§5. CONGRUENCES FOR THE HECKE-EISENSTEIN SERIES

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§5. Congruences for the Hecke-Eisenstein series

For K a totally real number field and $m \geqslant 1$, define

$$\bar{G}_{2m}^{K}(z) = \{(2\pi i)^{2m}/(2m-1)!\}^{-n} D^{2m-1/2} G_{2m}^{K}(z), \qquad (1)$$

where $n = [K:\mathbf{Q}]$ and $G_{2m}^{K}(z)$ (as in §1) is the restriction to the diagonal of the Hecke-Eisenstein series of weight $2m$. Then $\bar{G}_{2m}^{K}$ is a modular form of weight $h = 2mn$ whose Fourier expansion (cf. eqs. (22), (23), (24) and (6) of §1) is

$$\bar{G}_{2m}^{K}(z) = 2^{-n}\zeta_K(1-2m) + \sum_{l=1}^{\infty} s_l^K(2m)\, e^{2\pi i l z} \qquad (2)$$

with $s_l^K(2m) \in \mathbf{Z}$.

In the space $\mathfrak{M}_h$ of all modular forms of weight h, let

$$\mathfrak{M}_h^{\mathbf{Z}} = \{f \in \mathfrak{M}_h \mid f(z) = \sum_{n=0}^{\infty} a_n e^{2\pi i n z},\ a_n \in \mathbf{Z} \text{ for } n \geqslant 1\}$$

be the set of modular forms whose Fourier coefficients, apart from the constant term, are all integral. Then $\mathfrak{M}_h^{\mathbf{Z}}$ is a free $\mathbf{Z}$-module of rank $r = \dim_{\mathbf{C}} \mathfrak{M}_h$ and $\mathfrak{M}_h = \mathfrak{M}_h^{\mathbf{Z}} \otimes_{\mathbf{Z}} \mathbf{C}$. Write

$$c: \mathfrak{M}_h^{\mathbf{Z}} \to \mathbf{C}$$

for the map sending a modular form $f(z) = \Sigma\, a_n e^{2\pi i n z}$ to its constant term a_0. Then

$$c(\mathfrak{M}_h^{\mathbf{Z}}) = \frac{M_h}{N_h}\mathbf{Z}$$

for some coprime integers M_h and N_h, and N_h is then a universal bound for the denominators of the constant terms of forms in $\mathfrak{M}_h^{\mathbf{Z}}$ and in particular of $\bar{G}_{2m}^{K}$, i.e.

$$N_h\, 2^{-n}\zeta_K(1-2m) \in \mathbf{Z}. \qquad (3)$$

This is the essence of Siegel's theorem as discussed in §1.

But we know that (3) is not the best possible bound for the denominator of $\zeta_K(1-2m)$ (cf. the remarks at the end of §3), and this means that the modular forms $\bar{G}_{2m}^{K}$ must be contained in some *smaller* lattice than

$\mathfrak{M}_h^{\mathbf{Z}}$. For example, if K is a real quadratic field, then Serre's bound for the denominator of $\frac{1}{4}\zeta_K(1-2m)$, at least for K not in the set

$$\{\mathbf{Q}(\sqrt{2})\} \cup \{\mathbf{Q}(\sqrt{p}) \mid p \text{ prime}, \quad (p-1) \mid 4m\,, \quad (p-1) \nmid 2m\}\,, \tag{4}$$

is the number $j(m)$ defined in §3, eq. (40), and this is always smaller than $N_h = N_{4m}$ (for $m = 1, 2, 3, 4, 5$ the values of N_{4m} are $2^4.3.5$, $2^5.3.5$, $2^4.3^2.5.7.13$, $2^6.3.5.17$ and $2^4.3.5^2.11$, whereas those of $j(m)$ are $2^3.3$, $2^4.3.5$, $2^3.3^2.7$, $2^5.3.5$ and $2^3.3.11$). Therefore, if K is not one of the finitely many exceptional fields (4), the modular form $\bar{G}_{2m}^K$ lies in the proper sublattice

$$\mathfrak{M}_{4m}^{Se} = \mathfrak{M}_{4m}^{\mathbf{Z}} \cap c^{-1}\left(\frac{1}{j(m)}\,\mathbf{Z}\right) \tag{5}$$

of $\mathfrak{M}_{4m}^{\mathbf{Z}}$. We want to describe some numerical evidence that, although $j(m)$ is the best possible bound for the denominator of $\frac{1}{4}\zeta_K(1-2m)$, the modular forms $\bar{G}_{2m}^K$ are contained in a much smaller sublattice than (5). This means that the coefficients $s_l^K(2m)$ satisfy congruences (modulo certain powers of certain primes) above and beyond those required to obtain the correct bound for the denominator of ζ.

For $m = 1$ or $m = 2$, $\mathfrak{M}_{4m}$ is one-dimensional, so a modular form is completely determined by its constant term and (5) is best possible. Consider $m = 3$. A basis for $\mathfrak{M}_{12}$ is given by Q and R^2, where

$$Q = E_4(z) = 1 + 240 \sum_{n=1}^{\infty} \sigma_3(n)\, e^{2\pi i n z}$$

$$R = E_6(z) = 1 - 504 \sum_{n=1}^{\infty} \sigma_5(n)\, e^{2\pi i n z}$$

(Ramanujan's notation). The lattice $\mathfrak{M}_{12}^{\mathbf{Z}}$ has the basis $\frac{1}{720}\,Q^3$, $\frac{1}{156}\left(\frac{1}{720}\,Q^3 + \frac{1}{1008}\,R^2\right)$. We conjecture, however, that for all real quadratic fields K different from $\mathbf{Q}(\sqrt{2})$, $\mathbf{Q}(\sqrt{5})$ and $\mathbf{Q}(\sqrt{13})$, the modular form $\bar{G}_6^K$ lies in the sublattice generated by $\frac{1}{24}\,Q^3$ and $\frac{5}{504}\,R^2$ i.e. that if we write

$$\bar{G}_6^K = \frac{x}{24}\,Q^3 + \frac{5y}{504}\,R^2 \quad (x, y \in \mathbf{Q})\,,$$

TABLE 5

The modular form $G_6^K(z)$

$K = \mathbf{Q}(\sqrt{D})$, D = discriminant of K

$$\bar{G}_6^K(z) = \frac{225}{64\pi^{12}} D^{11/2} G_6^K(z)$$

$$= \frac{1}{4}\zeta_K(-5) + \sum_{l=1}^{\infty} s_l^K(6)\, q^l \quad (q = e^{2\pi i z})$$

$$E_4(z) = \frac{45}{\pi^4} G_4(z) = 1 + 240 \sum_{n=1}^{\infty} \sigma_3(n)\, q^n$$

$$E_6(z) = \frac{945}{2\pi^6} G_6(z) = 1 - 504 \sum_{n=1}^{\infty} \sigma_5(n)\, q^n$$

$$\bar{G}_6^K(z) = \frac{x}{24} E_4(z)^3 + \frac{5y}{504} E_6(z)^2$$

D	x	y
5	2/5	1
8	11/2	13
12	51	122
13	1018/13	2417/13
17	352	838
21	1092	2602
24	2313	5502
28	5404	12872
29	6438	15327
33	13536	32226
37	24650	58681
40	38437	91526
41	44608	106216
44	64757	154166

then the coefficients x and y will be integral for all quadratic fields K except the three mentioned. Some numerical evidence for this is presented in Table 5 (x and y were calculated for much larger discriminants and were always integers). Similar data for $m=4$ and $m=5$ leads to the conjectures

$$\bar{G}_8^K \in \frac{7Q^4}{480}\mathbf{Z} + \frac{5QR^2}{12}\mathbf{Z} \quad (K \neq \mathbf{Q}(\sqrt{2}),\ \mathbf{Q}(\sqrt{17})\ ,\ ..$$

$$\bar{G}_{10}^K \in \frac{147Q^5}{8}\mathbf{Z} + \frac{5Q^2R^2}{264}\mathbf{Z} \quad (K \neq \mathbf{Q}(\sqrt{2}),\ \mathbf{Q}(\sqrt{5}))\,.$$

These assertions imply highly non-trivial congruences for the coefficients $s_l^K(2m)$ of the Hecke-Eisenstein series, since (for example) the lattice generated by $\frac{147}{8}Q^5$ and $\frac{5}{264}Q^2R^2$ has index 7,938,000 in $\mathfrak{M}_{20}^{\mathbf{Z}}$ (whereas $[\mathfrak{M}_{20}^{\mathbf{Z}} : \mathfrak{M}_{20}^{Se}]$ is only 50). This leads to the following

CONJECTURE. *For each* $m \geqslant 1$, *define the "Hecke-Eisenstein lattice"* $\mathfrak{M}_{4m}^{HE}$ *as the sublattice of* $\mathfrak{M}_{4m}^{\mathbf{Z}}$ *generated by the modular forms* $\bar{G}_{2m}^K$, *where* K *runs over all real quadratic fields not in the finite set* (4). *Then*

(i) $\mathfrak{M}_{4m}^{HE}$ *has finite index in* $\mathfrak{M}_{4m}^{\mathbf{Z}}$.

(ii) *If we replace* (4) *by any larger finite set in the definition of* $\mathfrak{M}_{4m}^{HE}$, *we obtain the same lattice* (in other words, the only fields which are exceptional with respect to the congruence properties of their Hecke-Eisenstein series are those for which the denominator of $\zeta_K(1-2m)$ is exceptionally large).

(iii) *For* $m \leqslant 5$, $\mathfrak{M}_{4m}^{HE}$ *is as given in* Table 6.

(iv) $\mathfrak{M}_{4m}^{HE}$ *has a basis consisting of monomials in* Q *and* R.

(v) *For* $m > 2$, *the primes dividing* $[\mathfrak{M}_{4m} : \mathfrak{M}_{4m}^{HE}]$ *are: all primes* $\leqslant 2m$ *and* $4m+1$ *(if the latter is prime).*

It would be of interest to have numerical data on $\bar{G}_{2m}^K$ for $m > 5$ and for $[K:\mathbf{Q}] > 2$, especially to test the somewhat rash conjecture (iv). Particularly interesting would be to fix a prime p and study the behaviour at p of the sublattice $\mathfrak{M}_{4m}^{HE}$ for varying m, since this could give information about the p-adic analogue of the zeta-function of K.

TABLE 6

The "Hecke-Eisenstein lattice" for $m \leqslant 5$

(In the table, $Q = E_4(z)$, $R = E_6(z)$. The data for $m = 3, 4, 5$ is conjectural only.)

m	Basis for $\mathfrak{M}_{4m}^{\mathbf{Z}}$	Basis for $\mathfrak{M}_{4m}^{HE}$	$[\mathfrak{M}_{4m}^{\mathbf{Z}} : \mathfrak{M}_{4m}^{HE}]$	Exceptional discriminants
1	$\frac{1}{240} Q$	$\frac{1}{24} Q$	$2.5 = 10$	5, 8
2	$\frac{1}{480} Q^2$	$\frac{1}{240} Q^2$	2	8
3	$\frac{1}{720} Q^3$, $\frac{1}{156}\left(\frac{Q^3}{720} + \frac{R^2}{1008}\right)$	$\frac{1}{24} Q^3$, $\frac{5}{504} R^2$	$2^4.3^2.5^2.13 = 46800$	5, 8, 13
4	$\frac{1}{960} Q^4$, $\frac{1}{153}\left(\frac{Q^4}{240} + \frac{QR^2}{192}\right)$	$\frac{7}{480} Q^4$, $\frac{5}{12} QR^2$	$2^5.3^2.5.7.17 = 171360$	8, 17
5	$\frac{1}{1200} Q^5$, $\frac{1}{36}\left(\frac{Q^5}{1200} + \frac{Q^2R^2}{528}\right)$	$\frac{147}{8} Q^5$, $\frac{5}{264} Q^2R^2$	$2^4.3^4.5^3.7^2$ 7938000	5, 8

AFTERWORD

The original version of this paper was written three years ago. To bring it up to date, we must comment on two developments which have occurred in the intervening time.

1. The conjecture of Serre quoted at the end of Section 3 is now (almost) a theorem. In the original paper [6], Serre proved the partial result that,