§ 3. The construction when E is sequentially complete

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for every sequence (γ_n) satisfying (2.4) the series

$$
\sum_{n \in N} \gamma_n \lambda_n x_n \tag{2.14}
$$

is normally convergent in E ; and

$$
f^*(x) = \infty \tag{2.15}
$$

for every sum x of the series (2.14) .

In the sequel we shall denote by $l^1_+(N)$ the set of sequences (λ_n) satisfying $(2.13).$

PROOF. Define by recurrence a strictly increasing sequence (k_n) of positive integers, taking k_1 to the first $k \in N$ such that $f_k(x_k) > 1^3$ and k_{n+1} to be the first $k \in N$ such that $k > k_n$ and $f_k(x_k) > (n+1)^3$. Then apply 2.1 and 2.2 with x_n and f_n replaced by $n^{-2} x_{k_n}$ and f_{k_n} respectively. This furnishes at least one strictly increasing sequence (n_v) of positive integers such that (2.4) entails that the series

$$
\sum_{v \in N} \gamma_v n_v^{-2} x_{k_{n_v}}
$$
 (2.16)

is normally convergent in E and that (2.15) holds for every sum x of (2.16) . It thus suffices to define λ_n to be n_v^{-2} when $n = k_{n_v}$ for some $v \in N$ and to be zero for all other $n \in N$; it is obvious that (2.13) is then satisfied.

§ 3. The construction when E is sequentially complete

3.1 In this section we assume merely that E is a locally convex space which is sequentially complete. Again P will denote a set of bounded gauges on E , and f^* will denote its upper envelope. Suppose given sequences (x_n) in E and (f_n) in P such that (2.1) , $(2.2'')$ and (2.3) are satisfied. Then the conclusion of 2.4 remains valid.

PROOF. Consider the continuous linear map T of $l^{1}(N)$ into E defined by

$$
T\xi=\sum_{n\in N}\xi_n\,x_n.
$$

Evidently, $x_n = T\alpha_n$ for suitably chosen α_n such that $\{\alpha_n : n \in N\}$ is a bounded subset of $l^1(N)$. It therefore suffices to apply 2.4 with E replaced by $l^1(N)$, x_n by α_n , and f_n by $f_n \circ T$.

The following corollary will find application in §§ ⁵ and ⁶ below.

3.2 COROLLARY. Suppose that H is a Hausdorff topological linear space and that $(E_i)_{i \in I}$ is a family of linear subspaces of H such that

(i) E_i is a Banach space relative to a norm $|| \cdot ||_i$ and the injection $E_i \rightarrow H$ is continuous.

Let $\mathscr{E} = \bigcap \{E_i : i \in I\}$ be topologised as a topological linear space by taking a base at 0 in $\mathscr E$ formed of the sets $\{x \in \mathscr E : \sup_{i \in J} ||x||_i < \varepsilon\}$, where ε ranges over positive numbers and J over finite subsets of I . Let E be a sequentially closed linear subspace of $\mathscr E$ and $(f_n)_{n\in\mathbb N}$ a sequence of bounded gauges on E, and write f^* for the upper envelope of $(f_n)_{n \in N}$. Suppose finally that $(x_n)_{n\in\mathbb{N}}$ is a sequence of elements of E such that

- (ii) $f^*(x_n) < \infty$ for every $n \in N$;
- (iii) $\sup_{n \in \mathbb{N}} ||x_n||_i < \infty$ for every $i \in I$;
- (iv) $\sup_{n\in\mathbb{N}}f_n(x_n)=\infty$.

The conclusion is that, given real numbers $\beta > \alpha > 0$, a sequence $(\lambda_n)_{n\in\mathbb{N}} \in l^1_+(N)$ may be constructed such that, for every sequence $(\gamma_n)_{n\in\mathbb{N}}$ satisfying (2.4), the series (2.14) is normally convergent in E to a (unique) sum x satisfying (2.15) .

PROOF. In view of 3.1, it will suffice to verify that \mathscr{E} (which is obviously locally convex) is sequentially complete and Hausdorff. The latter property is evidently present. As to the former, suppose that $(y_n)_{n\in\mathbb{N}}$ is a Cauchy sequence in \mathscr{E} . Then, by definition of the topology on \mathscr{E} , (y_n) is Cauchy in E_i for every $i \in I$. Hence, by the first clause of (i), (y_n) is convergent in E_i to a limit $y_{(i)} \in E_i$. The second clause of (i), plus the fact that H is Hausdorff, entails that there exists $y \in H$ such that $y_{(i)} = y$ for every $i \in I$. Accordingly, $y \in \mathscr{E}$; and, since $\lim_{n\to\infty} y_n = y_{(i)} = y$ in E_i for every $i \in I$, $\lim_{n\to\infty}y_n = y$ in $\mathscr E$. This shows that $\mathscr E$ is sequentially complete.

3.3 REMARKS. (1) If the elements of P are seminorms (rather than merely gauges), we may everywhere permit (γ_n) to be a sequence taking values in the (real or complex) scalar field of E , replacing (2.4) by the condition

$$
\alpha \leq |\gamma_n| \leq \beta \quad \text{for every } n \in N. \tag{2.4'}
$$

This is easily seen by reverting to 2.2 and using the fact that now $f_n(\gamma x) = |\gamma| f_n(x)$ for every $x \in E$, every $n \in N$ and every scalar γ . No changes are needed in the choice of the $n_{\rm w}$.

(2) Local convexity is needed in the proof of 3.1 since otherwise $(2.2'')$, i.e., the boundedness of $S = \{x_n : n \in N\}$ in E, does not guarantee the existence of any continuous or bounded linear map T from $l^1(N)$ into E such that S is contained in the T-image of a bounded subset of $l^1(N)$. For it is plain that such a T can exist, only if the convex envelope S' of S is bounded in E . On the other hand, it is not difficult to verify that any first countable linear topological space E , in which the convex envelope of every bounded set (or of the range of every sequence converging to zero in E) is bounded, is necessarily locally convex.

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(3) Naturally, local convexity of E may be dropped from the hypotheses of 3.1, if one assumes in place of (2.2") that the convex envelope of $\{x_n : n \in N\}$ is a bounded subset of E.

§ 4. Deduction of boundedness principles

4.1 THEOREM. Suppose that E is a sequentially complete locally convex space and that P is a set of bounded gauges on E. If $f^*(x) = \sup \{f(x) :$ $f \in P$ < ∞ for every $x \in E$, then f^* is bounded.

PROOF. Suppose the contrary, that is, that $f^*(x) < \infty$ for every $x \in E$ and yet there exists a bounded subset B of E on which f^* is unbounded. Then we can choose $x_n \in B$, $f_n \in P$ such that $f_n(x_n) > n$ for every $n \in N$. Then (2.1), (2.2") and (2.3) are satisfied; hence, by 3.1, there exists $x \in E$ such that $f^*(x) = \infty$, which is the required contradiction.

4.2 REMARKS. (1) If we assume also that E is infrabarrelled and that each $f \in P$ is continuous, it follows that f^* is continuous, that is, that P is
equicontinuous if it is pointwise bounded: of [2] np. 47, 480,81. For it equicontinuous if it is pointwise bounded; cf. [2], pp. 47, 480-81. For, if V denotes the interval $[-\varepsilon, \varepsilon]$, where $\varepsilon > 0$, then

$$
f^{*-1}(V) = \bigcap \{ f^{-1}(V) : f \in P \}
$$

is closed, convex and balanced and absorbs bounded sets in E . Since E is infrabarrelled, $f^{*-1}(V)$ is therefore a neighbourhood of the origin in E and thus f^* is continuous, as asserted.

(2) If one drops the hypothesis that E be locally convex (the remaining assumptions of Theorem 4.1 remaining intact), the substance of Remark 3.3 (3) shows that one may still conclude that $f^*(B)$ is bounded whenever B is a subset of E whose convex envelope in E is bounded.