

# §1 Theta characteristics

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where  $0, I$  are respectively the zero, identity  $g \times g$  matrix. The datum  $(J_o, e_o)$  is a symplectic pair, and is standard in the sense that for a fixed  $(J, e)$ , giving a symplectic basis for  $(J, e)$  amounts to the same thing as giving a linear isomorphism  $J_o \simeq J$  compatible with  $e_o, e$ . By 0.4, this in turn defines an isomorphism  $Q(J_o, e_o) \simeq Q(J, e)$  with the properties stated there.

Going back to the standard situation, there is an obvious identification  $Q(J_o, e_o) \simeq (\mathbf{Z}/2\mathbf{Z})^{2g}$ , obtained associating with every quadratic form  $q$  its values on the canonical basis of  $J_o$ . With this identification in mind, the action of  $J_o$  on  $Q(J_o, e_o)$  defined at the end of 0.2 is the action of  $(\mathbf{Z}/2\mathbf{Z})^{2g}$  on itself by translations, and the Arf invariant is given by the mapping  $Q: (\varepsilon, \varepsilon') \mapsto \sum \varepsilon_i \varepsilon'_i$ , where  $\varepsilon, \varepsilon' \in (\mathbf{Z}/2\mathbf{Z})^g$ .

We will use the following notation,

$$\begin{aligned} J_o(g) &= (\mathbf{Z}/2\mathbf{Z})^{2g} \\ S_o(g) &= Q(J_o, e_o) \\ S_o^+(g) &= \{s \in S_o(g) / Q(s) = 0\} \\ S_o^-(g) &= \{s \in S_o(g) / Q(s) = 1\} \end{aligned}$$

## § 1 THETA CHARACTERISTICS

1.1 *On an algebraic curve.* Let  $C$  be a non-singular projective algebraic curve over an algebraically closed base field  $k$  of characteristic different from 2. The set  $S(C)$  of *theta characteristics* on  $C$  is the set of isomorphism classes of line bundles  $L$  on  $C$  whose tensor square is isomorphic to the canonical bundle. If  $J_2(C)$  is the group of points of order two in  $\text{Pic}(C)$ , i.e. the multiplicative group of isomorphism classes of line bundles on  $C$  whose square is the trivial line bundle  $\mathbf{O}_C$ , then clearly  $J_2(C)$  acts on the set  $S(C)$ , and this in a simply transitive way. In addition, there is a function

$$Q: S(C) \rightarrow \mathbf{Z}/2\mathbf{Z}$$

defined by

$$Q(L) = \dim \Gamma(C, L) \quad (2).$$

The following formula holds, where  $x, y \in J_2(C)$ ,  $s \in S(C)$ , and we use additive notation both for the group law in  $J_2(C)$  and the action of  $J_2(C)$  on  $S(C)$ :

$$Q(s) + Q(x+s) + Q(y+s) + Q(x+y+s) = e(x, y).$$

Here,  $e$  stands for the intersection pairing on  $J_2(C)$ . If  $g$  is the genus of  $C$ , it is proved that  $Q^{-1}(0)$  (resp.  $Q^{-1}(1)$ ) has  $2^{g-1}(2^g+1)$  (resp.  $2^{g-1}(2^g-1)$ ) elements.

The proof of these assertions goes back to Riemann in the case  $k = \mathbf{C}$ , and in the general case it may be found in Mumford [5].

1.2 *On a principally polarized abelian variety.* Let  $X$  be an abelian variety over  $k$ ,  $\theta: X \xrightarrow{\sim} \hat{X}$  a principal polarization. The set  $S(X, \theta)$  of *theta characteristics* on  $(X, \theta)$  is the subset of  $\text{Pic}^\theta(X)$  determined by the symmetric line bundles; i.e. the elements of  $S(X, \theta)$  are the isomorphism classes of line bundles  $L$  on  $X$  belonging to  $\theta$  and such that  $L \simeq i^*(L)$ , where  $i: X \rightarrow X$  sends  $x \in X$  into  $-x$ . Again, the group  $X_2$  of points of order two in  $X$  acts on  $S(X, \theta)$  through the induced isomorphism  $\theta: X_2 \xrightarrow{\sim} \hat{X}_2$ , and this in a simply transitive way. Now, for any symmetric line bundle  $L$  on  $X$ , there exists a unique isomorphism  $\varphi: L \xrightarrow{\sim} i^*(L)$  such that over the zero of  $X$ ,  $\varphi$  induces the identity on the fibers. Over any  $x \in X_2$ , the fibers of  $L, i^*(L)$  identify naturally, and  $\varphi$  induces the multiplication by some scalar that will be denoted  $e_*^L(x)$ . It is proved that  $e_*^L(x) = \pm 1$ , and indeed that  $e_*^L: X_2 \rightarrow \mathbf{Z}/2\mathbf{Z}$  is a quadratic form whose associated bilinear form is the intersection pairing  $e$  on  $X_2$ . Now we define a mapping

$$Q: S(X, \theta) \rightarrow \mathbf{Z}/2\mathbf{Z}$$

by

$$Q(s) = \text{Arf invariant of } e_*^s.$$

The following formula is valid, where additive notation is used both for group law and group action, and where  $s \in S(X, \theta), x, y \in X_2$

$$Q(s) + Q(x+s) + Q(y+x) + Q(x+y+s) = e(x, y).$$

It is also true that, if  $g = \dim X$ ,  $Q^{-1}(0)$  (resp.  $Q^{-1}(1)$ ) has  $2^{g-1}(2^g+1)$  (resp.  $2^{g-1}(2^g-1)$ ) elements.

All the preceding is proved in or follows easily from § 2 of Mumford [4] and from § 0 above. Note in addition that in  $\text{Pic}^{2\theta}(X)$  there is a unique totally symmetric line bundle  $L_0$  (i.e.  $L_0$  is symmetric and  $e_*^{L_0}(x) = 1$  for every  $x \in X_2$ ), and that the symmetric line bundles in  $\text{Pic}^\theta(X)$  are the line bundles  $L$  such that  $L^2$  is isomorphic with  $L_0$  (cf. Mumford [4], *loc. cit.*).