3. Proof of Theorem 1

Objekttyp: **Chapter**

Zeitschrift: **L'Enseignement Mathématique**

Band (Jahr): **21 (1975)**

Heft 1: **L'ENSEIGNEMENT MATHÉMATIQUE**

PDF erstellt am: **22.09.2024**

3. PROOF OF THEOREM 1

It is convenient to introduce approximations to π and li. We define

$$\Pi(x) = \sum_{p^\alpha \leqq x} \frac{1}{\alpha} = \pi(x) + \tfrac{1}{2}\pi(x^{\frac{1}{2}}) + \tfrac{1}{3}\pi(x^{\frac{1}{3}}) + \ldots (x \geqslant 1) .$$

The first sum extends over all prime powers not exceeding x. The second sum, which is formally infinite, is in fact terminating, since $\pi(x^{1/n}) = 0$ for $n > \log x / \log 2$. Thus we have

$$0 \leqslant \Pi(x) - \pi(x) \leqslant \tfrac{1}{2}\pi(x^{\frac{1}{2}}) + \frac{\log x}{3 \log 2}\pi(x^{\frac{1}{3}}) = O\left(\frac{x^{\frac{1}{2}}}{\log x}\right).$$

Also, for $x > 1$ we set

$$\tau(x) \overset{def}{=} \int_1^x \frac{1 - u^{-1}}{\log u}\, du = \operatorname{li} x + \tau(e) - \operatorname{li} e - \log \log x .$$

It follows from the preceding relations that

$$\frac{\pi(x) - \operatorname{li} x}{\sqrt{x}/\log x} = \frac{\Pi(x) - \tau(x)}{\sqrt{x}/\log x} + O(1) . \tag{3}$$

We shall establish Theorem 1 by proving that $x \mapsto x^{-\frac{1}{2}}\{\Pi(x) - \tau(x)\}\log x$ is unbounded from above and below.

The function Π occurs in the Mellin transform of the branch of $\log \zeta(s)$ which is real on the interval $(1, \infty)$. Indeed, the Euler product

$$\zeta(s) = \prod_p (1 - p^{-s})^{-1} \qquad (\sigma > 1)$$

yields

$$\log \zeta(s) = -\sum_p \log(1 - p^{-s}) = \sum_p \left(p^{-s} + \frac{p^{-2s}}{2} + \frac{p^{-3s}}{3} + \ldots\right).$$

Writing the last sum as a Stieltjes integral, we obtain

$$\log \zeta(s) = \int_1^\infty x^{-s}\, d\,\Pi(x) \qquad (\sigma > 1) .$$

The function τ has been introduced to exploit its simple Mellin transform:

$$\log \frac{s}{s-1} = \int_1^\infty x^{-s}\, d\tau(x) \qquad (\sigma > 1),$$

where the branch of $\log s/(s-1)$ is chosen which is real on $(1, \infty)$. One can verify this identity by showing that (*i*) each member of the equation tends to zero as $\sigma \to +\infty$ and (ii) the derivatives of the two sides are equal.

We form the difference of the two Mellin transforms and integrate by parts, obtaining

$$\frac{1}{s}\left\{\log \zeta(s) - \log \frac{s}{s-1}\right\} = \int_1^\infty x^{-s-1}\{\Pi(x) - \tau(x)\}\, dx\,.$$

We then differentiate this formula with respect to s to get

$$-\frac{1}{s^2}\left\{\log \zeta(s) - \log \frac{s}{s-1}\right\} + \frac{1}{s}\left\{\frac{\zeta'(s)}{\zeta(s)} - \frac{1}{s} + \frac{1}{s-1}\right\}$$

$$= -\int_1^\infty x^{-s-1} \log\, x\{\Pi(x) - \tau(x)\}\, dx \qquad (\sigma > 1)\,.$$

We have now succeeded in making a Mellin transform of $x^{-\frac{1}{2}} \log x$ $\{\Pi(x) - \tau(x)\}$. For convenience we shall denote the left hand side of this formula by $-G_1(s-\frac{1}{2})$. Then we have for $\sigma > \frac{1}{2}$

$$G_1(s) = \int_1^\infty x^{-s-\frac{3}{2}} \log\, x\{\Pi(x) - \tau(x)\}\, dx\,. \tag{4}$$

We shall apply Theorem 2 to this Mellin transform.

There are two possible cases to consider in proving Theorem 1, according to whether the Riemann hypothesis (R.H.) holds or not. (A form of the R.H. asserts that there exist no zeros of the Riemann zeta function with real part exceeding $\frac{1}{2}$. It is not known at present whether the R.H. is true.)

For each case we require the following theorem of Landau (cf.[4], pp. 88-89): *If $f(x)$ is a real valued right continuous function which is of one* *sign for all sufficiently large* x *and if* $\int_1^{\infty} x^{-s} f(x)\, dx$ *has abscissa of convergence* α, *then the analytic function*

$$s \mapsto \int_1^{\infty} x^{-s} f(x)\, dx$$

has a singularity at the real point $s = \alpha$.

If the conclusion of Theorem 1 were false, then for some real K

$$x^{-\frac{1}{2}} \{\Pi(x) - \tau(x)\} \log x + K$$

would be ultimately of one sign. We obtain from (4)

$$(5) \qquad G_1(s) + \frac{K}{s} = \int_1^{\infty} x^{-s-1} \left(x^{-\frac{1}{2}} \{\Pi(x) - \tau(x)\} \log x + K\right) dx\,.$$

According to Landau's theorem there must be a singularity of $G_1(x) + K/s$ at the real point on the abscissa of convergence, say α. Now $G_1(s) + K/s$ has no singularities on the half line $(0, \infty)$ because zeta is analytic and non zero on $(\frac{1}{2}, 1)$ [cf. remarks following the definition of $N(T)$ below] and $(1, \infty)$ [convergent Euler product!] and has a simple pole at $s = 1$. It follows that $\alpha \leqslant 0$.

Thus the integral in (5) converges and defines $G_1(s) + K/s$ as an analytic function on the half plane $\{s \colon \sigma > 0\}$. If we recall the definition of G_1, we see that zeta can have no zeros with real part exceeding $\frac{1}{2}$, i.e. the R.H. holds. This establishes the truth of Theorem 1 in case the R.H. does not hold.

Now we assume that the R.H. holds but Theorem 1 is false and deduce a contradiction.

The preceding argument with Landau's Theorem implies that (4) is valid for $\sigma > 0$. The function G_1 has two types of singularities on the line $\sigma = 0$, both arising from zeros of zeta. The following lemma will enable us to see that the logarithmic singularities are "negligible."

LEMMA 3. *Let a branch of log be fixed. Then*

$$\lim_{\sigma, \sigma' \to 0+} \int_{-1}^{1} |\log(\sigma + it) - \log(\sigma' + it)|\, dt = 0\,.$$

Proof. Let $0 < \varepsilon < 1/\sqrt{2}$ be given. The integral tends to zero uniformly for $|t| \geqslant \varepsilon$ as $\sigma, \sigma' \to 0+$. For $|t| < \varepsilon$ and $0 < \sigma < \sigma' < 1/\sqrt{2}$, say, we have

$$\int_{-\varepsilon}^{\varepsilon} \left| \log \frac{\sigma + it}{\sigma' + it} \right| dt < 4\pi\varepsilon + \int_{-\varepsilon}^{\varepsilon} \log \left| \frac{\sigma' + it}{\sigma + it} \right| dt$$

$$< 4\pi\varepsilon + \int_{-\varepsilon}^{\varepsilon} \log \left| \frac{1}{\sigma + it} \right| dt$$

$$< 4\pi\varepsilon + 2\int_{0}^{\varepsilon} \log (1/t)\, dt ,$$

and the last integral tends to zero as $\varepsilon \to 0+$. #

We can now show that the logarithmic terms in $G_1(\sigma + \mathrm{it})$ satisfy the Cauchy condition in $L^1[-T, T]$ for any fixed $T > 0$, as $\sigma \to 0+$. Indeed, let $[t_1, t_2] \subset [-T, T]$ and suppose that there exists at most one $\gamma \in [t_1, t_2]$ for which $\zeta(\frac{1}{2} + i\gamma) = 0$. For $0 < \sigma < 1/3$ and $t_1 \leqslant t \leqslant t_2$ we have

$$\log \zeta(s + \tfrac{1}{2}) = n \log (s - i\gamma) + \varphi(s) ,$$

where n is the order of the zero at $\frac{1}{2} + i\gamma$ and φ is analytic on the closure of the region. It follows from the preceding lemma and an estimate based on the triangle inequality that

$$\lim_{\sigma, \sigma' \to 0+} \int_{t_1}^{t_2} | f(\sigma + it) - f(\sigma' + it) | \, dt = 0 ,$$

where

$$f(s) = (s + \tfrac{1}{2})^{-2} \log \{\zeta(s + \tfrac{1}{2})(s - \tfrac{1}{2}) / (s + \tfrac{1}{2})\} .$$

Adding together a finite number of such estimates we see that the Cauchy condition applies for the logarithmic terms in G_1 on the whole interval $[-T, T]$.

It remains to consider the pole terms in $G_1(s)$. For given $T > 0$ set

$$H_1(s) = \sum_{|\gamma| < T} \frac{1}{(\frac{1}{2} + i\gamma)(s - i\gamma)} ,$$

where γ ranges over the imaginary parts of zeros of zeta on the line $\sigma = \frac{1}{2}$. A term is repeated n times in the sum in case $\frac{1}{2} + i\gamma$ is a zero of zeta of multiplicity n. It is convenient to assume that T is distinct from each of the γ's. The function $G_1 - H_1$ is analytic on the region

$$\{s = \sigma + it : 0 < \sigma \leqslant 1/3, \; - T \leqslant t \leqslant T\}$$

and has a finite number of logarithmic singularities in the closure of this region. Thus $G_1(\sigma + \mathrm{it}) - H_1(\sigma + \mathrm{it})$ is Cauchy in L^1 norm on $[-T, T]$ as $\sigma \to 0+$.

Let $F_1(x) = x^{-\frac{1}{2}} \{\Pi(x) - \tau(x)\} \log x$ for $1 \leqslant x < \infty$. Then (4) can be rewritten as $G_1(s) = \int x^{-s-1} F_1(x)\, dx$. We are assuming that Theorem 1 is false and hence F_1 is bounded by a constant from above or below. Under this assumption we have shown that the preceding integral converges for $\sigma > 0$.

Now the triple F_1, G_1, H_1 satisfies the conditions required of F, G, and H in Theorem 2. Thus we have the formula

$$\int_{x=1}^{\infty} x^{-\frac{3}{2}} \{\Pi(x) - \tau(x)\} \log x \, K_T(y - \log x)\, dx$$

$$= \sum_{|\gamma| < T} \frac{1}{\frac{1}{2} + i\gamma} \left(1 - \frac{|\gamma|}{T}\right) e^{i\gamma y} + o_T(1)$$

as $y \to \infty$.

For $T \geqslant 0$ let $N(T)$ denote the number of zeros of the Riemann zeta function in the rectangle $\{s = \sigma + \mathrm{it} : 0 < \sigma < 1, \;\; 0 \leqslant t \leqslant T\}$. Each zero is counted with its appropriate multiplicity. We observe that $N(0) = 0$. This follows from the identity

$$\zeta(\sigma)(1 - 2^{1-\sigma}) = 1 - 2^{-\sigma} + 3^{-\sigma} - 4^{-\sigma} + \ldots \qquad (\sigma > 0)$$

and the alternating series inequality

$$\zeta(\sigma)(1 - 2^{1-\sigma}) > 1 - 2^{-\sigma} > 0\,.$$

Moreover, the function N is continuous from the right and hence $N(T) = 0$ for some positive values of T also. (It is known that the first jump in $N(T)$ occurs near $T = 14.13$.)

For large T we have the asymptotic estimate $N(T) \sim \dfrac{1}{2\pi} T \log T$ (cf. [4], pp. 68-70). Actually, it would be enough for our purposes to have the weaker bounds

(6) $$N(T+1) - N(T) = O(\log T)$$

and

$$\overline{\lim_{T\to\infty}}\ N(T)/T = \infty . \tag{7}$$

We digress for a moment to indicate how one can establish (6) and (7). The first estimate can be made by applying Jensen's inequality [4, p. 49] to zeta. We use the bound $\zeta(s) = O(|t|^{\frac{3}{2}})$ for $|t| \geqslant 1$ and $\sigma > -1$. This bound follows from the functional equation for zeta [4, p. 41] and Stirling's formula for the gamma function [4, p. 57]. Another bound of this general type can be deduced from the representation

$$\zeta(s) = \frac{1}{s-1} + \tfrac{1}{2} + s(s+1)\int_{x=1}^{\infty} x^{-s-2}\int_{0}^{x}([t]-t+\tfrac{1}{2})\,dt\,dx \qquad (\sigma > -1),$$

which results from two integrations by parts of the Mellin transform for zeta.

For (7) we consider the formula

$$\frac{\zeta'}{\zeta}(\sigma) = -\tfrac{1}{2}\frac{\Gamma'}{\Gamma}(\tfrac{1}{2}\sigma+1) + \sum_{\rho}\left(\frac{1}{\sigma-\rho}+\frac{1}{\rho}\right) + O(1) \qquad (\sigma \geqslant 2)$$

[4, p. 58]. Here $\rho = \beta + i\gamma$ extends over all zeros of zeta satisfying $0 < \beta < 1$. As $\sigma \to +\infty$ we have $\zeta'(\sigma)/\zeta(\sigma) \to 0$ and by Stirling's formula $(\Gamma'/\Gamma)(\frac{1}{2}\sigma+1) \sim \log\sigma$. If $N(T)/T$ were bounded, then, as a short calculation shows, the sum over ρ would be bounded as $\sigma \to +\infty$. This is clearly impossible and hence (7) holds.

Applying (6) we have

$$\sum_{\gamma}\left|\frac{1}{\frac{1}{2}+i\gamma} - \frac{1}{i\gamma}\right| < \sum_{\gamma}\frac{1}{2\gamma^2} = \sum_{k=1}^{\infty}\ \sum_{k-1<\gamma\leqq k}\gamma^{-2}$$
$$= O\left\{\sum_{k=1}^{\infty}\frac{\log(k+1)}{k^2}\right\} = B < \infty .$$

We can thus rewrite the formula for $\Pi - \tau$ as

$$\int_{1}^{\infty} x^{-\frac{1}{2}}\{\Pi(x) - \tau(x)\}\log x\, K_T(y-\log x)\,d(\log x)$$
$$= \sum_{0<\gamma<T} 2\left(1-\frac{\gamma}{T}\right)\frac{\sin\gamma y}{\gamma} + B\theta + o_T(1), \tag{8}$$

where $\theta = \theta(y, T)$ is bounded by 1 in absolute value.

Let $\Sigma(y)$ denote the sum in (8). We have

$$\Sigma(1/T) = \frac{2}{T} \sum_{0<\gamma<T} \left(1 - \frac{\gamma}{T}\right) \frac{\sin \gamma/T}{\gamma/T}$$

$$\geqslant \frac{2}{T} \sum_{0<\gamma<T/2} \frac{1}{2} \cdot \frac{2}{\pi} = \frac{1}{\pi} \frac{N(T/2)}{T/2},$$

which can be made arbitrarily large by choosing suitable large values of T. (We have used the estimates $\sin x > 2x/\pi$ for $0 < x < \pi/2$ and (7).) Also, since sine is odd $\Sigma(-1/T)$ will be a negative number of large modulus for some positive values of T.

The evaluation of $\Sigma(\pm 1/T)$ does not appear to be of any direct benefit, since (8) applies only for large positive values of y. However, $\Sigma(y)$ is a trigonometric polynomial and as such has the following approximate periodicity property: For any $\varepsilon > 0$ and any $y_0 \in R$, there exists a sequence $y_n \to \infty$ such that

$$|\Sigma(y_n) - \Sigma(y_0)| < \varepsilon \qquad (n = 1, 2, 3, \ldots).$$

This assertion can be established by appealing to the theory of almost periodic functions [7, pp. 158-159]. Alternatively, we can apply Dirichlet's theorem on diophantine approximation [4, pp. 94-95]. Suppose we are given $\varepsilon > 0$ and $0 < \gamma_1 \leqslant \gamma_2 \ldots \leqslant \gamma_N$, the imaginary parts of the first $N = N(T)$ zeros of zeta arranged in ascending order. Then by Dirichlet's Theorem we can find arbitrarily large numbers t for which the inequalities

$$\|\gamma_n t / 2\pi\| < \varepsilon / \{4\pi \sum_{1 \leq j \leq N} \gamma_j^{-1}\} \qquad (1 \leqslant n \leqslant N)$$

hold. Here $\|x\|$ denotes the (non negative) distance from x to the nearest integer. Simple estimates show that

$$|\sin \gamma_n (y+t) - \sin \gamma_n y| \leqslant 2\pi \|\gamma_n t / 2\pi\|$$

for $1 \leqslant n \leqslant N$, and hence for all real y we have

$$|\Sigma(y+t) - \Sigma(y)| < \sum_{n=1}^{N} 4\pi \gamma_n^{-1} \|\gamma_n t / 2\pi\| < \varepsilon.$$

It follows from either of these methods that the values $\Sigma(1/T)$ and $\Sigma(-1/T)$ are nearly repeated by $\Sigma(y)$ on a sequence of values of y tending to infinity.

At the end of the proof of Theorem 2 we showed that $\int_{-\infty}^{\infty} K_T(u)\, du = 1$.

Also, (2) implies that $K_T(x) \geqslant 0$ for all x. Now formula (8) can be interpreted as expressing a certain average of $x^{-\frac{1}{2}} \{\Pi(x) - \tau(x)\} \log x$ as $\Sigma(x)$ plus a bounded error term.

It follows that there exist sequences $\{x_n\}$ and $\{y_n\}$ tending to infinity for which

$$x_n^{-\frac{1}{2}} \{\Pi(x_n) - \tau(x_n)\} \log x_n > c$$
$$y_n^{-\frac{1}{2}} \{\Pi(y_n) - \tau(y_n)\} \log y_n < -c$$

for any given number $c > 0$. Thus

$$x^{-\frac{1}{2}} \{\Pi(x) - \tau(x)\} \log x$$

is unbounded from above and below. If we recall (3), we have completed our proof that $\pi(x) - \mathrm{li}\, x$ changes sign infinitely often.

4. Further results

Littlewood actually showed a bit more than we have. He proved that

$$x^{-\frac{1}{2}} \{\pi(x) - \mathrm{li}\, x\} \log x / \log \log \log x$$

has a positive limit superior and negative limit inferior. The best account of this estimate is probably that given in [5].

It appears that our arguments can be extended to achieve this estimate. The contradiction arguments can be reorganized, exploiting more fully the hypothesized one sided bound in Theorem 2. However, we would also require an explicit estimate in place of the $o_T(1)$ in the conclusion of this theorem. Such estimation would cancel out the economy we have achieved.

It is reasonable to ask for an $x > 3/2$ for which $\pi(x) - \mathrm{li}\, x > 0$. The first person to provide an estimate of such a number x was Skewes [13]. He showed that there exists an $x < \exp \exp \exp \exp (7.705)$ for which $\pi(x) - \mathrm{li}\, x > 0$. This enormous bound was reduced to a more modest $1.65 \cdot 10^{1165} \approx \exp \exp (7.895)$ by R. S. Lehman [10]. Each of these authors combined theoretical arguments with extensive numerical calculations using the position of many zeros of the Riemann zeta function. The case in which the Riemann hypothesis is assumed false requires much more work than we had to do.