FORGOTTEN GEOMETRICAL TRANSFORMATION

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A FORGOTTEN GEOMETRICAL TRANSFORMATION

by Daniel PEDOE

In 1882 Laguerre published *Transformations par les semi-droites réciproques* (Laguerre [3]). Thirty years earlier, while still a student, he had become famous for his discovery of the cross-ratio interpretation of Euclidean angle. In this paper Laguerre describes a geometrical transformation between oriented lines (which we shall call *rays*), using a given oriented

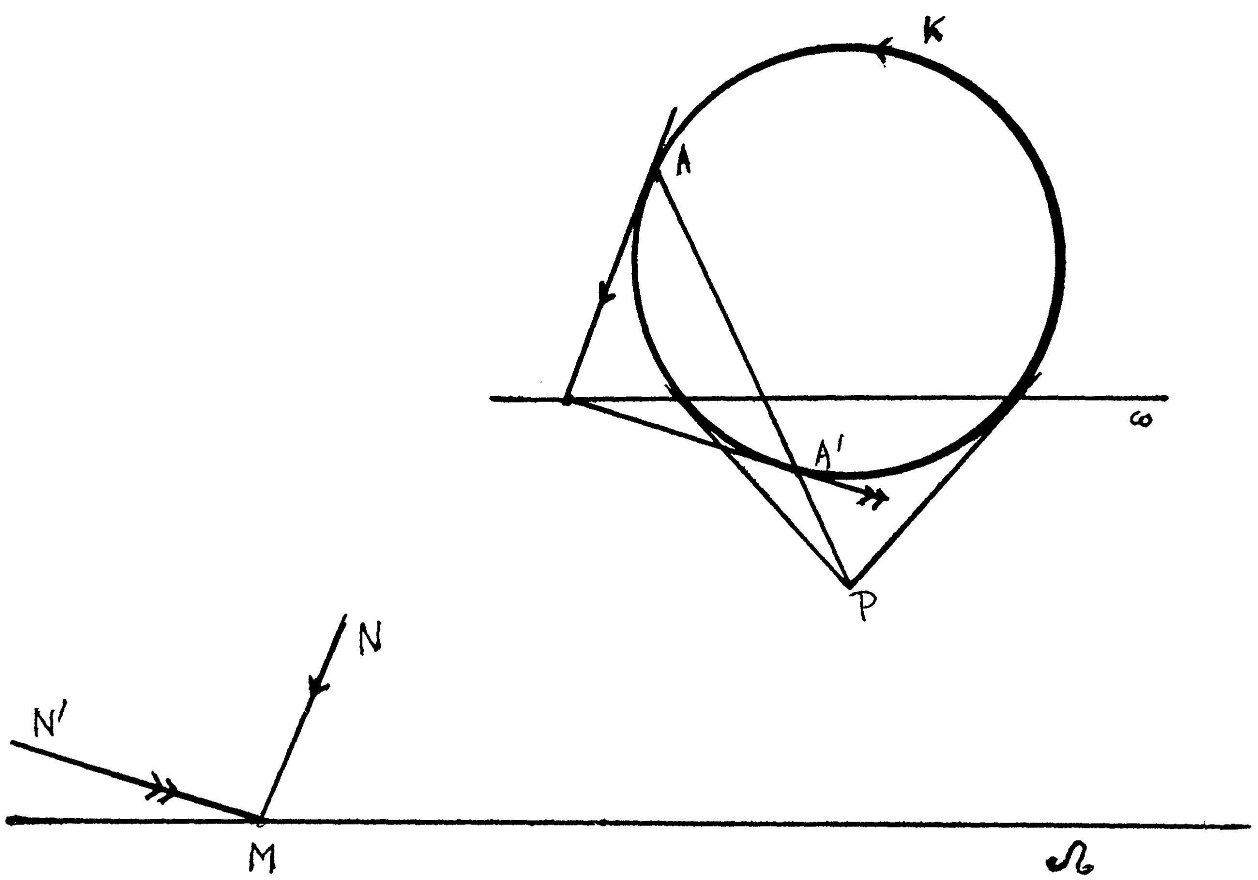

FIG. 1

circle (or *cycle*), a point outside the circle, and a line, called the *axis*, as the machinery of the transformation. If the cycle is called K, the point P, the axis Ω, and a ray NM meets Ω in M, there is a unique parallel ray which touches the cycle K. Let A be its point of contact (Fig. 1), and let PA intersect K again at A'. There is a ray tangent to K at A'. Then the ray $N'M$ through M parallel to this tangent ray is the transform of the ray NM.

Laguerre choses P so that its polar line with respect to K is parallel to Ω, and the transformation can also be described by saying that corresponding tangents to K intersect on this polar line, which we call *the auxiliary axis* ω. The variables in setting up this transformation, called *Laguerre inversion,* are K, Ω and ω. It is evidently an *involutory* transformation; that is, if the map of NM is $N'M$, then the map of $N'M$ is NM. Laguerre proves that the tangent rays belonging to a given cycle map onto the tangent rays to a cycle, and this theorem at once makes Laguerre inversion interesting. The origin of the transformation can be traced to a previous paper, *Sur la géométrie de direction* (Laguerre [**3**], 592-603), where Laguerre considers tangent rays to a cycle which are in involution, and in both papers he stresses the correspondence of his transformation with ordinary inversion (transformation by reciprocal radii). The final theorem of this paper is: *three given cycles may be simultaneously transformed into three points,* and he then derives the eight Apollonius contact circles of the three original circles by a most elegant device, from this theorem.

Geometers are familiar with ordinary inversion, but a number with whom I discussed Laguerre inversion had never come across it. Laguerre inversion is as powerful a method for solving certain kinds of problem as ordinary inversion, and should certainly be a part of every geometer's tool-kit. It is mentioned by Coolidge [**2**], and Yaglom [**6**] discusses it in detail, using dual complex numbers. Blaschke [**1**] devotes a whole chapter to Laguerre transformations but, as far as I can discover, none of these three writers mentions Laguerre's final theorem, which is surely a remarkable one.

Sophus Lie, around 1870, interpreted the group generated by composition of Laguerre inversions in the plane as a subgroup of a group of transformations in E_3 which map a certain geometric construct onto itself. Blaschke gives all this, and more. So the development given below, to show how Laguerre inversion arises quite naturally when we consider a dual aspect of ordinary inversion cannot claim to be entirely original, but it is perhaps more geometrical than Blaschke's treatment, and, in any case, was conceived independently.

At the end of this paper we shall use Laguerre inversion to obtain a rapid proof of a theorem on chains of contact circles, first remarking that the theorem is essentially one on chains of cycles. *Non Angli, sed angeli* may often apply to *circles* and *cycles.*

1. If the circle $\Sigma \equiv X^2 + Y^2 - 2pX - 2qY + r = 0$ in the plane $Z = 0$ is represented in E_3 by the point $V = (p, q, r)$, then point-circles (circles of zero radius) are mapped onto points which lie on the quadric surface $\Phi \equiv X^2 + Y^2 - Z = 0$. If the points P, P' in $Z = 0$ are inverse points in Σ, and these points, considered as point-circles, are mapped onto the points Q, Q' of Φ, then the points V, Q, Q' are collinear. If P describes a circle $\mathscr{C}$ in $Z = 0$, the point Q describes a plane section of Φ. The theorem that the inverse of a circle $\mathscr{C}$ in Σ is a circle or a line $\mathscr{C}'$ is equivalent to the theorem in E_3 that the cone of lines joining V to the points of a plane section of Φ intersects the quadric again in the points of a plane section, and so we have a transformation between plane sections of a quadric. All this is described in detail in Pedoe ([**4**], Chapter IV).

For the dual transformation in E_3 we consider a tangential quadric Ψ and a plane v. The planes q, q' both touch Ψ, and the line $q \cap q'$ lies in v. The dual theorem corresponding to the one just given above is: if the planes q pass through a point, the planes q' also pass through a point, and so we have a transformation between cones of tangent planes to the quadric Ψ.

To obtain a transformation between *cycles* in $Z = 0$, we map the circle $(X-p)^2 + (Y-q)^2 - R^2 = 0$ onto the point (p, q, R) of E_3 if the circle is traversed positively, and onto the point $(p, q, -R)$ if the circle is traversed negatively. The tangent planes to the cone with vertex $(p, q, \pm R)$ which passes through the circle in the plane $Z = 0$ all touch the conic $X^2 + Y^2 - Z^2 = 0$ in the plane at infinity, and this conic is our (degenerate) tangential quadric Ψ. If cycles (oriented circles) $\mathscr{C}$, $\mathscr{D}$ are represented by points P, Q, then the two common tangent rays to $\mathscr{C}$ and $\mathscr{D}$ are given by the intersection with $Z = 0$ of the two tangent planes to Ψ which pass through the line PQ. The points of the line PQ represent the set of cycles which also touch the common tangent rays of $\mathscr{C}$ and $\mathscr{D}$. The points of a plane π in E_3 which intersects the plane $Z = 0$ in a line Ω represent the set of cycles which cut Ω at the same angle α. This angle α is zero (so that the cycles all touch Ω) if and only if the plane π makes an angle of 45° with the plane $Z = 0$, in which case the plane touches Ψ. All this is also described in Pedoe ([**4**], pp. 426-431).

We now interpret this dual transformation in terms of cycles in the plane $Z = 0$. The planes q and q' touch Ψ, and the line $q \cap q'$ lies in v. If v meets $Z = 0$ in the line Ω, the intersections of the planes q and q' with $Z = 0$ are lines p and p' which intersect on Ω. The points of the plane v represent cycles which all cut Ω at some fixed angle α. The lines p and p', with a suitable orientation, are rays which touch a set of such cycles, because the line $q \cap q'$ in E_3 represents a set of cycles with the same two common tangent rays, and these common tangents are given by the intersections with $Z = 0$ of the two tangent planes to Ψ which pass through the line $q \cap q'$.

Our transformation is therefore described thus: let NM be any ray intersecting a given line Ω in the point M. Draw *any* cycle K' which cuts Ω at the given angle α, and also touches the ray NM (Fig. 2). Then the other tangent from M to the cycle K', MN', gives the ray which is the transform of NM.

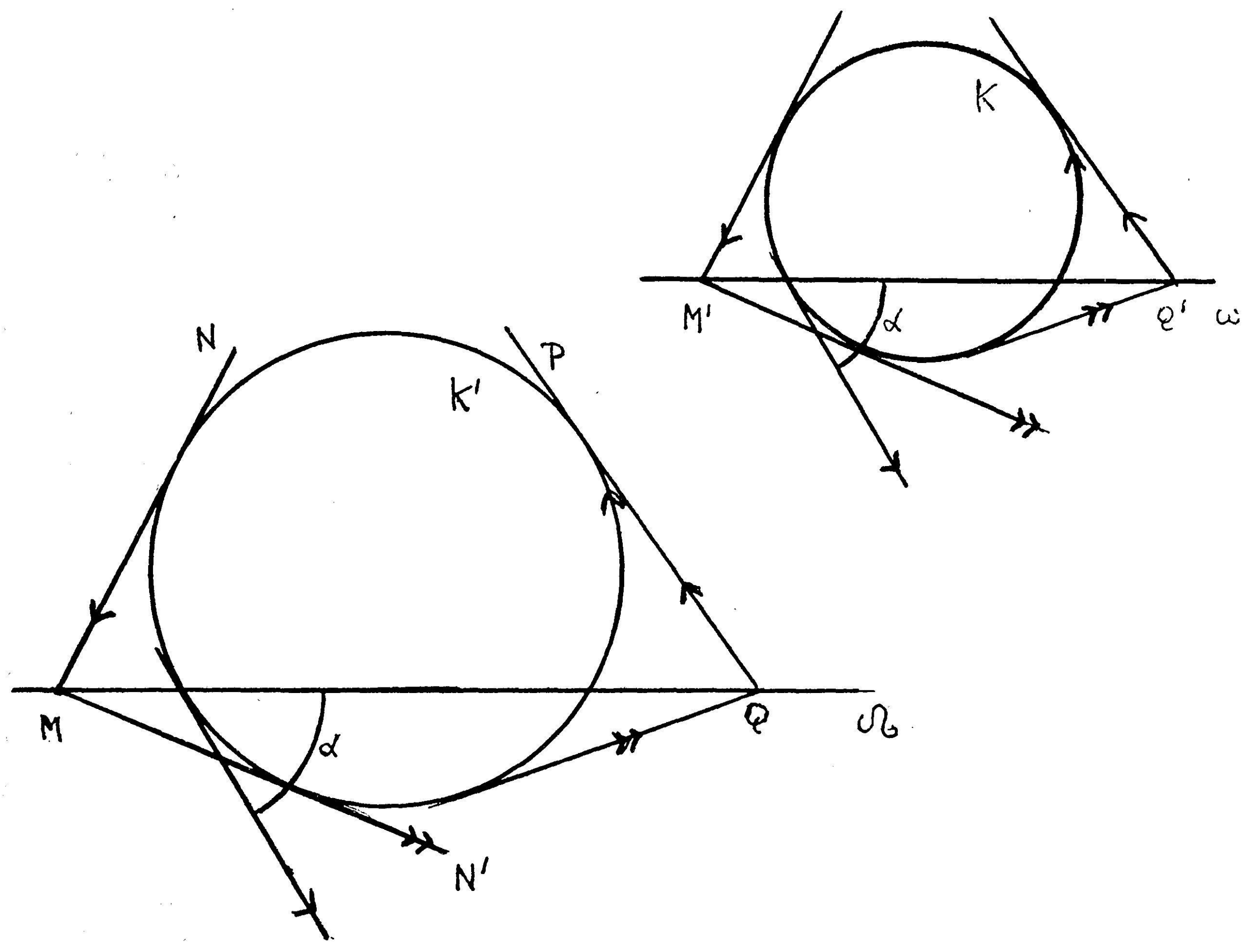

FIG. 2

We may use as a model for the cycle K' a fixed cycle K cutting a line ω at an angle α anywhere in the plane, where ω is parallel to Ω, and since the figures shown in Figure 2 are similar figures, we have rediscovered Laguerre inversion. But the fact that the reference cycle K is not necessarily in a fixed position in the plane is not explicitly stated by Laguerre, although he makes use of this fact in the proofs of theorems. To give an immediate demonstration of this assertion, let us consider Laguerre's theorem that any two pairs of corresponding rays touch a cycle. Any two pairs of tangent rays to K, intersecting on ω, naturally touch K, and this leads to the theorem for any two pairs of corresponding rays intersecting on Ω. To obtain the cycle K' which these corresponding rays touch, all that we have to do is to map M' on M and Q' on Q in a direct similarity transformation, and the map K' of K is the required cycle. What Laguerre does is to begin with the pair of corresponding rays MN' and NM and to suppose that PQ is any other ray, meeting Ω at Q. There is a unique cycle which touches three given rays, and, if this be K', Laguerre suddenly announces that K' is the fundamental cycle of the transformation, and that the tangent ray through Q to K' is the ray which corresponds to the ray PQ in the transformation. Of course this is correct, but the reader is not warned that the fundamental cycle to be used can be any one of a class of similar and similarly situated cycles.

2. Laguerre inversion maps the tangent rays of a cycle onto the tangent rays of a cycle. Our dual approach gives this theorem immediately, since we saw that if the planes q pass through a point, so do the planes q'. But when planes q, which touch Ψ, pass through a point of E_3, they all touch the cycle in $Z = 0$ which is represented by the point, and so the rays which touch a cycle in $Z = 0$ are mapped into the rays which touch a cycle in $Z = 0$. Laguerre proves more than this, that the cycle and the transformed cycle have the axis Ω as their radical axis, and his proof is worth giving.

We note in the first instance that the tangent rays to K where K meets ω (Fig. 1) map onto themselves under Laguerre inversion, so every Laguerre inversion is blessed with a pair of self-corresponding (fixed) rays. Now let K^* be the cycle we wish to transform, cutting the axis Ω at A and B (Fig. 3). Draw tangent rays MN, $N'M'$ to K^* parallel to the fixed rays of the given Laguerre inversion, touching K^* at M and M' respectively. Let K' be the cycle through A and B which touches MN and $N'M'$. Then Laguerre proves that K' is the transform of K^*.

Laguerre uses K^* as the fundamental cycle of the transformation, the auxiliary axis being the line $M'M = \omega$. Let P be the intersection of the lines NM and $N'M'$, and let a line through P cut K^* in β and γ, and K' in α, where α corresponds to β in the similarity which exists between K' and K^*, centre P. The tangents to K^* at β and γ meet at U on ω, the polar of P with respect to K^*, and since the tangent to K' at α is parallel to the tangent to K^* at β, the intersection T of the tangent γU and the tangent at α to K' is such that $|T\gamma| = |T\alpha|$. Therefore T is on the radical axis AB of K^* and K'.

It now follows immediately that the transform of the ray $U\gamma$, using K^* as fundamental cycle, is the ray αT, which is parallel to βU, and inter-

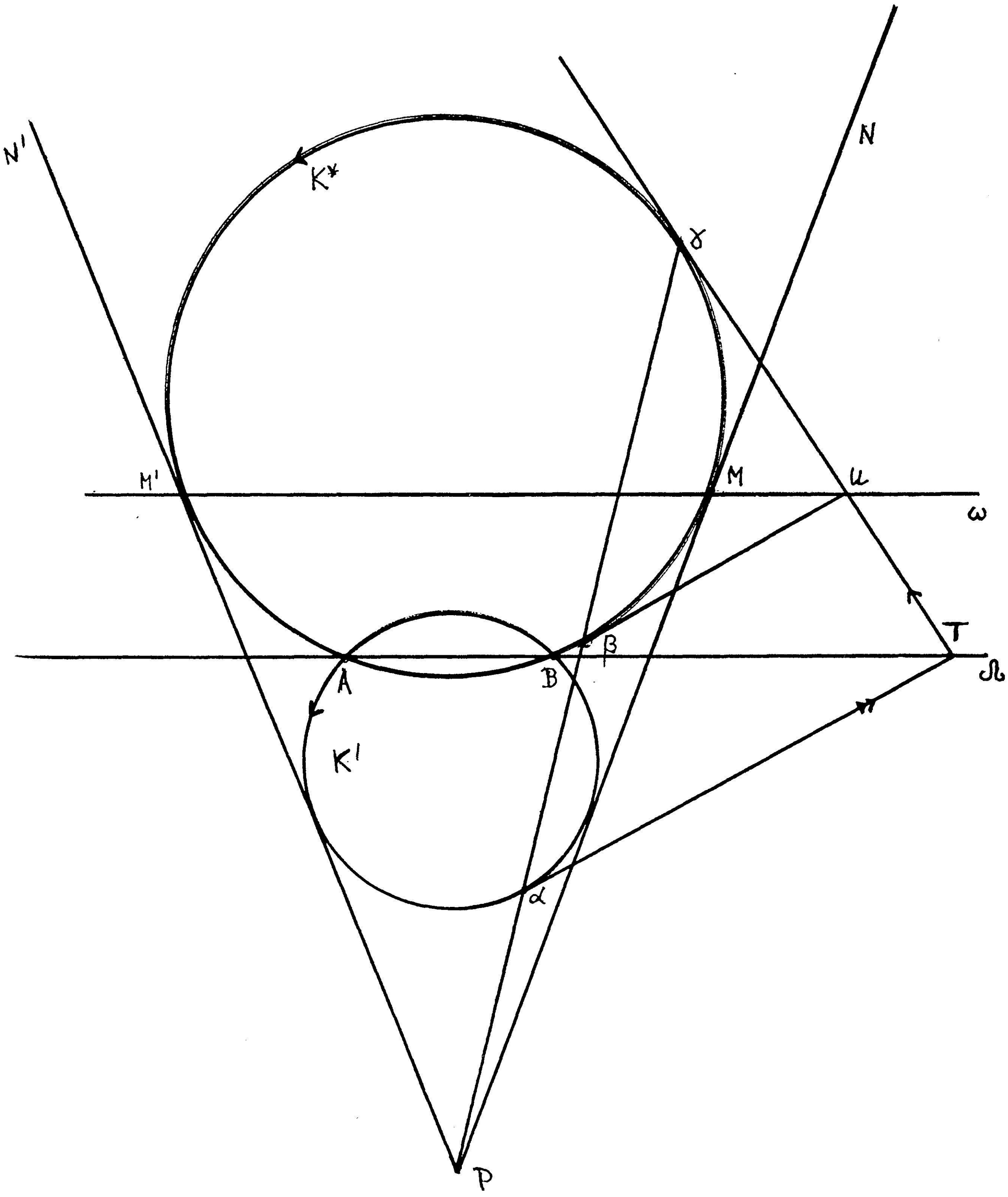

FIG. 3

sects $\Omega = AB$ at the point T. Hence rays which touch K^* are mapped onto ıays which touch K'.

The proof is still valid if K^* does not cut Ω. We also note that K^* and its transform K' not only have Ω as their radical axis, but their common tangent rays are parallel to the fixed rays of the transformation.

If $\mathscr{C}$ and $\mathscr{D}$ are two cycles which map onto the cycles $\mathscr{C}'$ and $\mathscr{D}'$, let a common tangent ray to $\mathscr{C}$ and to $\mathscr{D}$ touch $\mathscr{C}$ at P, touch $\mathscr{D}$ at Q, and meet the axis Ω at T. Then this ray maps onto a common tangent ray to the maps

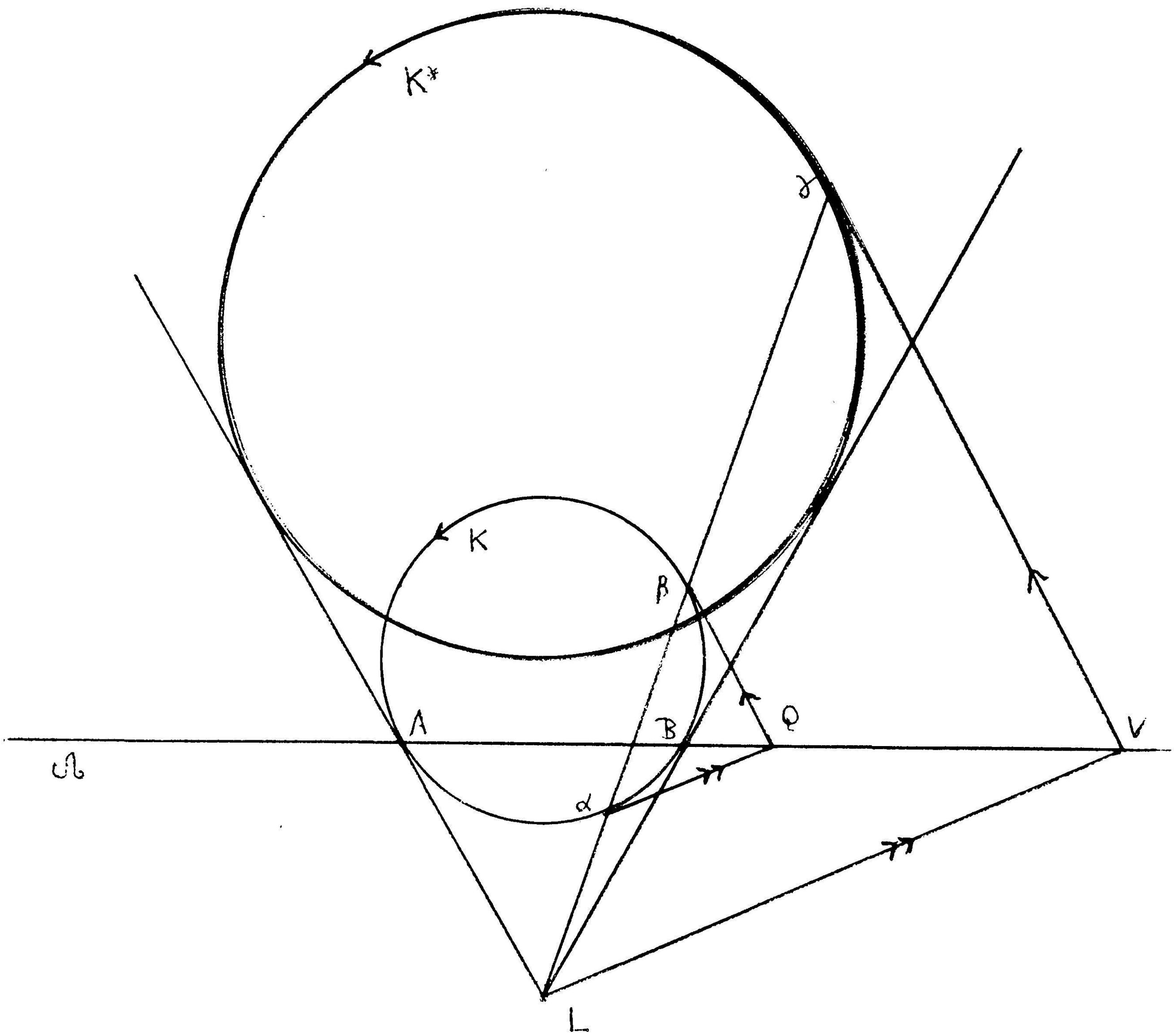

FIG. 4

$\mathscr{C}'$ and $\mathscr{D}'$ of $\mathscr{C}$ and $\mathscr{D}$ respectively, with points of contact P' and Q'. Since Ω is the radical axis of $\mathscr{C}$ and $\mathscr{C}'$, $|TP| = |TP'|$, and since Ω is also the radical axis of $\mathscr{D}$ and $\mathscr{D}'$, $|TQ| = |TQ'|$. Hence $|PQ| = |P'Q'|$, and if we call $|PQ|$ *the tangential distance* between the cycles $\mathscr{C}$ and $\mathscr{D}$, we have the important theorem: *Laguerre inversion preserves the tangential distance between two cycles.*

3. At this stage Laguerre introduces algebraic considerations, but not *the power of a ray with respect to a cycle.* We introduce this concept later (§5). He obtains the theorem that an axial transformation may be found such that *a given cycle is mapped onto a given point lying outside the cycle.* We give a direct proof of this theorem, which leads to the final theorem.

Let K^* be the given cycle, and L a given point lying outside K^*. Let Ω be the radical axis of the coaxal system determined by the circle K^* and the point-circle L (Fig. 4). Let the tangents from L to K^* intersect Ω at the points A and B. Let K be the cycle which goes through A and B and touches LA and LB. Then we assert that with K as fundamental cycle and Ω as axis, the cycle K^* maps onto the point L.

If a line through L meets K at α, β and K^* at γ, where γ and β correspond in the similarity which has L as centre, then if the tangent at γ to K^* meets Ω at V, $|V\gamma| = |VL|$, since Ω is the radical axis of K^* and L. If the tangents to K at α and β meet at Q on Ω, then βQ is parallel to γV, and since $|\beta Q| = |\alpha Q|$ it follows that VL is parallel to $Q\alpha$. Hence LV is the transform of γV, and therefore the cycle K^* is mapped onto the point L.

4. Laguerre's final theorem, that *three cycles may be simultaneously transformed into points* follows almost visibly, and is worth reproducing. We note that if two rays touching a given cycle map onto the same line traversed in opposite directions, then the map of the given cycle must be a point.

Let the cycles be K_1, K_2 and K_3, with common tangent rays as shown, and centres of similitude P, Q and R (Fig. 5). These lie on a line, and we shall assume that this line does not intersect any of the cycles. We choose PQR as axis of transformation. Then we may transform the cycle K_1 into the point ω_1, where ω_1 is the limiting point of the coaxal system, determined by the circle K_1 and the radical axis PQR, which lies outside the circle K_1. The rays AP and PB tangent to K_1 will transform into opposite rays lying along the line $P\omega_1$. But the rays AP and BP also touch the cycle K_2. Since they are transformed into opposite rays along the same line $P\omega_1$, the cycle K_2 is transformed into a point ω_2, and this must lie at the intersection of $P\omega_1$ and the perpendicular from the centre of K_2 onto the line PQR. The rays EQ and QF which touch K_2 are transformed into opposite rays along the line $Q\omega_2$, and since EQ and QF also touch K_3, the cycle K_3 is transformed into a point ω_3 at the intersection of $Q\omega_2$ and the perpendicular from the centre of K_3 onto the line PQR.

This is the final theorem in Laguerre's paper, and he uses it to construct a cycle to touch the three given cycles K_1, K_2 and K_3. After transforming the cycles into three points ω_1, ω_2 and ω_3, he considers the circle which passes through the three points. This circle determines two opposite cycles, K and K', say. Applying the same Laguerre inversion which mapped K_1,

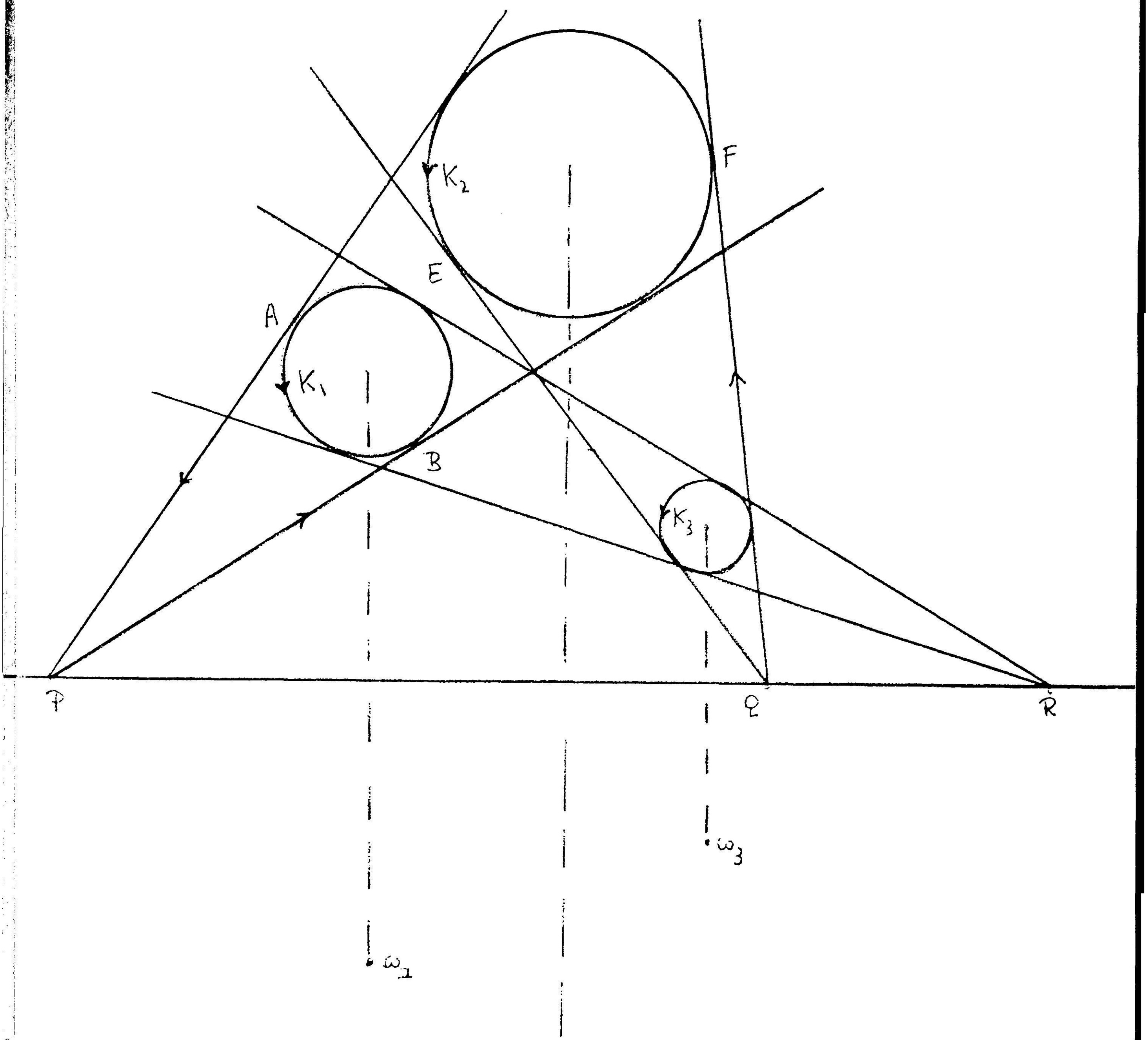

FIG. 5

K_2 and K_3 onto the points ω_1, ω_2 and ω_3, the points map back onto the cycles, and the cycles K and K' map onto cycles which touch the three given cycles K_1, K_2 and K_3. If we begin with three circles, one circle can be given an arbitrary sense, and the other two can then be oriented in four distinct ways. Each set of three cycles obtained in this way produces two contact cycles, and so we find the eight circles which touch each of three given circles, and solve the Apollonius problem. This is surely one of the most elegant methods ever applied to this problem !

5. We conclude our discussion of the technical aspect of Laguerre inversion by introducing *the power of a ray with respect to a given cycle.* It is well known that if a chord through a fixed point V cuts a given circle in the points P and Q, the product $\overline{VP}.\ \overline{VQ}$ of directed segments is constant for all positions of the chord through V, and this number is called *the power of the point V with respect to the circle.* Now let PN in Figure 6 be a given ray, K a given cycle, and PT a ray tangent to the cycle. With evident

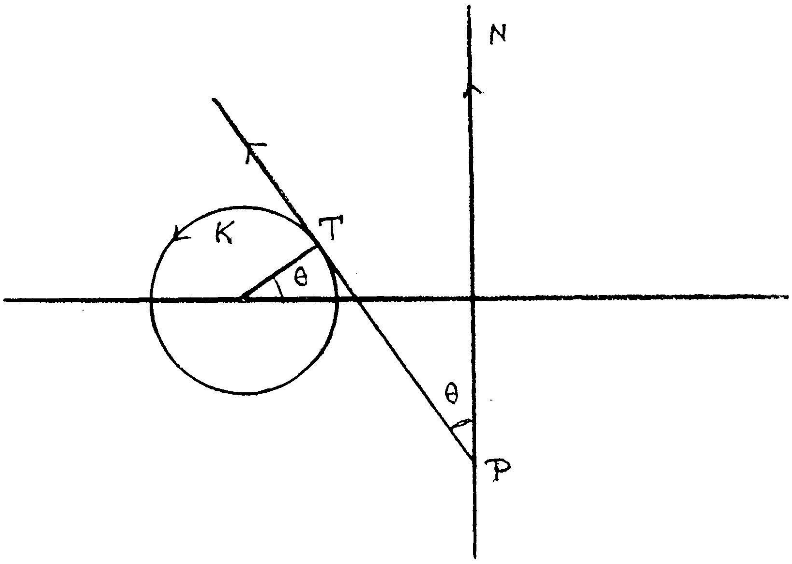

FIG. 6

axes of coordinates let the given line have equation $X = h$, and let the given cycle have radius r. Then the equation of the tangent ray is $X \cos \theta + Y \sin \theta - r = 0$, where θ is the angle shown. If P be the point (h, k), then $h\cos \theta + k\sin \theta - r = 0$, and expressing $\cos \theta$ and $\sin \theta$ in terms of $\tan (\theta/2)$, we obtain the equation:

$$(h+r)\, t^2 - 2kt + (r-h) = 0,$$

where $t = \tan (\theta/2)$, so that if the two tangent rays from P to the cycle make angles θ_1 and θ_2 with the ray PN,

$$\tan (\theta_1/2) \tan (\theta_2/2) = (r-h)/(r+h),$$

and this is independent of k, that is of the position of P on the ray. This number is called *the power of the ray with respect to the cycle.*

If Ω is a given ray, and NM, MN' are rays meeting on Ω and making angles θ and φ respectively with Ω, then the equation $\tan(\theta/2)\tan(\varphi/2) = k$, where k is a constant, is the defining equation of Laguerre inversion, and it is tempting to think that Laguerre arrived at his transformation this way, but the power of a ray with respect to a cycle is not mentioned in the paper we have been discussing. The theorems obtained by Laguerre may be obtained by the use of the power concept, but unless Laguerre's geometrical methods are followed the algebra becomes very tedious.

6. We conclude this paper with a discussion of a recent publication on contact circles in which two theorems are proved (Tyrrell and Powell [5]). The first, described in terms of circles, is essentially a theorem on cycles, and Laguerre inversion yields a very rapid proof. We introduce the theorem, as the authors do in their paper.

Let A, B and C be three circles of general position in the plane, and let S_1 be any circle touching A and B. Consider the following chain of circles: S_2 is a circle touching B, C and S_1; S_3 is a circle touching C, A and S_2; S_4 touches A, B and S_3; S_5 touches B, C and S_4; S_6 touches C, A and S_5; and S_7 touches A, B and S_6. There are a finite number of choices for each of the successive circles $S_2, \ldots, S_7$, but *if the choice at each stage is appropriately made* (in a manner to be described later) *then the last circle S_7 coincides with the first circle S_1.*

The " appropriate choices " are now described.

(i) Three circles of general position in the plane have eight contact circles. When, however, two of the three circles already touch, as is the case at every stage in constructing the chain described above, this number is reduced to six. For example, there are two circles of the contact coaxal family determined by B and S_1 which also touch C, and these are two of the six possible choices for S_2. These are called *the special choices* for S_2, and the other four possible choices for S_2 are called *the general choices.* A similar distinction between special and general choices arises at each stage in the construction of the chain.

(ii) If a circle S is drawn to touch two given circles, the line joining the two points of contact necessarily passes through one of the two centres of similitude of the given circles. The circle S is said *to belong to* that centre of similitude. Thus, of the six possible choices for S_2 in the chain, three choices (one special and two general) belong to each centre of similitude of B and C.

With these observations Tyrrell and Powell now state their theorem more precisely:

A set of three collinear centres of similitude is chosen, once and for all, for the circles A, B and C, and at each stage in the construction of the chain the circle S_i must belong to the appropriate one of these fixed centres of similitude. If the special choice for S_i is always made, the chain will close up. On the other hand, if at each stage a general choice is made, and if S_2, S_3 and S_4 are chosen at will (two choices for each), then it is always possible to choose S_5 and S_6 so that the chain closes up.

Laguerre inversion can be used to give a rapid proof of the " special choice " part of this theorem. The three circles can be oriented so that the three collinear centres of similitude chosen are precisely the centres of similitude of the three cycles taken in pairs. For special choices of cycles the centres of similitude need no longer be considered at all. If we use Laguerre inversion to transform the three cycles A, B and C into points, which we shall still call A, B and C, the theorem reads as follows:

A, B and C are three given points, and S_1 is a cycle through A and B. The cycle S_2 is drawn to touch S_1 at B and to pass through C. The cycle S_3 is drawn to touch S_2 at C and to pass through A. The cycle S_4 is drawn to touch S_3 at A and to pass through B. The cycle S_5 is drawn to touch S_4 at B and to pass through C. The cycle S_6 is drawn to touch S_5 at C and to pass through A. Finally, the cycle S_7 is drawn to touch S_6 at A and to pass through B. Then S_7 coincides with S_1.

This theorem can be proved easily by a number of methods, but if we use ordinary inversion (as Laguerre advocates, if it is necessary) and invert with respect to A as centre of inversion, noting that circles which touch each other at A invert into parallel lines, the theorem becomes the following, the proof of which is immediate:

Cycles $\mathscr{C}$ and $\mathscr{D}$ intersect at points B and C, and the tangent to $\mathscr{C}$ at B is parallel to the tangent to $\mathscr{D}$ at C. Then the tangent to $\mathscr{D}$ at B is also parallel to the tangent to $\mathscr{C}$ at C.

The part of the Tyrrell-Powell theorem which involves a *general* choice of contact circle does not seem to be a theorem on cycles, and so a simple proof by Laguerre inversion does not seem to be possible. The proof given in the authors' paper involves elliptic functions. It is a remarkable fact that both the special and the general parts of the theorem were conjectured as the result of a series of accurate drawings made by amateurs of geometry, J. Evelyn and G. B. Money-Coutts.

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