3. Dirichlet's fundamental theorems

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3. Dirichlet's fundamental theorems

Theorem 3.1. If p is a prime with $p \equiv 3 \pmod 4$, then (1.3) holds.

Proof. Let M denote the left side of (1.3). We shall first show that

$$M = \frac{1}{2} p^{1/2} \sum_{k=1}^{p-1} \left(\frac{k}{p}\right) \cot (\pi k/p) . \tag{3.1}$$

Formula (3.1) is quite ancient, and several references to it can be found in Dickson's history [22, Chapter 6]. For references to more recent proofs and generalizations, see [7, section 5]. For completeness, we shall reproduce the following argument of Whiteman [60]. Since

$$\sum_{j=1}^{p-1} j \sin (2\pi jk/p) = - \frac{1}{2} p \cot (\pi k/p) , \tag{3.2}$$

we have, upon the use of (3.2) and then (2.2),

$$\begin{aligned} \sum_{k=1}^{p-1} \left(\frac{k}{p}\right) \cot (\pi k/p) &= - \frac{2}{p} \sum_{j=1}^{p-1} j \sum_{k=1}^{p-1} \left(\frac{k}{p}\right) \sin (2\pi jk/p) \\ &= - \frac{2}{p} \sum_{j=1}^{p-1} j \left(\frac{j}{p}\right) p^{1/2} , \end{aligned}$$

and (3.1) immediately follows.

Thus, to show that M is positive, it suffices to show that the right side of (3.1) is positive. As

$$M = \sum_{j=1}^{p-1} j - 2 \sum_{1 \leqq r \leqq p-1} r \equiv p(p-1)/2 \equiv 1 \pmod 2 ,$$

since $p \equiv 3 \pmod 4$, it suffices to show that the right side of (3.1) is non-negative.

Using the partial fraction decomposition

$$\pi \cot (\pi x) = \lim_{N -> \infty} \sum_{m=-N}^{N} 1/(m+x) ,$$

where x is non-integral, we have

$$\begin{aligned} \sum_{k=1}^{p-1} \left(\frac{k}{p}\right) \cot (\pi k/p) &= \frac{1}{\pi} \lim_{N -> \infty} \sum_{k=1}^{p-1} \left(\frac{k}{p}\right) \sum_{m=-N}^{N} \frac{1}{m + k/p} \\ &= \frac{p}{\pi} \lim_{N -> \infty} \sum_{j=-Np}^{(N+1)p} \left(\frac{j}{p}\right) \frac{1}{j} \\ &= \frac{2p}{\pi} L(1, \chi_p) , \end{aligned} \tag{3.3}$$

where in the penultimate step we put $j = mp + k$ and lastly use the fact that $\left(\frac{j}{p}\right)$ is an odd function of j. Thus, from (3.3), it suffices to show that $L(1, \chi_p)$ is non-negative.

Now, for $s > 1$,

$$L(s, \chi_p) = \prod_q \left\{1 - \left(\frac{q}{p}\right) q^{-s}\right\}^{-1}, \tag{3.4}$$

where the product is over all primes q. Each factor on the right side of (3.4) is positive for $s > 1$. Thus, $L(s, \chi_p) > 0$ for $s > 1$. Since the infinite series in (2.3) converges uniformly for $\varepsilon \leqq s < \infty$, where $0 < \varepsilon < 1$, $L(s, \chi_p)$ is continuous at $s = 1$. Hence, $L(1, \chi_p) \geqq 0$, and the proof of Theorem 3.1 is complete.

Apparently, Chung [19] was the first person to give a proof of Theorem 3.1 that was independent of the consideration of binary quadratic forms and class numbers. Subsequent proofs of (1.1) and (1.3) were given by Chowla [18], Whiteman [60], Moser [47] and Carlitz [16]. Moser also discusses (1.1) in [48]. There is also a nice proof of (1.3) in Davenport's book [20, p. 10]. All of these proofs use Fourier series. Now, in fact, the proofs of Chung, Chowla, Whiteman, Moser, and Carlitz are essentially no different from the proofs given by Dirichlet [24] in 1840 and later by Berger [5] in 1884 and Lerch [44] in 1905. The only difference is that the five aforementioned authors avoid the language of class numbers.

Perhaps our proof above is a modicum more elementary in that it does not use Fourier series but instead employs the partial fraction decomposition of cot (πx), which can be derived by quite elementary means [49]. Of course, our method above is applicable to any odd real primitive character.

Next, we show that very short proofs of (1.1) and (1.3) may be given by the use of contour integration.

THEOREM 3.2. If χ is odd, then

$$S_{21} = \frac{iG(\chi)}{\pi} \{\bar{\chi}(2) - 2\} L(1, \bar{\chi}).$$

Proof. Let

$$f(z) = \frac{\pi F(z, \chi)}{z \cos(\pi z)},$$

where

$$F(z, \chi) = \sum_{0<j<k/2} \chi(j) \cos(\pi z - 4\pi j z/k).$$

Observe that f has a simple pole at $z = 0$ with

$$(3.5) \qquad R(f, 0) = \pi F(0, \chi) = \pi S_{21} .$$

Also, f has simple poles at $z = (2n-1)/2$, $-\infty < n < \infty$, with

$$(3.6) \qquad \begin{aligned} R(f, (2n-1)/2) &= \frac{2(-1)^n}{2n-1} F((2n-1)/2, \chi) \\ &= \frac{i}{2n-1} G(2n-1, \chi) \\ &= \frac{i}{2n-1} \bar{\chi}(2n-1) G(\chi), \end{aligned}$$

by (2.1).

Let C_N denote the positively oriented rectangle with center at the origin, horizontal sides of length $2N$, and vertical sides of length $N^{1/2}$, where N is a positive integer. Applying the residue theorem with the aid of (3.5) and (3.6), we get

$$(3.7) \quad I_N \equiv \frac{1}{2\pi i} \int_{C_N} f(z)\, dz = \pi S_{21} + iG(\chi) \sum_{n=-N+1}^{N} \frac{\bar{\chi}(2n-1)}{2n-1} .$$

From the definition of $F(z, \chi)$, we see that there exists a positive constant A, independent of N, such that for all $z = x + iy$ on the horizontal sides of C_N, $| F(z, \chi)/\cos(\pi z) | \leqq A \exp(-2\pi |y|/k)$. Also, $F(z, \chi)/\cos(\pi z)$ has period $2k$. Thus, there is a positive constant B, independent of N, such that for all z on the vertical sides of C_N, $| F(z, \chi)/\cos(\pi z) | \leqq B$. Hence we find that as N tends to ∞,

$$(3.8) \qquad I_N = 0(e^{-\pi N^{1/2}/k}) + 0(N^{-1/2}) = o(1) .$$

Letting N tend to ∞, we deduce from (3.7) and (3.8) that

$$S_{21} = -\frac{iG(\chi)}{\pi} \sum_{n=-\infty}^{\infty} \frac{\bar{\chi}(2n-1)}{2n-1} = -\frac{2iG(\chi)}{\pi} \left\{1 - \frac{1}{2}\bar{\chi}(2)\right\} L(1, \bar{\chi}),$$

which completes the proof.

A direct proof of Theorem 3.1, or, more properly, an obvious generalization thereof, may also be achieved by contour integration. Integrate

$$\frac{1}{z(e^{2\pi i z} - 1)} \sum_{0<j<p} \chi(j)\, e^{2\pi i j z/p}$$

over a rectangle C_N like that of the previous proof, but with the horizontal sides of length $2N + 1$.

A short proof of Theorem 3.2 using the character Poisson formula can be found in [7, section 4].

From the classical theory of L-functions, it can be shown that if χ is a real primitive character, then $L(1, \chi) > 0$ [2, pp. 27-28]. We shall repeatedly use this fact without comment in the sequel. Hence, the following is immediate from Theorem 3.2.

COROLLARY 3.3. If χ is real and odd, then $S_{21} > 0$.

The following corollary is an immediate consequence of Theorem 3.2 and (2.4) and is one of Dirichlet's famous class number formulas [23].

COROLLARY 3.4. If $d < 0$, then

$$S_{21} = \left\{2 - \left(\frac{d}{2}\right)\right\} h(d).$$

COROLLARY 3.5. If $p \equiv 3 \pmod 4$, then $S_{21}(\chi_p)$ is odd; if, furthermore, $p \equiv 3 \pmod 8$, then $3 \mid S_{21}(\chi_p)$.

COROLLARY 3.6. If $p \equiv 3 \pmod 4$, then $h(-p)$ is odd.

We now will give two proofs of (1.2) below. The first, in essence, is due to Dirichlet [24].

THEOREM 3.7. Let χ be even. Then if $\chi_{4k}(n) = \chi_4(n)\chi_k(n)$,

$$S_{41} = \frac{G(\chi)}{\pi} L(1, \bar{\chi}_{4k}).$$

First proof. Let

$$f(x) = \begin{cases} 1, & 0 \leqslant x < \pi/2, \\ 0, & x = \pi/2, \\ -1, & \pi/2 < x \leqslant \pi, \end{cases}$$

be an even function with period 2π. Calculating the Fourier series of f, we find that

$$(3.9) \qquad f(x) = -\frac{4}{\pi} \sum_{n=1}^{\infty} \frac{(-1)^n \cos(2n-1)x}{2n-1} \qquad (-\infty < x < \infty).$$

Next, in (2.1), replace n by $2n - 1$. Then multiply both sides by $(-1)^n/(2n-1)$ and sum on n, $1 \leqq n < \infty$, to get

$$(3.10) \quad -G(\chi)\,L(1, \bar{\chi}_{4k}) = \sum_{j=1}^{k-1} \chi(j) \sum_{n=1}^{\infty} \frac{(-1)^n}{2n-1} \cos\{2\pi j(2n-1)/k\}$$
$$= -\frac{\pi}{4} \sum_{j=1}^{k-1} \chi(j) f(2\pi j/k),$$

by (3.9). Since χ is even, $S_{41} = -S_{42} = -S_{43} = S_{44}$. Using the definition of f, we see then that (3.10) reduces to

$$G(\chi)\,L(1, \bar{\chi}_{4k}) = \pi S_{41},$$

which completes the proof.

Second proof. Let

$$f(z) = \frac{\pi F(z, \chi)}{z \cos(\pi z)},$$

where

$$F(z, \chi) = \sum_{0<j<k/4} \chi(j) \cos(4\pi j z/k)$$
$$- \sum_{k/4<j<k/2} \chi(j) \cos(2\pi z - 4\pi j z/k).$$

Note that

$$(3.11) \qquad R(f, 0) = \pi F(0, \chi) = \pi(S_{41} - S_{42}) = 2\pi S_{41}$$

and that, for $-\infty < n < \infty$,

$$(3.12) \qquad R(f, (2n-1)/2) = \frac{2(-1)^n}{2n-1} F((2n-1)/2, \chi)$$
$$= \frac{(-1)^n}{2n-1} G((2n-1)/2, \chi)$$
$$= \frac{(-1)^n}{2n-1} \bar{\chi}(2n-1)\, G(\chi),$$

by (2.1).

We integrate f over the same rectangle C_N as in the proof of Theorem 3.2. By an argument similar to that in that proof, we find that

$$(3.13) \qquad I_N \equiv \frac{1}{2\pi i} \int_{C_N} f(z)\,dz = o(1),$$

as N tends to ∞. Hence, applying the residue theorem to I_N, using (3.11) and (3.12), letting N tend to ∞, and employing (3.13), we find that

$$0 = 2\pi S_{41} + G(\chi) \sum_{n=-\infty}^{\infty} \frac{(-1)^n \bar{\chi}(2n-1)}{2n-1},$$

from whence Theorem 3.7 follows.

A proof of Theorem 3.7 using the character Poisson formula may be found in [7, section 4].

COROLLARY 3.8. If χ is real and even, then $S_{41} > 0$.

Additional class number formulas of Dirichlet are immediate consequences of Theorem 3.7.

COROLLARY 3.9. If $4 \nmid d$, then

(3.14) $$S_{41}(\chi_d) = \frac{1}{2} h(-4d), \qquad d > 0,$$

$$S_{41}(\chi_{-4d}) = 2h(d), \qquad d < 0,$$

(3.15) $$S_{41}(\chi_{8d}) = h(-8d), \qquad d > 0,$$

and

(3.16) $$S_{41}(\chi_{-8d}) = h(8d), \qquad d < 0.$$

COROLLARY 3.10. If $p \equiv 1 \pmod 8$, then $h(-4p) \equiv 0 \pmod 4$; if $p \equiv 5 \pmod 8$, then $h(-4p) \equiv 2 \pmod 4$. If p is odd, then $h(-8p)$ is even.

Proof. The number of summands in $S_{41}(\chi_p)$ is even if $p \equiv 1 \pmod 8$ and odd if $p \equiv 5 \pmod 8$. Thus, the congruences for $h(-4p)$ readily follow from (3.14). For all odd primes p, $S_{41}(\chi_{8p})$ has $2p$ terms and, thus, $p - 1$ non-zero summands. Hence, $S_{41}(\chi_{8p})$ is even, and (3.15) and (3.16) show that $h(-8p)$ is even.

The congruences for $h(-4p)$ in Corollary 3.10 appear to have been first stated by Lerch [45, p. 224], although they were, no doubt, known to Dirichlet. For other proofs of the congruences in Corollary 3.10, for equivalent formulations, and for some refinements, see the papers of Brown [10], [11], [14], Hasse [32], [33], [34], and Barrucand and Cohn [4].