§ 3. The construction when E is sequentially complete

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for every sequence (γ_n) satisfying (2.4) the series

$$\sum_{n\in N} \gamma_n \lambda_n x_n \tag{2.14}$$

is normally convergent in E; and

$$f^*(x) = \infty \tag{2.15}$$

for every sum x of the series (2.14).

In the sequel we shall denote by $l^1_+(N)$ the set of sequences (λ_n) satisfying (2.13).

PROOF. Define by recurrence a strictly increasing sequence (k_n) of positive integers, taking k_1 to the first $k \in N$ such that $f_k(x_k) > 1^3$ and k_{n+1} to be the first $k \in N$ such that $k > k_n$ and $f_k(x_k) > (n+1)^3$. Then apply 2.1 and 2.2 with x_n and f_n replaced by $n^{-2}\, x_{k_n}$ and f_{k_n} respectively. This furnishes at least one strictly increasing sequence (n_ν) of positive integers such that (2.4) entails that the series

$$\sum_{\nu\in N} \gamma_\nu \, n_\nu^{-2} \, x_{k_{n_\nu}} \tag{2.16}$$

is normally convergent in E and that (2.15) holds for every sum x of (2.16). It thus suffices to define λ_n to be n_ν^{-2} when $n = k_{n_\nu}$ for some $\nu \in N$ and to be zero for all other $n \in N$; it is obvious that (2.13) is then satisfied.

§ 3. *The construction when* E *is sequentially complete*

3.1 In this section we assume merely that E is a locally convex space which is sequentially complete. Again P will denote a set of bounded gauges on E, and f^* will denote its upper envelope. Suppose given sequences (x_n) in E and (f_n) in P such that (2.1), (2.2″) and (2.3) are satisfied. Then the conclusion of 2.4 remains valid.

PROOF. Consider the continuous linear map T of $l^1(N)$ into E defined by

$$T\xi = \sum_{n\in N} \xi_n \, x_n.$$

Evidently, $x_n = T\alpha_n$ for suitably chosen α_n such that $\{\alpha_n : n \in N\}$ is a bounded subset of $l^1(N)$. It therefore suffices to apply 2.4 with E replaced by $l^1(N)$, x_n by α_n, and f_n by $f_n \circ T$.

The following corollary will find application in §§ 5 and 6 below.

3.2 COROLLARY. Suppose that H is a Hausdorff topological linear space and that $(E_i)_{i \in I}$ is a family of linear subspaces of H such that

(i) E_i is a Banach space relative to a norm $\| \cdot \|_i$ and the injection $E_i \to H$ is continuous.

Let $\mathscr{E} = \bigcap \{E_i : i \in I\}$ be topologised as a topological linear space by taking a base at 0 in $\mathscr{E}$ formed of the sets $\{x \in \mathscr{E} : \sup_{i \in J} \| x \|_i < \varepsilon\}$, where ε ranges over positive numbers and J over finite subsets of I. Let E be a sequentially closed linear subspace of $\mathscr{E}$ and $(f_n)_{n \in N}$ a sequence of bounded gauges on E, and write f^* for the upper envelope of $(f_n)_{n \in N}$. Suppose finally that $(x_n)_{n \in N}$ is a sequence of elements of E such that

(ii) $f^*(x_n) < \infty$ for every $n \in N$;

(iii) $\sup_{n \in N} \| x_n \|_i < \infty$ for every $i \in I$;

(iv) $\sup_{n \in N} f_n(x_n) = \infty$.

The conclusion is that, given real numbers $\beta > \alpha > 0$, a sequence $(\lambda_n)_{n \in N} \in l^1_+(N)$ may be constructed such that, for every sequence $(\gamma_n)_{n \in N}$ satisfying (2.4), the series (2.14) is normally convergent in E to a (unique) sum x satisfying (2.15).

PROOF. In view of 3.1, it will suffice to verify that $\mathscr{E}$ (which is obviously locally convex) is sequentially complete and Hausdorff. The latter property is evidently present. As to the former, suppose that $(y_n)_{n \in N}$ is a Cauchy sequence in $\mathscr{E}$. Then, by definition of the topology on $\mathscr{E}$, (y_n) is Cauchy in E_i for every $i \in I$. Hence, by the first clause of (i), (y_n) is convergent in E_i to a limit $y_{(i)} \in E_i$. The second clause of (i), plus the fact that H is Hausdorff, entails that there exists $y \in H$ such that $y_{(i)} = y$ for every $i \in I$. Accordingly, $y \in \mathscr{E}$; and, since $\lim_{n \to \infty} y_n = y_{(i)} = y$ in E_i for every $i \in I$, $\lim_{n \to \infty} y_n = y$ in $\mathscr{E}$. This shows that $\mathscr{E}$ is sequentially complete.

3.3 REMARKS. (1) If the elements of P are seminorms (rather than merely gauges), we may everywhere permit (γ_n) to be a sequence taking values in the (real or complex) scalar field of E, replacing (2.4) by the condition

$$\alpha \leqq | \gamma_n | \leqq \beta \quad \text{for every } n \in N. \tag{2.4'}$$

This is easily seen by reverting to 2.2 and using the fact that now $f_n(\gamma x) = | \gamma | f_n(x)$ for every $x \in E$, every $n \in N$ and every scalar γ. No changes are needed in the choice of the n_ν.

(2) Local convexity is needed in the proof of 3.1 since otherwise (2.2″), i.e., the boundedness of $S = \{x_n : n \in N\}$ in E, does not guarantee the existence of any continuous or bounded linear map T from $l^1(N)$ into E such that S is contained in the T-image of a bounded subset of $l^1(N)$. For it is plain that such a T can exist, only if the convex envelope S' of S is bounded in E. On the other hand, it is not difficult to verify that any first countable linear topological space E, in which the convex envelope of every bounded set (or of the range of every sequence converging to zero in E) is bounded, is necessarily locally convex.

(3) Naturally, local convexity of E may be dropped from the hypotheses of 3.1, if one assumes in place of (2.2″) that the convex envelope of $\{x_n : n \in N\}$ is a bounded subset of E.

§ 4. *Deduction of boundedness principles*

4.1 Theorem. Suppose that E is a sequentially complete locally convex space and that P is a set of bounded gauges on E. If $f^*(x) = \sup\{f(x) : f \in P\} < \infty$ for every $x \in E$, then f^* is bounded.

Proof. Suppose the contrary, that is, that $f^*(x) < \infty$ for every $x \in E$ and yet there exists a bounded subset B of E on which f^* is unbounded. Then we can choose $x_n \in B$, $f_n \in P$ such that $f_n(x_n) > n$ for every $n \in N$. Then (2.1), (2.2″) and (2.3) are satisfied; hence, by 3.1, there exists $x \in E$ such that $f^*(x) = \infty$, which is the required contradiction.

4.2 Remarks. (1) If we assume also that E is infrabarrelled and that each $f \in P$ is continuous, it follows that f^* is continuous, that is, that P is equicontinuous if it is pointwise bounded; cf. [2], pp. 47, 480-81. For, if V denotes the interval $[-\varepsilon, \varepsilon]$, where $\varepsilon > 0$, then

$$f^{*-1}(V) = \bigcap \{f^{-1}(V) : f \in P\}$$

is closed, convex and balanced and absorbs bounded sets in E. Since E is infrabarrelled, $f^{*-1}(V)$ is therefore a neighbourhood of the origin in E and thus f^* is continuous, as asserted.

(2) If one drops the hypothesis that E be locally convex (the remaining assumptions of Theorem 4.1 remaining intact), the substance of Remark 3.3 (3) shows that one may still conclude that $f^*(B)$ is bounded whenever B is a subset of E whose convex envelope in E is bounded.