§4. The Circle Method and the Numbers $e_{2m-1}(n)$

Define an integer $j(m)$ for $m = 1, 2, \ldots$ by

$$j(m) = G.C.D.\{n^{m+2}(n^{2m}-1), n \in \mathbf{Z}\}. \tag{40}$$

Thus

$$j(1) = 24, j(2) = 240, j(3) = 504, j(4) = 480, \ldots$$

Then it is easy to check that, for K a quadratic field, $w_m(K) = j(m)$ (independent of K!) unless K is one of the finitely many fields $\mathbf{Q}(\sqrt{p})$ with p a prime such that $(p-1) \mid 4m$, $(p-1) \nmid 2m$, in which case $w_m(K)$ $p^{v+1} j(m)$, where p^v is the largest power of p dividing m. This is interesting because the numbers $j(m)$ occur in topology: it is known (now that the Adams conjecture has been proved) that $j(m)$ is precisely the order of the group $J(S^{4m})$. This may be just a coincidence, of course, but could conceivably reflect some deeper connection between the values of zeta-functions and topological K-theory (the conjectured connection between these values and algebraic K-theory was mentioned in the introduction).

§4. The Circle Method and the Numbers $e_{2m-1}(n)$

In §3 we defined

$$e_r(n) = \sum_{\substack{k^2 \equiv n \pmod 4 \\ |k| \leq \sqrt{n}}} \sigma_r\left(\frac{n-k^2}{4}\right), \tag{1}$$

where r and n are positive integers and, for b a positive integer, $\sigma_r(b)$ is defined as the sum of the r-th powers of the positive divisors of b. Since (1) was only needed for n not a perfect square, we are still at liberty to define $\sigma_r(0)$; we set

$$\sigma_r(0) = \frac{1}{2}\zeta(-r) = -\frac{1}{2}\frac{B_{r+1}}{r+1}. \tag{2}$$

This defines $\sigma_r(b)$ for $b = 0, 1, 2, \ldots$; we extend the definition to all real b by setting $\sigma_r(b) = 0$ if $b < 0$ or $b \notin \mathbf{Z}$. Then (1) can be rewritten

$$e_r(n) = \sum_{k=-\infty}^{\infty} \sigma_r\left(\frac{n-k^2}{4}\right). \tag{3}$$

We were led to consider these numbers by Siegel's theorem, which, for real quadratic fields K, expresses the value of $\zeta_K(2m)$ or $\zeta_K(1-2m)$ in terms of the numbers $e_{2m-1}(n)$ with $K = \mathbf{Q}(\sqrt{n})$. In this section we

follow a different course, and study the numbers (3) directly by the techniques of analytic number theory—specifically, by means of the Hardy-Littlewood circle method. This will lead to the following formula for $e_{2m-1}(n)$:

THEOREM 1. *Let m and n be positive integers, n not a perfect square. If $n \equiv 2$ or $3 \pmod 4$ then $e_{2m-1}(n) = 0$. If $n \equiv 0$ or $1 \pmod 4$, write*

$$n = f^2 D \tag{4}$$

with

$$D = \text{discriminant of } K, \quad K = \mathbf{Q}(\sqrt{n}). \tag{5}$$

Then

$$e_{2m-1}(n) = \frac{\zeta_K(1-2m)}{2\zeta(1-4m)} T_{2m}^{\chi}(f) + 0(n^{m+1/4}), \tag{6}$$

where χ is the character associated to K (cf. §2) and $T_{2m}^{\chi}(f)$ is the multiplicative function given by

$$T_{2m}^{\chi}(f) = \sum_{t|f} t^{4m-1} \sum_{a|t} \frac{\mu(a)\chi(a)}{a^{2m}} \tag{7}$$

$$= \sum_{a|f} \mu(a)\chi(a) a^{2m-1} \sigma_{4m-1}(f/a) \tag{8}$$

($\mu(a)$ denotes the Möbius function).

Note that the first term in (6) really is of bigger order than the error term, since one easily checks that $T_{2m}^{\chi}(f) > c_1 f^{4m-1}$ and $\zeta_K(1-2m) > c_2 D^{2m-1/2}$ with constants $c_1, c_2 > 0$, and hence the first term is $> c\, n^{2m-1/2}$.

Before turning to the proof of this theorem by means of the Hardy-Littlewood method, we consider its relationship to the results discussed in Sections 1 and 3. We saw in §1 that the Hecke-Eisenstein series $G_{2m}^K(z)$ of K has the Fourier expansion

$$G_{2m}^K(z) \sim a_0 + a_1 q + a_2 q^2 + \ldots \quad (q = e^{2\pi i z}) \tag{9}$$

with

$$a_0 = \zeta_K(2m), \tag{10}$$

$$a_l = k_m s_l^K(2m) = k_m \sum_{j|l} \chi(j) j^{2m-1} e_{2m-1}\left(\frac{l^2}{j^2} D\right), \tag{11}$$

where $k_m = (2\pi)^{4m} D^{-2m+1/2} / (2m-1)!^2$. Since $G_{2m}^K(z)$ is a modular form of weight $4m$, the form $G_{2m}^K(z) - a_0 G_{4m}(z) / 2\zeta(4m)$ is a cusp form

of weight $4m$, where $G_{4m}(z)$ is the ordinary Eisenstein series (we have used 1 (20)). But a very well-known theorem of Hecke asserts that the n-th Fourier coefficient of a cusp form of weight $2k$ is $0(n^k)$. Therefore (using 1 (21) for the Fourier coefficients of G_{4m})

$$\begin{aligned} s_l^K(2m) &= \frac{1}{k_m}\frac{a_0}{2\zeta(4m)}\frac{2^{4m+1}\pi^{4m}}{(4m-1)!}\sigma_{4m-1}(l) + 0(l^{2m}) \\ &= D^{2m-1/2}\frac{(2m-1)!^2}{(4m-1)!}\frac{\zeta_K(2m)}{\zeta(4m)}\sigma_{4m-1}(l) + 0(l^{2m}) \\ &= \frac{\zeta_K(1-2m)}{2\zeta(1-4m)}\sigma_{4m-1}(l) + 0(l^{2m}), \end{aligned}$$

where in the last line we have used the functional equations of ζ_K and ζ. Substituting (11) and inverting gives

$$e_{2m-1}(f^2D) = \sum_{a|f}\mu(a)\chi(a)a^{2m-1}s_{f/a}^K(2m) \tag{12}$$

$$= \frac{\zeta_K(1-2m)}{2\zeta(1-4m)}\sum_{a|f}\mu(a)\chi(a)a^{2m-1}\sigma_{4m-1}(a) + 0(f^{2m}), \tag{13}$$

and this is essentially the same as (6)—indeed with a better error term $0(n^m)$ rather than $0(n^{m+1/4})$.

Nevertheless, there is some point to proving Theorem 1 by the circle method. First of all, it provides a direct proof of the relationship between the arithmetic function $e_{2m-1}(n)$ and the value at $s = 2m$ of the zeta-function of $\mathbf{Q}(\sqrt{n})$. Secondly, the evaluation of the "singular series"—which yields the first term of eq. (6)—involves an evaluation of certain Gauss sums and of a Dirichlet series with such Gauss sums as coefficients which are of interest in their own right. Namely, we will prove the following two theorems.

THEOREM 2. *For positive integers a and c, let*

$$\lambda(a,c) = \begin{cases} i^{(1-c)/2}\left(\dfrac{a}{c}\right) & \text{if } c \text{ is odd, } a \text{ even,} \\ i^{a/2}\left(\dfrac{c}{a}\right) & \text{if } a \text{ is odd, } c \text{ even,} \\ 0 & \text{otherwise,} \end{cases} \tag{14}$$

where $\left(\frac{p}{q}\right)$ *(q odd) is the Legendre-Jacobi symbol and* $i^{a/2} = e^{\pi i a/4}$. *Thus* $\lambda(a, c)$ *is* 0 *if* a *and* c *have a common factor or are both odd, and is an* 8th *root of unity otherwise; furthermore,* $\gamma(a, c)$ *is periodic in* a *with period* $2c$. *We define a Gauss sum* $\gamma_c(n)$ *by*

$$\gamma_c(n) = \frac{1}{\sqrt{c}} \sum_{a=1}^{2c} \lambda(a, c)\, e^{-\pi i n a/c}. \tag{15}$$

Then $\lambda_c(n)$ *is given as follows:*

If c *is odd, write* $c = ld^2$ *with* l *square-free. Then*

$$\gamma_c(n) = \begin{cases} 0 & \text{if } d \nmid n, \\ \displaystyle\sum_{t \mid \left(d, \frac{n}{d}\right)} t\, \mu\left(\frac{d}{t}\right) \left(\frac{n/t^2}{l}\right) & \text{if } d \mid n. \end{cases} \tag{16}$$

If c *is even, write* $c = 2^r c_1$ *with* c_1 *odd,* $r \geqslant 1$. *Then*

$$\gamma_c(n) = Q_r(n)\, \gamma_{c_1}(n), \tag{17}$$

where

$$Q_r(n) = \begin{cases} 2^{r/2} (-1)^{(m-1)/4} & \text{if } r \text{ is even}, \\ & n = 2^{r-2} m, \\ & m \equiv 1 \pmod 4, \\ 2^{\frac{r-1}{2}} (-1)^{m(m-1)/2} & \text{if } r \text{ is odd}, \\ & n = 2^{r-1} m, \\ 0 & \text{otherwise}. \end{cases} \tag{18}$$

THEOREM 3. *Let* n *be a non-zero integer and define a Dirichlet series* $E_n(s)$ *by*

$$E_n(s) = \frac{1}{2} \sum_{\substack{c=1 \\ c \text{ odd}}}^{\infty} \frac{\gamma_c(n)}{c^s} + \frac{1}{2} \sum_{\substack{c=2 \\ c \text{ even}}}^{\infty} \frac{\gamma_c(n)}{(c/2)^s} \tag{19}$$

(i.e. $E_n(s) = \Sigma\, a_m\, m^{-s}$ with $a_m = \frac{1}{2}(\gamma_m(n) + \gamma_{2m}(n))$ for m odd, $a_m = \frac{1}{2}\gamma_{2m}(n)$ for m even. Clearly $|\gamma_c(n)| \leqslant 2c^{1/2}$, so the series in (19) converge for $Re\ s > \frac{3}{2}$; in fact, $\gamma_c(n) = 0(1)$ as $c \to \infty$ by Theorem 2.

so they even converge for $Re\, s > 1$). *Let* $K = \mathbf{Q}(\sqrt{n})$, D = *discriminant of* K, χ = *character of* K, $L(s, \chi)$ = *L-series of* χ *(if* n *is a perfect square,* $\chi(m) = 1$ *for all* m *and* $L(s, \chi) = \zeta(s)$*). Then*

$$E_n(s) \equiv 0 \quad \text{if } n \equiv 2 \text{ or } 3 \pmod 4, \tag{20}$$

while, if $n \equiv 0$ *or* $1 \pmod 4$, *then*

$$E_n(s) = \frac{L(s,\chi)}{\zeta(2s)} \sum_{\substack{a,c \geqq 1 \\ ac \mid f}} \frac{\mu(a)\chi(a)}{c^{2s-1} a^s} = \frac{L(s,\chi)}{\zeta(2s)} \frac{T_s^{\chi}(f)}{f^{2s-1}}, \tag{21}$$

where $n = f^2 D$.

As corollaries to Theorem 3, we see that $E_n(s)$ has a meromorphic continuation to the whole s-plane, and that $E_n(s)$ possesses an Euler product whose p-factor is $1 + \chi(p) p^{-s}$ if $p \nmid n$ and is a polynomial in p^{-s} in any case.

We will now show how the Dirichlet series (19) arises in connection with the numbers $e_{2m-1}(n)$, deferring to the end of the section the proofs of the two theorems on Gauss sums just enunciated.

Let $G_{2m}(z)$ be the Eisenstein series of weight $2m$, defined in 1 (18), and $\theta(z)$ the theta series

$$\theta(z) = \sum_{k=-\infty}^{\infty} e^{\pi i k^2 z} \qquad (z \in \mathfrak{H}), \tag{22}$$

where $\mathfrak{H}$ is the upper half-plane $\{z \in \mathbf{C} \mid Im\, z > 0\}$.

We define

$$F_m(z) = \frac{(-1)^m (2m-1)!}{2^{2m+1} \pi^{2m}} G_{2m}(2z)\, \theta(z) \quad (z \in \mathfrak{H}). \tag{23}$$

Clearly $F_m(z+2) = F_m(z)$, so $F_m(z)$ has a Fourier expansion. From (22) and the Fourier expansion of G_{2m} (eqs. 1 (19)-1 (21)), together with eq. (3) and the functional equation of $\zeta(s)$, we obtain

$$\begin{aligned} F_m(z) &= \left(\frac{(-1)^m (2m-1)!}{2^{2m} \pi^{2m}} \zeta(2m) + \sum_{a=1}^{\infty} \sigma_{2m-1}(a)\, e^{4\pi i a z} \right) \\ &\quad \times \sum_{k=-\infty}^{\infty} e^{\pi i k^2 z} \\ &= \sum_{a=0}^{\infty} \sigma_{2m-1}(z)\, e^{4\pi i a z} \sum_{k} e^{\pi i k^2 z} \\ &= \sum_{n=0}^{\infty} e_{2m-1}(n)\, e^{\pi i n z}. \end{aligned} \tag{24}$$

Thus the numbers $e_{2m-1}(n)$ are precisely the Fourier coefficients of $F_m(z)$. By Cauchy's theorem, therefore,

$$e_{2m-1}(n) = \frac{1}{2}\int_{i\varepsilon}^{2+i\varepsilon} e^{-\pi i n z} F_m(z)\, dz \tag{25}$$

for any $\varepsilon > 0$.

The idea of the Hardy-Littlewood method is to replace the integrand in the neighbourhood of each rational point of the interval [0, 2] by an elementary function, integrate this function, and then sum up the contributions obtained in this way from all rational points; this sum, the so-called "singular series," should then be an approximation to the integral. To apply this to (25), we first use the transformation laws of the theta and Eisenstein series under modular transformations to obtain

$$\theta\left(\frac{a}{c}+iy\right) = \lambda(a,c)(cy)^{-1/2} + 0(y^{-1/2}e^{-\pi/4c^2y}), \tag{26}$$

$$G_{2m}\left(\frac{a}{c}+iy\right) = 2(-1)^m \zeta(2m)(cy)^{-2m} + 0(y^{-2m}e^{-\pi/c^2y}) \tag{27}$$

as $y \to 0$ with $Re(y) > 0$, where $\frac{a}{c}$ is a rational number in lowest terms. Therefore

$$F_m\left(\frac{a}{c}+iy\right) = \frac{(2m-1)!}{2^{4m}\pi^{2m}}\zeta(2m)\frac{(c,2)^{2m}}{c^{2m+1/2}}\lambda(a,c)\,y^{-2m-1/2} + 0(y^{-2m-1/2}e^{-\pi/4c^2y}) \tag{28}$$

as $y \to 0$, where a and c are relatively prime and $(c, 2)$ is the greatest common divisor of c and 2. To obtain the contribution from the rational point a/c to the singular series, therefore, we replace F_m by the first member of (28) and integrate over y. Since

$$\frac{1}{2}\int_{-i\infty+\varepsilon}^{i\infty+\varepsilon} e^{-\pi i n(iy+a/c)} y^{-2m-1/2}\, dy = \pi^{2m+1/2} n^{2m-1/2} e^{-\pi i n a/c}/\Gamma(2m+1/2) \tag{29}$$

(this is just the standard integral representation for $1/\Gamma(s)$), we obtain as the contribution from a/c

$$C(m)\, n^{2m-1/2}\frac{(c,2)^{2m}}{c^{2m+1/2}}\lambda(a,c)\, e^{-\pi i n a/c} \tag{30}$$

with

$$C(m) = \frac{\pi^{1/2}(2m-1)!}{2^{4m}\,\Gamma(2m+1/2)}\,\zeta(2m)\,. \tag{31}$$

Summing this over all rational points $\frac{a}{c} \in [0, 2)$, we obtain the following formula for the singular series:

$$\bar{e}_{2m-1}(n) = C(m)\,n^{2m-1/2} \sum_{c=1}^{\infty} \frac{(c,2)^{2m}}{c^{2m}}\,\gamma_c(n) \tag{32}$$

$$= 2\,C(m)\,n^{2m-1/2}\,E_n(2m)\,, \tag{33}$$

where $E_n(s)$ is the Dirichlet series of Theorem 3.

We wish to estimate the difference between $e_{2m-1}(n)$ and $\bar{e}_{2m-1}(n)$. To do this, we define a function having the same behaviour in the neighbourhood of each rational point $\frac{a}{c}$ as that described by the leading term of (28):

$$\bar{F}_m(z) = \frac{(2m-1)!}{2^{4m}\,\pi^{2m}}\,\zeta(2m)$$

$$\times \sum_{c=1}^{\infty} \frac{(c,2)^{2m}}{c^{2m-1/2}} \sum_{a=-\infty}^{\infty} \lambda(a,c) \left(\frac{z-a/c}{i}\right)^{-2m-1/2}. \tag{34}$$

The series is convergent for $z \in \mathfrak{H}$, and

$$F_m(z) - \bar{F}_m(z) = 0\,(y^{-2m-1/2}\,e^{-\pi/4c^2y}) \tag{35}$$

for $z = \frac{a}{c} + iy$, $y \to 0$. On the other hand, $\bar{F}_m(z)$ is evidently periodic with period 2, and one easily finds (using the Cauchy integral for the Fourier coefficients and the contour integral (29)) that its Fourier expansion is

$$\bar{F}_m(z) = \sum_{n=1}^{\infty} \bar{e}_{2m-1}(n)\,e^{\pi i n z} \tag{36}$$

with $\bar{e}_{2m-1}(n)$ given by (32). The analysis given by Hardy [2] now permits us to deduce from (35) that

$$e_{2m-1}(n) - \bar{e}_{2m-1}(n) = 0\,(n^{m+1/4}) \tag{37}$$

as $n \to \infty$. We will not reproduce this analysis here, since our main interest is not in a rigorous proof of (6) with error term (in any case, as pointed out

above, this error term is not best possible) but in the evaluation of the singular series obtained in the Hardy-Littlewood approach. To see that (37) and (6) are the same, we use equation (33) and Theorem 3 to get

$$\bar{e}_{2m-1}(f^2D) = 2C(m)f^{4m-1}D^{2m-1/2}\frac{L(2m,\chi)}{\zeta(4m)}\sum_{ac|f}\frac{\mu(a)\chi(a)}{c^{4m-1}a^{2m}}$$

$$= \frac{\zeta_K(1-2m)}{2\zeta(1-4m)}T_{2m}^{\chi}(f), \tag{38}$$

where in the last line we have used (7) and the functional equations of ζ and ζ_K.

It remains to prove Theorems 2 and 3.

Proof of Theorem 2: We first suppose c is odd. Then the standard Gauss sum

$$\tau_c(n) = \sum_{b=1}^{c}\left(\frac{b}{c}\right)e^{2\pi inb/c} \tag{39}$$

is related to $\gamma_c(n)$ by

$$\gamma_c(n) = \sum_{\substack{a=1\\ a \text{ even}}}^{2c} c^{-1/2}\, i^{\frac{1-c}{2}}\left(\frac{a}{c}\right)e^{-\pi ina/c}$$

$$= c^{-1/2}\, i^{(1-c)/2}\left(\frac{-2}{c}\right)\tau_c(n), \tag{40}$$

as one sees by setting $a = 2b$. If c is square-free, then the value of (39) is well known to be

$$\tau_c(n) = \begin{cases} \left(\frac{n}{c}\right)\sqrt{c} & \text{if } c \equiv 1 \pmod 4, \\ \left(\frac{n}{c}\right)i\sqrt{c} & \text{if } c \equiv 3 \pmod 4, \end{cases} \tag{41}$$

or

$$\tau_c(n) = i^{\frac{c-1}{2}}\left(\frac{-2n}{c}\right)c^{1/2} \quad (c \text{ square-free}) \tag{42}$$

Therefore $\gamma_c(n) = \left(\frac{n}{c}\right)$ if c is square-free, in agreement with (16) (since in this case $d = 1$, $l = c$). Now let $c = ld^2$ with l square-free. Then

$$\left(\frac{b}{c}\right) = \left(\frac{b}{l}\right)\left(\frac{b}{d}\right)^2 = \begin{cases} \left(\frac{b}{l}\right) & \text{if } (b,d) = 1, \\ 0 & \text{if } (b,d) > 1 \end{cases}$$

$$= \left(\frac{b}{l}\right) \sum_{\substack{j \mid b \\ j \mid d}} \mu(j),$$

where $\mu(j)$ is the Möbius function, so

$$\tau_c(n) = \sum_{j \mid d} \mu(j) \sum_{\substack{b=1 \\ j \mid b}}^{c} \left(\frac{b}{l}\right) e^{2\pi i n b/c}$$

$$= \sum_{j \mid d} \mu(j) \left(\frac{j}{l}\right) \sum_{k=1}^{c/j} \left(\frac{k}{l}\right) e^{2\pi i n k j/c}, \tag{43}$$

where we have written $b = jk$. Since $\left(\frac{k}{l}\right)$ only depends on $k \pmod l$, the inner sum in (43) equals

$$\sum_{r=1}^{l} \left(\frac{r}{l}\right) \sum_{m=1}^{c/jl} e^{2\pi i n (r+ml) j/c} = \begin{cases} 0 & \text{if } \frac{c}{jl} \nmid n, \\ \frac{c}{jl}\, \tau_l\left(\frac{njl}{c}\right) & \text{if } \frac{c}{jl} \mid n. \end{cases} \tag{44}$$

Write t for d/j, so $\frac{c}{jl} = dt$. Then, substituting (44) into (43), we find that $\tau_c(n) = 0$ if $d \nmid n$, while if $d \mid n$

$$\tau_c(n) = \sum_{\substack{t \mid d \\ dt \mid n}} \mu\left(\frac{d}{t}\right)\left(\frac{d/t}{l}\right) dt\, \tau_l\left(\frac{n}{dt}\right)$$

Since l is square-free, we can now use (42) to get

$$\tau_c(n) = dl^{1/2}\left(\frac{-2}{l}\right) i^{(l-1)/2} \sum_{t \mid \left(d, \frac{d}{n}\right)} t\mu\left(\frac{d}{t}\right)\left(\frac{d/t}{l}\right)\left(\frac{n/dt}{l}\right). \tag{45}$$

The factor preceding the sum is precisely $c^{1/2}\left(\frac{-2}{c}\right) i^{(c-1)/2}$, since $c = ld^2 \equiv l \pmod 8$, so combining (45) and (40) yields precisely equation (16).

Now suppose that c is even, $c = 2^r c_1$ ($r \geqslant 1$, c_1 odd). For a odd, we have

$$\lambda(a,c) = i^{a/2}\left(\frac{c}{a}\right) = \left[i^{a/2}\left(\frac{2}{a}\right)^r\left(\frac{-1}{c_1}\right)^{(a-1)/2}\right]\left(\frac{a}{c_1}\right), \tag{46}$$

where we have used the law of quadratic reciprocity. The factor in square brackets has period 8 and the factor $\left(\frac{a}{c_1}\right)$ has period c_1, so

$$\lambda(a+8c_1,c) = \lambda(a,c). \tag{47}$$

It follows easily that $\gamma_c(n)$ is 0 unless $e^{-8\pi i n c_1/c}$ equals 1, i.e. unless 2^{r-2} divides n (this condition is empty if $r = 1$). Write

$$n = 2^{r-2}\nu \tag{48}$$

with ν an integer. Then

$$\gamma_c(n) = \frac{2^{r-2}}{\sqrt{c}} \sum_{\substack{a=1\\ a \text{ odd}}}^{8c_1} \lambda(a,c)\, e^{-\pi i \nu a/4c_1}. \tag{49}$$

Now write

$$a = k c_1^{\,2} + 8jy \tag{50}$$

where

$$8y \equiv 1 \pmod{c_1} \tag{51}$$

(e.g. $y = (1-c_1^2)/8$). Then $a \equiv j \pmod{c_1}$ and $a \equiv k \pmod 8$, so a runs over all odd residue classes (mod $8c_1$) when j runs over the values 1, 2, ..., c_1 and k over the values 1, 3, 5, 7. Therefore (46) and (49) give

$$\gamma_c(n) = \frac{2^{r-2}}{\sqrt{c}} \sum_{j=1}^{c_1}\left(\frac{j}{c_1}\right) e^{-2\pi i \nu y j/c_1} \times \sum_{\substack{k=1\\ k \text{ odd}}}^{8} i^{k/2}\left(\frac{2}{k}\right)^r\left(\frac{-1}{c_1}\right)^{(k-1)/2} e^{-\pi i \nu c_1 k/4}. \tag{52}$$

The first sum is $\tau_{c_1}(-\nu y)$, and by virtue of (51), (48) and (40).

$$\tau_{c_1}(-\nu y) = \left(\frac{-2}{c_1}\right)\left(\frac{2}{c_1}\right)^r \tau_{c_1}(n) = \sqrt{c_1}\left(\frac{2}{c_1}\right)^r i^{\frac{c_1-1}{2}}\, \gamma_{c_1}(n). \tag{53}$$

The second sum in (52) is

$$i^{1/2}\, e^{-\pi i \nu c_1/4} + (-1)^r\left(\frac{-1}{c_1}\right) i^{3/2}\, e^{-3\pi i \nu c_1/4}$$

$$+(-1)^r i^{5/2} e^{-5\pi i v c_1/4} + \left(\frac{-1}{c_1}\right) i^{7/2} e^{-7\pi i v c_1/4}$$

$$= i^{1/2} e^{-\pi i v c_1/4} \left(1-(-1)^{r+v}\right)\left(1+i(-1)^r \left(\frac{-1}{c_1}\right) e^{-\pi i v c_1/2}\right).$$

Putting this all into (52), we obtain

$$\gamma_c(n) = 2^{\frac{r}{2}-2} \left(\frac{2}{c_1}\right)^r i^{\frac{c_1}{2}} e^{-\pi i v c_1/4} \left(1-(-1)^{r+v}\right)$$
$$\times \left(1+(-1)^r i^{c_1(1-v)}\right) \gamma_{c_1}(n). \tag{54}$$

Clearly this is 0 if $r \equiv v \pmod 2$, while if $v \equiv r-1 \pmod 2$ we obtain

$$\gamma_c(n) = 2^{r/2} \left(\frac{2}{c_1}\right)^r \cos\frac{\pi c_1 (v-1)}{4} \gamma_{c_1}(n). \tag{55}$$

If r is even, therefore, v must be odd, and then the cosine in (55) is 0 if $v \equiv 3 \pmod 4$ and $(-1)^{(v-1)/4}$ if $v \equiv 1 \pmod 4$. Thus for r even, $\gamma_c(n)$ is 0 unless $n = 2^{r-2} m$ with $m \equiv 1 \pmod 4$ and is then $2^{r/2} (-1)^{(m-1)/4} \times \gamma_{c_1}(n)$. If r is odd, then v is even, say $v = 2m$, and then the cosine in (55) $= (-1)^{m(m-1)/2} (2/c_1)/\sqrt{2}$. Thus for r odd, $\gamma_c(n)$ is 0 unless $n = 2^{r-1} m$ and is then $2^{(r-1)/2} (-1)^{m(m-1)/2} \gamma_{c_1(n)}$. This proves equation (18).

Proof of Theorem 3. According to eq. (17), we can write

$$E_n(s) = E_n^{\text{odd}}(s) R_n(s), \tag{56}$$

with

$$E_n^{\text{odd}}(s) = \sum_{\substack{c=1 \\ c \text{ odd}}}^{\infty} \frac{\gamma_c(n)}{c^s} \tag{57}$$

and

$$R_n(s) = \frac{1}{2} + \frac{1}{2} \sum_{r=1}^{\infty} \frac{Q_r(n)}{(2^{r-1})^s}. \tag{58}$$

We first evaluate (57). Substituting (16) gives

$$E_n^{\text{odd}}(s) = \sum_{\substack{d \mid n \\ d \text{ odd}}} \sum_{\substack{l=1 \\ l \text{ odd} \\ l \text{ square-free}}}^{\infty} \frac{\gamma_{ld^2}(n)}{l^s d^{2s}}$$

$$= \sum_{\substack{d \mid n \\ d \text{ odd}}} d^{-2s} \sum_{t \mid \left(d, \frac{n}{d}\right)} t\mu\left(\frac{d}{t}\right) \sum_{\substack{l \text{ odd} \\ l \text{ square-free}}} \left(\frac{n/t^2}{l}\right) l^{-s}. \tag{59}$$

Now let r^2 be the largest odd square dividing n, and write $n = Nr^2$. Then $t \mid r$, and N and D differ by an even power of 2, so for any odd l

$$\left(\frac{n/t^2}{l}\right) = \left(\frac{r/t}{l}\right)^2 \left(\frac{D}{l}\right) = \left(\frac{r/t}{l}\right)^2 \chi(l),$$

where χ is the character of $K = \mathbf{Q}(\sqrt{n})$. Also $\left(\frac{r/t}{l}\right)^2$ is 1 or 0 depending whether l is or is not relatively prime to r/t. Therefore (59) can be rewritten

$$E_n^{\text{odd}}(s) = \sum_{\substack{d \mid n \\ d \text{ odd}}} d^{-2s} \sum_{t \mid \left(d, \frac{n}{d}\right)} t\mu\left(\frac{d}{t}\right) \prod_{p \nmid \frac{2r}{t}} (1 + \chi(p)\, p^{-s}), \tag{60}$$

where the final product extends over primes p not dividing the even integer $2r/t$. Let $u = \frac{r}{t}$, $e = \frac{d}{t}$ then

$$\begin{aligned} E_n^{\text{odd}}(s) &= \sum_{u \mid r} (u/r)^{2s-1} \sum_{\substack{e \mid Nu^2 \\ e \text{ odd}}} \frac{\mu(e)}{e^{2s}} \prod_{p \nmid 2u} (1 + \chi(p)\, p^{-s}) \\ &= \prod_{p \neq 2} \left(1 + \frac{\chi(p)}{p^s}\right) \sum_{u \mid r} \frac{u^{2s-1}}{r^{2s-1}} \prod_{\substack{p \mid Nu^2 \\ p \neq 2}} (1 - p^{-2s}) \prod_{p \mid u} (1 + \chi(p)\, p^{-s})^{-1} \\ &= \prod_{p \neq 2} \left(1 + \frac{\chi(p)}{p^s}\right) \prod_{\substack{p \mid N \\ p \neq 2}} \left(1 - \frac{1}{p^{2s}}\right) \sum_{u \mid r} \frac{u^{2s-1}}{r^{2s-1}} \prod_{p \mid u} \left(1 - \frac{\chi(p)}{p^s}\right) \\ &= \prod_{p \neq 2} \frac{1 - p^{-2s}}{1 - \chi(p)\, p^{-s}}\, r^{1-2s}\, T_s^{\chi}(r). \end{aligned} \tag{61}$$

We now evaluate the factor (58) of $E_n(s)$ corresponding to the prime 2. Comparing (61) and (20), (21), we see that it remains to prove

$$R_n(s) = \begin{cases} 0 & \text{if } n \equiv 2, 3 \pmod 4, \\ \dfrac{1 - 2^{-2s}}{1 - \chi(2)\, 2^{-s}}\, 2^{q(1-2s)}\, T_s^{\chi}(2^q) & \text{if } n = f^2 D, \end{cases} \tag{62}$$

where in the latter case we have set $f = 2^q r$, r odd.

The first line of (62) follows immediately from (18), since we see that

$$n \equiv 2, 3 \pmod 4 \Rightarrow Q_1(n) = -1,\ Q_r(n) = 0\ (r > 1). \tag{63}$$

We thus suppose $n = f^2 D$, $f = 2^q r$, r odd. We distinguish two cases, according to the parity of D:

Case 1. $D \equiv 0 \pmod 4$, $\chi(2) = 0$. Then either $D = 8d$ with d odd or $D = 4d$ with $d \equiv 3 \pmod 4$. In either case, we deduce easily from (18) that $Q_r(n) = 0$ if r is even or if r is odd and greater than $2q+3$, that $Q_r(n) = 2^{(r-1)/2}$ if r is odd and less than $2q+3$, and that $Q_{2q+3}(n) = -2^{q+1}$. Therefore

$$\begin{aligned} R_n(s) &= \frac{1}{2}\left[1 + \sum_{\substack{r=1 \\ r \text{ odd}}}^{2q+1} \frac{2^{(r-1)/2}}{2^{(r-1)s}} - \frac{2^{q+1}}{2^{(q+1)s}}\right] \\ &= \frac{1}{2}\left[1+1+x^2+x^4+\ldots+x^{2q}-x^{2q+2}\right] \\ &= (1-x^2/2)(1+x^2+\ldots+x^{2q}) \\ &= (1-2^{-2s})\,2^{-q(2s-1)}(1+2^{2s-1}+\ldots+2^{q(2s-1)}) \\ &= (1-2^{-2s})\,2^{-q(2s-1)}\,T_s^{\chi}(2^q), \end{aligned}$$

in agreement with (62); in this calculation we have set $x = 2^{-s+\frac{1}{2}}$ for convenience.

Case 2. $D \equiv 1 \pmod 4$, $\chi(2) = (-1)^{(D-1)/4}$. In this case, equation (18) tells us that $Q_r(n) = 2^{(r-1)/2}$ if r is odd and $1 \leqslant r \leqslant 2q+1$, that $Q_{2q+2}(n) = 2^{q+1}\chi(2)$, and that $Q_r(n) = 0$ for all other values of r. Therefore

$$\begin{aligned} R_n(s) &= \frac{1}{2}\left[1 + \sum_{\substack{r=1 \\ r \text{ odd}}}^{2q+1} \frac{2^{(r-1)/2}}{2^{(r-1)s}} + \frac{2^{q+1}\chi(2)}{2^{(2q+1)s}}\right] \\ &= \frac{1}{2}\left[1+1+x^2+x^4+\ldots+x^{2q}+\chi(2)\sqrt{2}\,x^{2q+1}\right] \\ &= \left[1+\frac{\chi(2)}{\sqrt{2}}x\right]\left[1+x^2+\ldots+x^{2q}-\frac{\chi(2)}{\sqrt{2}}(x+x^3+\ldots+x^{2q-1})\right] \\ &= 2^{-q(2s-1)}\left[1+\frac{\chi(2)}{2^s}\right]\left[1+2^{2s-1}+\ldots+2^{q(2s-1)}\right. \\ &\qquad \left. -\frac{\chi(2)}{2^s}(2^{2s-1}+2^{2(2s-1)}+\ldots+2^{q(2s-1)})\right] \\ &= 2^{-q(2s-1)}\,\frac{1-2^{-2s}}{1-\chi(2)\,2^{-s}}\,T_s^{\chi}(2^q). \end{aligned}$$

This proves (62) in this case also, and completes the evaluation of $E_n(s)$.