# §1 Symplectic torsors

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# II. THE ABSTRACT THEORY OF CHARACTERISTICS

## § 1 SYMPLECTIC TORSORS

1.1 *Definitions.* Recall that, if $\Gamma$ is a group, a $\Gamma$-torsor (or torsor over $\Gamma$) is a non-void set endowed with a simply transitive action of $\Gamma$ on it. Let $(J, e)$ be a symplectic pair, a *symplectic torsor* over $(J, e)$ is a pair $(S, Q)$ of a $J$-torsor $S$ and a mapping $Q: S \to \mathbf{Z}/2\mathbf{Z}$ having the property

$$(1.1.1) \qquad Q(s) + Q(x+s) + Q(y+s) + Q(x+y+s) = e(x, y)$$

where $s \in S$, $x, y \in J$. It is clearly equivalent to ask this property for a fixed $s \in S$ or for all $s \in S$, and it may be thought of as meaning that $Q$ "is a quadratic form." Indeed, any $s \in S$ sets an identification $J \xrightarrow{\sim} S$ $(x \mapsto x+s)$, and through this identification $Q$ becomes the map $x \mapsto Q(x+s)$. The above property means that the map $q_s: J \to \mathbf{Z}/2\mathbf{Z}$ defined by

$$(1.1.2) \qquad q_s(x) = Q(x+s) + Q(s)$$

is a quadratic form whose associated bilinear form is $e$. According to 0.4, two possibilities may and do arise for $Q$: either $Q^{-1}(0)$ has $2^{g-1}(2^g+1)$ or $2^{g-1}(2^g-1)$ elements, where $g = \dim J/2$ will be called the *genus* of $(S, Q)$. In the first case, $(S, Q)$ will be said to be *even*, *odd* in the second. In what follows, *all symplectic torsors will be even* unless otherwise stated. This because the symplectic torsors that will appear most often will be even and because of the following simple construction. If $(S, Q)$ is an even (resp. odd) symplectic torsor over $(J, e)$, and $\bar{Q}$ is defined by $\bar{Q}(s) = Q(s) + 1$, then $(S, \bar{Q})$ is an odd (resp. even) symplectic torsor over $(J, e)$.

For a given $(S, Q)$ the following notation will be used

$$S^+ = Q^{-1}(0) \quad S^- = Q^{-1}(1).$$

The elements of $S$ will be often called *characteristics*, those in $S^+$ are *positive*, those in $S^-$ are *negative*.

1.2 *Morphisms.* Let $(S, Q)$, $(S', Q')$ be symplectic torsors respectively over $(J, e)$, $(J', e')$. For any map $f: S \to S'$ we define a map $\sigma_f: J \times S \to J'$ by the property

$$f(x+s) = \sigma_f(x, s) + f(s);$$

this can be done because $S'$ is a $J'$-torsor. Now, the following cocycle-type property for $\sigma_f$ is immediately checked, where $x, y \in J$, $s \in S$

$$\sigma_f(x+y+s) = \sigma_f(x, y+s) + \sigma_f(y, s),$$

and from it one infers the equivalence of the following statements:

(i) For any $s, s' \in S$, $x \in J$

$$\sigma_f(x, s) = \sigma_f(x, s').$$

(ii) For some $s \in S$, any $x, y \in J$

$$\sigma_f(x+y, s) = \sigma_f(x, s) + \sigma_f(y, s)$$

(iii) For any $s \in S$, $x, y \in J$

$$\sigma_f(x+y, s) = \sigma_f(x, s) + \sigma_f(y, s).$$

So, when these statements hold, one gets a group homomorphism $\sigma_f: J \to J'$ and has $f(x+s) = \sigma_f(x) + f(s)$.

An *isomorphism* of $(S, Q)$ onto $(S', Q')$ is a bijection $f: S \to S'$ verifying statements (i) to (iii) above, and also the condition

$$Q' \circ f = Q.$$

It is clear in this case that $\sigma_f: J \to J'$ is an isomorphism compatible with $e, e'$. The group of automorphisms of $(S, Q)$ will be denoted $Sp(S, Q)$, so the mapping $f \to \sigma_f$ is a group homomorphism $Sp(S, Q) \to Sp(J, e)$.

1.3 *An example.* For any given $(J, e)$ there is a canonical example of an even symplectic torsor, namely $\big(Q(J, e), Q_e\big)$. The $J$-torsor $Q(J, e)$ was introduced in 0.2, the map $Q_e$ in 0.3 where it was also remarked that it has property (1.1.1) and that $Q_e^{-1}(0)$ has $2^{g-1}(2^g+1)$ elements.

If $(J, e)$, $(J', e')$ are two symplectic pairs, and if $\sigma: J \to J'$ is a linear isomorphism compatible with $e, e'$, a map $Q(\sigma): Q(J, e) \to Q(J', e')$ was defined in 0.4, where it was shown that it is an isomorphism of symplectic torsors. Clearly $Q(\sigma)$ is canonical in any conceivable way.

Indeed, if one still dares in these days to use the language of category theory, what I just did was to define a functor from the category of symplectic pairs to the category of even symplectic torsors (morphisms = isomorphisms, in both cases). In section 1.4 we will see that this is an equivalence of categories.

1.4 *Uniqueness of symplectic torsors.* It will be shown here, that for a given symplectic pair $(J, e)$ there is essentially only one symplectic torsor over it. Let $(S, Q)$ be such an object; then there is a map

$$f_s\colon S \to Q(J, e),$$

defined by the rule $s \mapsto q_s$, where $q_s$ was defined in (1.1.2). Let us prove that $f_s$ is an isomorphism of symplectic torsors inducing the identity $id_J\colon J \to J$. The formula

$$q_{x+s}(y) = (x + q_s)(y)$$

is a mere restatement of condition (1.1.1), and the formula

$$Q_e \circ f_s = Q$$

follows from the fact that $(S, Q)$ is even and from the meaning of the Arf invariant recalled in 0.3.

The isomorphisms $f_s$ are canonical, in the following sense. If $(S, Q)$, $(S', Q')$ are symplectic torsors over $(J, e)$, $(J', e')$, $f\colon S \to S'$ is an isomorphism of symplectic torsors inducing an isomorphism $\sigma\colon J \to J'$, then the following square commutes

$$\begin{array}{ccc} S & \xrightarrow{\;f\;} & S' \\ {\scriptstyle f_s}\downarrow & & \downarrow{\scriptstyle f_{s'}} \\ Q(J, e) & \xrightarrow{\;Q(\sigma)\;} & Q(J', e') \end{array}$$

Recalling the definitions, one has to check for $s \in S$, $x \in J$ that

$$Q(\sigma(x) + f(s)) + Q(f(s)) = Q(x + s) + Q(s)$$

which is immediate from the definition of isomorphism in 1.2.

It comes out of this that for any isomorphism $\sigma\colon J \to J'$ there exists one and only one isomorphism $f\colon S \to S'$ inducing it. In particular, the group homomorphism at the end of 1.2.

$$Sp(S, Q) \to Sp(J, e)$$

is an isomorphism. A useful application of this is the following: If by some unspecified means one is able to construct two symplectic torsors over a pair $(J, e)$, there is a unique isomorphism between them inducing the identity of $J$.

1.5 *Some notation.* a) Let $J$ be a vector space over $\mathbf{Z}/2\mathbf{Z}$, $S$ a $J$-torsor. Let's put

$$E(S) = J \amalg S$$

the disjoint union of $J$, $S$; on this set there is a structure of vector space over $\mathbf{Z}/2\mathbf{Z}$. In fact there is an exact sequence

$$0 \to J \to E(S) \to \mathbf{Z}/2\mathbf{Z} \to 0$$

where $J$ is sent identically onto itself, and the inverse image of 0 (resp. 1) in $E(S)$ is $J$ (resp. $S$). The addition law in $E(S)$ reduces to the given one on $J$ when both elements are in $J$, is the action of $J$ on $S$ when one element is in $J$ and the other in $S$, and finally $s + s'$ (for $s, s' \in S$) is the unique element $x \in J$ such that $x + s = s'$ (or equivalently $x + s' = s$).

b) Given the standard pair $(J_o, e_o)$, as in 0.5. I will write $S_o = Q(J_o, e_o)$, $Q_o = Q_{e_o}$. Both $J_o$, $S_o$ identify to $(\mathbf{Z}/2\mathbf{Z})^{2g}$, but the following notations will be used in compliance with tradition, where $u_1, \ldots, u_{2g}$ is the canonical basis. An element of the form

$$\sum_{i=1}^{g} (\varepsilon_i u_i + \varepsilon_i' u_{i+g})$$

will be written $\begin{pmatrix}\varepsilon\\ \varepsilon'\end{pmatrix}$ or $\begin{bmatrix}\varepsilon\\ \varepsilon'\end{bmatrix}$ whether it is seen in $J_o$ or $S_o$ respectively, where $\varepsilon$, $\varepsilon'$ are row vectors. In particular, the addition law in $E(S_o)$ is the following:

$$\begin{pmatrix}\varepsilon\\ \varepsilon'\end{pmatrix} + \begin{pmatrix}\eta\\ \eta'\end{pmatrix} = \begin{pmatrix}\varepsilon + \eta\\ \varepsilon' + \eta'\end{pmatrix}$$

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## § 2 Finite geometries on sets of characteristics

2.0 Let's fix for paragraph § 2 a symplectic torsor $(S, Q)$ over a symplectic pair $(J, e)$ of genus $g$. The letter $\Sigma$ will stand for either the set $S^+$ of $S^-$, its cardinality is $2^{g-1}(2^g \pm 1)$ (recall that according to 1.1 we assume