5. Other proofs of Wedderburn's theorem

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number of conjugates of a subgroup is the index of its normalizer, so H has at most $|G:H|$ conjugates in G and hence the union of the conjugates contains at most $|G:H|(|H|-1)+1 = |G| - |G:H| + 1$ elements. This number is less than $|G|$ except when $G = H$. Hence $D = F$ is a field.

4. Frobenius' theorem

Let $\mathbf{R}$ denote the field of real numbers, $\mathbf{C}$ the field of complex numbers and $\mathbf{H}$ the division ring of quaternions. The following proof makes use of the fundamental theorem that every polynomial with coefficients in $\mathbf{C}$ has a root in $\mathbf{C}$.

THEOREM. *Let D be a division ring which contains the real numbers* $\mathbf{R}$ *in its centre and suppose that every element of D satisfies a polynomial with coefficients in* $\mathbf{R}$. *Then D is isomorphic to one of* $\mathbf{R}$, $\mathbf{C}$ *or* $\mathbf{H}$.

Proof. Suppose that D is not isomorphic to $\mathbf{R}$ or $\mathbf{C}$. It follows that the maximal subfield F of D is isomorphic to $\mathbf{C}$, the centre K of D is isomorphic to $\mathbf{R}$ and $F = K(i)$ where $i^2 = -1$. Let j be an eigenvector of T_i corresponding to the eigenvalue $-i$. Then $ji = -ij$ and j^2 commutes with j and F. From (2.2) and (2.3) the elements 1 and j form an F-basis for D and therefore $j^2 = \alpha$ belongs to K. If $\alpha = \beta^2$ for some $\beta \in K$ then $(j-\beta)(j+\beta) = 0$ and j belongs to K, which is not the case; hence $\alpha = -\beta^2$ for some $\beta \in K$. Replacing j by $j\beta^{-1}$ we obtain a K-basis $1, i, j, ij$ for D such that $i^2 = j^2 = -1$ and $ij = -ji$. That is, D is isomorphic to $\mathbf{H}$.

An almost identical argument shows that if the dimension of D over its centre K is 4 and the characteristic is not 2, then D has a K-basis $1, i, j, ij$ where $i^2 = \alpha$, $j^2 = \beta$ and $ij = -ji$ for some $\alpha, \beta \in K$.

5. Other proofs of Wedderburn's theorem

The original proofs of the theorem of §3 were given first by Wedderburn [15] in 1905 and then by Dickson [5] in the same year; they depend on certain divisibility properties of the integers. The neatest proof along these lines is that of Witt [16]. Elementary proofs which avoid the use of such number theory have been given by Artin [1] and Herstein [7]. And proofs which deduce the theorem using finite group theory have been given by Zassenhaus [17], Brandis [3] and Scott [11, p. 426].

Perhaps the most interesting proofs are those which present the result as a consequence of a more general theory. There are two such proofs in the book of van der Waerden [14]: the first (on p. 203) uses the theory of central simple algebras, the second (sketched on p. 215) relates the theorem to cohomology and the Brauer group (see also, Serre [12, p. 170]). The theorem is also a consequence of the work of Tsen [13] and Chevalley [4]. Further comments on the history of the theorem can be found in an article by Artin [2] and in the book by Herstein [8] where many interesting generalisations are also given. One such generalization is a theorem of Jacobson: a division ring in which $x^{n(x)} = x$ for all x is commutative. Laffey [10] has recently given an elementary proof of this using Wedderburn's theorem and linear algebra similar to that used here. See also [18].

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