

§4. The case p5

Objektyp: **Chapter**

Zeitschrift: **L'Enseignement Mathématique**

Band (Jahr): **22 (1976)**

Heft 1-2: **L'ENSEIGNEMENT MATHÉMATIQUE**

PDF erstellt am: **26.09.2024**

Nutzungsbedingungen

Die ETH-Bibliothek ist Anbieterin der digitalisierten Zeitschriften. Sie besitzt keine Urheberrechte an den Inhalten der Zeitschriften. Die Rechte liegen in der Regel bei den Herausgebern. Die auf der Plattform e-periodica veröffentlichten Dokumente stehen für nicht-kommerzielle Zwecke in Lehre und Forschung sowie für die private Nutzung frei zur Verfügung. Einzelne Dateien oder Ausdrucke aus diesem Angebot können zusammen mit diesen Nutzungsbedingungen und den korrekten Herkunftsbezeichnungen weitergegeben werden. Das Veröffentlichen von Bildern in Print- und Online-Publikationen ist nur mit vorheriger Genehmigung der Rechteinhaber erlaubt. Die systematische Speicherung von Teilen des elektronischen Angebots auf anderen Servern bedarf ebenfalls des schriftlichen Einverständnisses der Rechteinhaber.

Haftungsausschluss

Alle Angaben erfolgen ohne Gewähr für Vollständigkeit oder Richtigkeit. Es wird keine Haftung übernommen für Schäden durch die Verwendung von Informationen aus diesem Online-Angebot oder durch das Fehlen von Informationen. Dies gilt auch für Inhalte Dritter, die über dieses Angebot zugänglich sind.

(3) E has split multiplicative reduction at 3 $\Leftrightarrow a_1^2 + a_2 \equiv 1 \pmod{3}$.

(4) E has non-split multiplicative reduction at 3 $\Leftrightarrow a_1^2 + a_2 \equiv -1 \pmod{3}$.

Proof:

$$c_4 \equiv b_2^2 - 24b_4 \equiv b_2^2 \equiv (a_1^2 + 4a_2)^2 \equiv (a_1^2 + a_2)^2 \pmod{3}.$$

The theorem then follows immediately from formula (3.1) and Corollary 1.2.

Remark. $C_2^2 \equiv c_4 \pmod{3}$. Note that $C_2 = a_1^2 + a_2$ is a more sensitive invariant than c_4 in that the residue class of C_2 modulo 3 allows us to distinguish between split and non-split multiplicative reduction, while c_4 does not allow us to separate these two possibilities.

§4. THE CASE $p \geq 5$

Assume $p \geq 5$. Then there exists a minimal Weierstrass equation for E at p of the form

$$(4.1) \quad Y^2 = X^3 + AX + B$$

with $A, B \in \mathbf{Z}$. The coefficient C_{p-1} modulo p is given by Deuring's classical formula [1]

$$(4.2) \quad C_{p-1} \equiv \sum_{2h+3i=P} \frac{P!}{i! h! (P-h-i)!} A^h B^i \pmod{p}$$

where $P = (1/2)(p-1)$.

Let $S = (x, y)$ be the singular point on the reduced curve with $x, y \in \mathbf{Z}/p\mathbf{Z}$. The tangents at S are given by a quadratic polynomial $R(T)$ as follows: Transform the curve by $X \rightarrow (X+x)$, $Y \rightarrow (Y+y)$ so that the singularity is now at $(0, 0)$. The tangents are given by a homogeneous form of degree 2 in X and Y which we can consider as a quadratic polynomial

$R(T)$ with $T = Y/X$. Let D be the discriminant of $R(T)$, and let $\left(\frac{-}{p}\right)$

denote the Legendre symbol with respect to p . We have the following results directly from the definitions.

PROPOSITION 4.1. Assume E has bad reduction at p .

(1) E has additive reduction at $p \Leftrightarrow f_p = 0 \Leftrightarrow S$ is a cusp $\Leftrightarrow R(T)$ has two identical roots over $\mathbf{Z}/p\mathbf{Z} \Leftrightarrow D = 0 \Leftrightarrow \left(\frac{D}{p}\right) = 0$.

(2) E has split multiplicative reduction at $p \Leftrightarrow f_p = 1 \Leftrightarrow S$ is a node with rational tangents $\Leftrightarrow R(T)$ has two distinct roots rational over $\mathbf{Z}/p\mathbf{Z} \Leftrightarrow \left(\frac{D}{p}\right) = 1$.

(3) E has non-split multiplicative reduction at $p \Leftrightarrow f_p = -1 \Leftrightarrow S$ is a node with irrational tangents $\Leftrightarrow R(T)$ has two distinct roots not rational over $\mathbf{Z}/p\mathbf{Z} \Leftrightarrow \left(\frac{D}{p}\right) = -1$.

COROLLARY 4.2. $f_p = \left(\frac{D}{p}\right)$.

In this case, H reduces to

$$(4.3) \quad H = Y^2 - X^3 - AX - B$$

Then we have

$$(4.4) \quad \partial H / \partial X = -3X^2 - A$$

$$(4.5) \quad \partial H / \partial Y = 2Y$$

From (4.5) we must have $y = 0$. From (4.4) we must have $x^2 = -A/3$ in $\mathbf{Z}/p\mathbf{Z}$, so that $-A/3$ is either a quadratic residue modulo p or 0 modulo p . Note that $x = 0 \Leftrightarrow A \equiv 0 \pmod{p}$. Let $X^3 + AX + B = (X - \alpha_1)(X - \alpha_2)(X - \alpha_3)$ be a factorization over $\mathbf{Z}/p\mathbf{Z}$. At least two of $\alpha_1, \alpha_2, \alpha_3$ must coincide with x , let us say $x = \alpha_2 = \alpha_3$. Then

$$(4.6) \quad X^3 + AX + B = X^3 + (-\alpha_1 - 2\alpha_2)X^2 + (2\alpha_1\alpha_2 + \alpha_2^2)X - \alpha_1\alpha_2^2$$

Thus comparing coefficients, we have

$$(4.7) \quad 0 = -\alpha_1 - 2\alpha_2$$

$$(4.8) \quad A = 2\alpha_1\alpha_2 + \alpha_2^2$$

$$(4.9) \quad B = -\alpha_1\alpha_2^2$$

Hence

$$(4.10) \quad \alpha_1 = -2\alpha_2$$

$$(4.11) \quad A = 2\alpha_1\alpha_2 + \alpha_2^2 = -3\alpha_2^2 = -3x^2$$

$$(4.12) \quad B = -\alpha_1\alpha_2^2 = 2\alpha_2^3 = 2x^3$$

From (4.12) we see that $B/2$ is either a cubic residue modulo p or 0 modulo p . Note that $x = 0 \Leftrightarrow B \equiv 0 \pmod{p}$ from (4.12).

Transform the curve by $X \rightarrow (X + \alpha_2)$, $Y \rightarrow Y$ so that the singular point $S = (x, y) = (x, 0) = (\alpha_2, 0)$ goes to $(0, 0)$. We obtain

$$(4.13) \quad Y^2 - (X + \alpha_2)^3 - A(X + \alpha_2) - B = Y^2 - X^3 - 3\alpha_2 X^2$$

The tangents to $(0, 0)$ on the transformed curve are given by

$$(4.14) \quad Y^2 - 3\alpha_2 X^2 = 0$$

so that the polynomial $R(T)$ is $R(T) = T^2 - 3\alpha_2$. $D = 12\alpha_2 = 12x$.

$$c_4 = b_2^2 - 24b_4 = (a_1^2 + 4a_2)^2 - 24(a_1 a_3 + 2a_4) = -48A.$$

Since

$$x = 0 \Leftrightarrow A \equiv 0 \pmod{p}, \quad D = 0 \Leftrightarrow A \equiv 0$$

and so the invariant c_4 is enough to distinguish between additive and multiplicative reduction. However, as we shall see below it does not separate split and non-split multiplicative reduction.

THEOREM 4.3. Assume that E has bad reduction at p .

(1) E has additive reduction at $p \Leftrightarrow A \equiv 0 \pmod{p} \Leftrightarrow B \equiv 0 \pmod{p}$

$$\Leftrightarrow \left(\frac{-2AB}{p} \right) = 0.$$

(2) E has split multiplicative reduction at $p \Leftrightarrow \left(\frac{-2AB}{p} \right) = 1$.

(3) E has non-split multiplicative reduction at $p \Leftrightarrow \left(\frac{-2AB}{p} \right) = -1$.

Proof: (1) We have seen that $A \equiv 0 \pmod{p} \Leftrightarrow x = 0 \Leftrightarrow B \equiv 0 \pmod{p}$. E has additive reduction at $p \Leftrightarrow D = 12x = 0 \Leftrightarrow x = 0$

$$\Leftrightarrow A \equiv B \equiv 0 \pmod{p} \Leftrightarrow \left(\frac{-2AB}{p} \right) = 0.$$

(2) and (3). Assume E has multiplicative reduction at p . Then $3\alpha_2 \neq 0$. From (4.14) we see that E has split multiplicative reduction at $p \Leftrightarrow 3\alpha_2$ is a square in $\mathbf{Z}/p\mathbf{Z}$. From formulas (4.11) and (4.12) we have that $3\alpha_2 = (-9/2)B/A$. Thus $3\alpha_2$ is a square $\Leftrightarrow (-9/2)B/A$ is a square modulo p

$$\Leftrightarrow -2AB \text{ is a square modulo } p \Leftrightarrow \left(\frac{-2AB}{p} \right) = 1.$$

COROLLARY 4.4. $f_p = \left(\frac{-2AB}{p} \right)$.