant smooth functions (see [21], [22]). Kolmogorov [23] showed that the same result can be obtained using only estimates of the number of elements of $\varepsilon$-nets of functional compacts.

We denote by $F_S^n$ the set of functions $f(x_1, x_2, \ldots, x_n)$ defined on an $n$-dimensional cube, whose partial derivatives up to order $S$ inclusive are all continuous and bounded by some constant $C$. Let $N_\varepsilon(F_S^n)$ be the minimum number of spheres of radius $\varepsilon$ in the space of all continuous functions by which the set $F_S^n$ can be covered.

It turns out that

$$\lim_{\varepsilon \to 0} \frac{\log \log N_\varepsilon(F_s^n)}{\log\left(\frac{1}{\varepsilon}\right)} = \frac{n}{s}.$$

Hence it follows that if $n/S > n'/S'$, then the set of functions $F_S^n$ is, in a certain sense, "more massive" than $F_{S'}^{n'}$.

If a consideration of the massivity of functional compacts does not give the answer then the problems remain open. For example, there is no answer to the question: is it possible to represent any analytic function of several variables by means of a superposition of smooth functions of a smaller number of variables.

The topic of superpositions led to a large number of papers in approximation theory. Here we formulate two results concerning non-linear approximations.

Let $\mathcal{I}^n$ be a cube $0 \leqslant x_i \leqslant 1$ $(i = 1, \ldots, n)$; $C$—the space of all realvalued continuous functions defined on $\mathcal{I}^n$ with the uniform norm; $F$—a compact subset of $C$, $\Phi$—a surface in $C$ which consists of the functions represented in the form

$$\varphi = \frac{\sum\limits_{\alpha_1+\alpha_2+\ldots+\alpha_p \leq k} a_1^{\alpha_1} \cdot a_2^{\alpha_2} \ldots a_p^{\alpha_p} f_{\alpha_1,\ldots,\alpha_p}}{\sum\limits_{\beta_1+\beta_2+\ldots+\beta_p \leq k} b_1^{\beta_1} \cdot b_2^{\beta_2} \ldots b_p^{\beta_p} g_{\beta_1,\ldots,\beta_p}} .$$

where the natural numbers $p$ and $k$ and the collections $\{ f_{\alpha_1, \ldots, \alpha_p} \in C \}$ and $\{ g_{\beta_1, \ldots, \beta_p} \in C \}$ are fixed in advance and independent of $\varphi$, $\{ a_i \}$ and $\{ \beta_i \}$ are positive integers and the coefficients $\{ a_i \}$ and $\{ b_i \}$ defining the function $\varphi$ can take arbitrary real values.

We remark that for $k = 1$ the class $\Phi$ can be turned into any of the usual classes in approximation theory by means of an appropriate choise of the number $p$ and collections $\{ f_{\alpha_1, \ldots, \alpha_p} \}$ and $\{ g_{\beta_1, \ldots, \beta_p} \}$. For example it can be turned into the classes of polynomials or rational functions of a fixed degree.

We put $e_{pk}(F) = \sup\limits_{f \in F} \inf\limits_{\varphi \in \Phi} \| f - \varphi \|$. Estimates of $e_{pk}$ for some functional compacts can be found in [21], [22], [24], [25]. Here are two examples of such estimates

$$1. \quad e_{pk}(F_s^n) \geqslant a \left( \frac{1}{p \log (k+1)} \right)^{s/n},$$

where $a > 0$ does not depend on $p$ and $k$.

2. For the set $F_{dc}$ consisting of all functions which have an analytic extension to some domain $d$ in $n$-dimensional complex space bounded in modulus by some constant $C$ the following inequality is valid

$$e_{pk}(F_{dc}) \geqslant b q^{n\sqrt{p \log (k+1)}},$$

where $b > 0$ and $0 < q < 1$ are constants independent of $p$ and $k$.

Now there are more elementary proofs of these inequalities for $k = 1$ with precise estimates of the constant (see Erohin [26], Lorentz [24], Tihomirov [27], Shapiro [25]).

Let us clarify the meaning of these inequalities. We agree to characterize the complexity of any algorithm for the approximate calculation of functions firstly by the number of parameters used in the algorithm, and secondly by the complexity of the scheme of the calculation, for example, by the number of arithmetic operations required for the approximate calculation of functions by means of the given algorithm.

In the above-mentioned method of approximation of functions by functions from $\Phi$ the parameters are the numbers $\{ a_i \}$ and $\{ b_i \}$, and the number of arithmetic operations increases very rapidly as $k$ increases. At the same time, from the inequalities mentioned above it follows that an increase in $k$ leads to an insignificant improvement in the accuracy of the approximation. Hence, in a certain sense, a more economical approximation of functions is by means of expressions of the given form with $k = 1$, that is, by fractions of the form

$$\frac{\sum_{i=0}^{p} a_i f_i(x)}{\sum_{j=0}^{p} b_j g_j(x)}$$

The same inequalities with $k = 1$ show that there are no methods of approximating functions by fractions of the given form essentially better than the standard methods of approximating functions by algebraic (or trigonometric) polynomials.

## § 4. *Superpositions of continuous functions*

Kolmogorov's theorem on the possibility of representing continuous functions of $n$ variables as superpositions of continuous functions of three variables was highly unexpected (see [7]).

In this paper Kolmogorov proves that on the $n$-dimensional cube $\mathscr{I}^n$ we can construct continuous functions $\varphi_i(x)$ $(i = 1, 2, \ldots, n+1)$ such that any continuous function $f(x)$, defined on the cube $\mathscr{I}^n$, can be represented in the form

$$f(x) = \sum_{i=1}^{n+1} f_i(d_i(x)),$$

where $d_i(x)$ is a continuous mapping of $\mathscr{I}^n$ onto the one-dimensional tree [1]) $D$ if the components of the level sets of the functions $\varphi_i(x)$, and $f_i(d_i)$ is a continuous function on the tree $D_i$. Since the trees $\{D_i\}$ can be embedded homeomorphically in the plane (see [30]), the functions $\{f_i(d_i(x))\}$ can be thought of as superpositions

$$\{f_i(u_i(x_1, x_2, \ldots, x_n), v_i(x_1, x_2, \ldots, x_n))\}$$

[1]) Kronrod [29] has shown that the components of all possible level sets of any continuous function defined on $\mathscr{I}^n$ in a certain natural topology, form a tree, that is, a one-dimensional locally connected continuum, not containing homeomorphic images of circles. Kronrod calls this the "one-dimensional tree of the function".