

5. Other proofs of Wedderburn's theorem

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number of conjugates of a subgroup is the index of its normalizer, so H has at most $|G : H|$ conjugates in G and hence the union of the conjugates contains at most $|G : H| (|H| - 1) + 1 = |G| - |G : H| + 1$ elements. This number is less than $|G|$ except when $G = H$. Hence $D = F$ is a field.

4. FROBENIUS' THEOREM

Let \mathbf{R} denote the field of real numbers, \mathbf{C} the field of complex numbers and \mathbf{H} the division ring of quaternions. The following proof makes use of the fundamental theorem that every polynomial with coefficients in \mathbf{C} has a root in \mathbf{C} .

THEOREM. *Let D be a division ring which contains the real numbers \mathbf{R} in its centre and suppose that every element of D satisfies a polynomial with coefficients in \mathbf{R} . Then D is isomorphic to one of \mathbf{R} , \mathbf{C} or \mathbf{H} .*

Proof. Suppose that D is not isomorphic to \mathbf{R} or \mathbf{C} . It follows that the maximal subfield F of D is isomorphic to \mathbf{C} , the centre K of D is isomorphic to \mathbf{R} and $F = K(i)$ where $i^2 = -1$. Let j be an eigenvector of T_i corresponding to the eigenvalue $-i$. Then $ji = -ij$ and j^2 commutes with j and F . From (2.2) and (2.3) the elements 1 and j form an F -basis for D and therefore $j^2 = \alpha$ belongs to K . If $\alpha = \beta^2$ for some $\beta \in K$ then $(j - \beta)(j + \beta) = 0$ and j belongs to K , which is not the case; hence $\alpha = -\beta^2$ for some $\beta \in K$. Replacing j by $j\beta^{-1}$ we obtain a K -basis 1, i , j , ij for D such that $i^2 = j^2 = -1$ and $ij = -ji$. That is, D is isomorphic to \mathbf{H} .

An almost identical argument shows that if the dimension of D over its centre K is 4 and the characteristic is not 2, then D has a K -basis 1, i , j , ij where $i^2 = \alpha$, $j^2 = \beta$ and $ij = -ji$ for some $\alpha, \beta \in K$.

5. OTHER PROOFS OF WEDDERBURN'S THEOREM

The original proofs of the theorem of §3 were given first by Wedderburn [15] in 1905 and then by Dickson [5] in the same year; they depend on certain divisibility properties of the integers. The neatest proof along these lines is that of Witt [16]. Elementary proofs which avoid the use of such number theory have been given by Artin [1] and Herstein [7]. And

proofs which deduce the theorem using finite group theory have been given by Zassenhaus [17], Brandis [3] and Scott [11, p. 426].

Perhaps the most interesting proofs are those which present the result as a consequence of a more general theory. There are two such proofs in the book of van der Waerden [14]: the first (on p. 203) uses the theory of central simple algebras, the second (sketched on p. 215) relates the theorem to cohomology and the Brauer group (see also, Serre [12, p. 170]). The theorem is also a consequence of the work of Tsen [13] and Chevalley [4]. Further comments on the history of the theorem can be found in an article by Artin [2] and in the book by Herstein [8] where many interesting generalisations are also given. One such generalization is a theorem of Jacobson: a division ring in which $x^{n(x)} = x$ for all x is commutative. Laffey [10] has recently given an elementary proof of this using Wedderburn's theorem and linear algebra similar to that used here. See also [18].

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