§1 Theta characteristics

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where $0, I$ are respectively the zero, identity $g \times g$ matrix. The datum (J_o, e_o) is a symplectic pair, and is standard in the sense that for a fixed (J, e), giving a symplectic basis for (J, e) amounts to the same thing as giving a linear isomorphism $J_o \simeq J$ compatible with e_o, e. By 0.4, this in turn defines an isomorphism $Q(J_o, e_o) \simeq Q(J, e)$ with the properties stated there.

Going back to the standard situation, there is an obvious identification $Q(J_o, e_o) \simeq (\mathbf{Z}/2\mathbf{Z})^{2g}$, obtained associating with every quadratic form q its values on the canonical basis of J_o. With this identification in mind, the action of J_o on $Q(J_o, e_o)$ defined at the end of 0.2 is the action of $(\mathbf{Z}/2\mathbf{Z})^{2g}$ on itself by translations, and the Arf invariant is given by the mapping Q: $(\varepsilon, \varepsilon') \mapsto \Sigma\, \varepsilon_i \varepsilon_i'$, where $\varepsilon, \varepsilon' \in (\mathbf{Z}/2\mathbf{Z})^g$.

We will use the following notation,

$$\begin{aligned} J_0(g) &= (\mathbf{Z}/2\mathbf{Z})^{2g} \\ S_0(g) &= Q(J_0, e_0) \\ S_0^+(g) &= \{s \in S_0(g)/Q(s) = 0\} \\ S_0^-(g) &= \{s \in S_0(g)/Q(s) = 1\} \end{aligned}$$

§ 1 Theta characteristics

1.1 *On an algebraic curve.* Let C be a non-singular projective algebraic curve over an algebraically closed base field k of characteristic different from 2. The set $S(C)$ of *theta characteristics* on C is the set of isomorphism classes of line bundles L on C whose tensor square is isomorphic to the canonical bundle. If $J_2(C)$ is the group of points of order two in Pic (C), i.e. the multiplicative group of isomorphism classes of line bundles on C whose square is the trivial line bundle $\mathbf{O}_C$, then clearly $J_2(C)$ acts on the set $S(C)$, and this in a simply transitive way. In addition, there is a function

$$Q: S(C) \to \mathbf{Z}/2\mathbf{Z}$$

defined by

$$Q(L) = \dim \Gamma(C, L) \quad (2).$$

The following formula holds, where $x, y \in J_2(C)$, $s \in S(C)$, and we use additive notation both for the group law in $J_2(C)$ and the action of $J_2(C)$ on $S(C)$:

$$Q(s) + Q(x+s) + Q(y+s) + Q(x+y+s) = e(x, y).$$

Here, e stands for the intersection pairing on $J_2(C)$. If g is the genus of C, it is proved that $Q^{-1}(0)$ (resp. $Q^{-1}(1)$) has $2^{g-1}(2^g+1)$ (resp. $2^{g-1}(2^g-1)$) elements.

The proof of these assertions goes back to Riemann in the case $k = \mathbf{C}$, and in the general case it may be found in Mumford [5].

1.2 *On a principally polarized abelian variety*. Let X be an abelian variety over k, $\theta: X \xrightarrow{\sim} \hat{X}$ a principal polarization. The set $S(X, \theta)$ of *theta characteristics* on (X, θ) is the subset of $\mathrm{Pic}^{\theta}(X)$ determined by the symmetric line bundles; i.e. the elements of $S(X, \theta)$ are the isomorphism classes of line bundles L on X belonging to θ and such that $L \simeq i^*(L)$, where $i: X \to X$ sends $x \in X$ into $-x$. Again, the group X_2 of points of order two in X acts on $S(X, \theta)$ through the induced isomorphism $\theta: X_2 \xrightarrow{\sim} \hat{X}_2$, and this in a simply transitive way. Now, for any symmetric line bundle L on X, there exists a unique isomorphism $\varphi: L \xrightarrow{\sim} i^*(L)$ such that over the zero of X, φ induces the identity on the fibers. Over any $x \in X_2$, the fibers of L, $i^*(L)$ identify naturally, and φ induces the multiplication by some scalar that will be denoted $e_*^L(x)$. It is proved that $e_*^L(x) = \pm 1$, and indeed that $e_*^L: X_2 \to \mathbf{Z}/2\mathbf{Z}$ is a quadratic form whose associated bilinear form is the intersection pairing e on X_2. Now we define a mapping

$$Q: S(X, \theta) \to \mathbf{Z}/2\mathbf{Z}$$

by

$$Q(s) = \text{Arf invariant of } e_*^s .$$

The following formula is valid, where additive notation is used both for group law and group action, and where $s \in S(X, \theta)$, $x, y \in X_2$

$$Q(s) + Q(x+s) + Q(y+x) + Q(x+y+s) = e(x, y) .$$

It is also true that, if $g = \dim X$, $Q^{-1}(0)$ (resp. $Q^{-1}(1)$) has $2^{g-1}(2^g+1)$ (resp. $2^{g-1}(2^g-1)$) elements.

All the preceding is proved in or follows easily from § 2 of Mumford [4] and from § 0 above. Note in addition that in $\mathrm{Pic}^{2\theta}(X)$ there is a unique totally symmetric line bundle L_o (i.e. L_o is symmetric and $e_*^{L_o}(x) = 1$ for every $x \in X_2$), and that the symmetric line bundles in $\mathrm{Pic}^{\theta}(X)$ are the line bundles L such that L^2 is isomorphic with L_o (cf. Mumford [4], *loc. cit.*).