2. Historical background: D = 7, p = 2 and D = 11, p = 3

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Ramanujan in 1913 [12], [13] asked whether there were other solutions to the diophantine equation

$$x^2 + 7 = 2^k \tag{2.1}$$

besides the known ones, namely when $k = 3, 4, 5, 7, 15$. This problem was again posed by Ljunggren [7] in 1943, and it was finally shown by Nagell [10] that the k above were the only five solutions. Nagell's paper was written in Norwegian and few knew about its existence although he had posed it shortly afterwards as an exercise in his book [11] on elementary number theory. Hence, thereafter, there were a large number of papers proving the same result (see [5] for an up-to-date list).

It is also interesting to note that equation (2.1) has interesting applications to binary error-correcting codes [3], [14].

Equation (2.1) is of the form

$$x^2 + D = [(D+1)/4]^k . \tag{2.2}$$

So the next equation of interest might seem to be

$$x^2 + 11 = 3^k . \tag{2.3}$$

This equation was solved by Ljunggren and the author [5]. Its only solution occurs when $k = 3$.

3. The main theorem

Theorem 3.1. Let $D \equiv 3 \pmod 8$. Let $p = (D+1)/4$ be a prime $\geqslant 5$. Then the diophantine equation

$$x^2 + D = p^k \tag{3.1}$$

has no solutions.

The remainder of this section will be devoted to a proof of this theorem, a sketch of which was presented in [4]. The last section will be devoted to corollaries and other related results.

Lemma 3.2. There are no solutions to the equation (3.1) when k is even.

Proof. When k is even, the equation can be written as

$$(p^{k/2})^2 - x^2 = D . \tag{3.2}$$

Therefore,