

# §3. Applications to theorems of the Schnee-Landau type

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exactly as (2.2) (a) implies (2.7). Since now  $\gamma > \sigma_r \geq \rho$ , the above condition in its turn implies

$$\overline{\lim}_{n \rightarrow \infty} \max_{l_n \leq l_m < l_n + \varepsilon l_n} (b_n + b_{n+1} + \dots + b_m) = o_R(1), \quad \varepsilon \rightarrow 0.$$

By Theorem A with hypothesis (1.2) (a) and  $a = b = 0$ , it follows that  $\Sigma a_n l_n^{-s}$  is convergent for any  $\sigma$  such that  $\sigma \geq \gamma > \sigma_r$  and therefore  $\sigma_0 \leq \sigma_r$ . But, in any case,  $\sigma_0 \geq \sigma_k \geq \sigma_r$  for  $0 \leq k < r$  and so we have the conclusion (2.5).

In the preceding argument we have supposed that  $\sigma_r < \infty$  since  $\sigma_r = \infty$  implies trivially  $\sigma_k = \infty$ .

### § 3. APPLICATIONS TO THEOREMS OF THE SCHNEE-LANDAU TYPE

Theorem II given next is the simplest of the theorems of the type mentioned above and it is a direct combination of Theorems I, B. Theorems V, VI are generalizations, respectively of Ananda-Rau's and Ganapathy Iyer's extensions of the Schnee-Landau theorem ([2], Theorem 9; [7], Theorem 10), as given by Chandrasekharan and Minakshisundaram ([6], pp. 88-9, Corollaries 3.73, 3.74). Theorems III, IV are apparently new counterparts of Theorems V, VI, the newness consisting in the replacement of the two-sided Tauberian conditions of the latter pair of theorems by analogous one-sided conditions suitably supplemented.

THEOREM II. *Suppose that (i) the Dirichlet series,*

$$\sum_1^{\infty} \frac{a_n}{l_n^s}, \quad s = \sigma + i\tau,$$

*is summable  $(R, l_n, q)$  for some  $q \geq 0$  when  $\sigma > \rho$ , (ii) the sum-function  $f(s)$  thus defined is regular for  $\sigma > \eta$  when  $\eta < \rho$ , and satisfies the condition*

$$f(s) = O(|\tau|^r), \quad r > 0, \quad \text{uniformly for } \sigma \geq \eta + \varepsilon > \eta,$$

*(iii) the coefficients  $a_n$  of the Dirichlet series satisfy ONE of the two alternatives (a), (b) of (2.2), but with  $\theta(x) \equiv x^{1 - (\rho - \eta)/r}$ . Then the Dirichlet series is summable  $(R, l_n, k)$ ,  $0 \leq k < r$ , for*

$$\sigma \geq \frac{(r - k)\rho + k\eta}{r}.$$

*Proof.* By Theorem B, the Dirichlet series is summable  $(R, l_n, r')$ ,  $r' > r$ , for  $\sigma > \eta$  and hence  $\sigma_r \leq \eta < \rho$ . Therefore it is evident from the proof of

Theorem I (A) ending with (2.10) that the Dirichlet series is summable  $(R, l_n, k)$ ,  $0 \leq k < r'$ , for

$$\sigma \geq \frac{(r' - k)\rho + k\eta}{r'}$$

whence the desired conclusion follows when we let  $r' \rightarrow r$ .

THEOREM III. In Theorem II, let  $\rho$  be replaced by  $\alpha + 1$  in hypotheses (i) and (ii); also let hypothesis (iii) be replaced by

$$a_n = O_R [l_n^\alpha (l_n - l_{n-1})], \quad l_n - l_{n-1} = O \left( l_n^{\frac{r-\alpha+\eta}{r+1}} \right). \quad (3.1)$$

Then the conclusion is that  $\Sigma a_n l_n^{-s}$ ,  $s = \sigma + i\tau$ , is summable  $(R, l_n, k)$ ,  $0 \leq k < r$ , for

$$\sigma > \frac{(r - k)(\alpha + 1) + (k + 1)\eta}{r + 1}. \quad (3.2)$$

*Proof.* As in the proof of Theorem II, the series  $\Sigma a_n l_n^{-s}$  is summable  $(R, l_n, r')$ ,  $r' > r$ , for  $\sigma > \eta$  where now  $\eta < \alpha + 1$ , so that  $\sigma_r \leq \eta < \alpha + 1$ . We begin by choosing  $\gamma$  and correspondingly  $\theta(x)$  as follows:

$$\eta < \gamma < \alpha + 1, \quad \theta(x) \equiv x^{(r' - \alpha + \gamma)/(r' + 1)}. \quad (3.3)$$

Then, since  $r' > r$  and  $\gamma > \eta$ , we have

$$\frac{r' - \alpha + \gamma}{r' + 1} > \frac{r - \alpha + \gamma}{r + 1} > \frac{r - \alpha + \eta}{r + 1}.$$

And so (3.1) gives us, as  $n \rightarrow \infty$ ,

$$a_n = O_R \left[ l_n^\alpha l_n^{\frac{r-\alpha+\eta}{r+1}} \right] = O_R \left[ l_n^\alpha l_n^{\frac{r'-\alpha+\gamma}{r'+1}} \right] = O_R [l_n^\alpha \theta(l_n)]. \quad (3.4)$$

Also, if  $l_n \leq l_m < l_n + \varepsilon \theta(l_n)$ , (3.1) again gives us as  $n \rightarrow \infty$ ,

$$a_{n+1} + a_{n+2} + \dots + a_m = \begin{cases} O_R [l_m^\alpha (l_m - l_n)] & \text{if } \alpha \geq 0, \\ O_R [l_n^\alpha (l_m - l_n)] & \text{if } \alpha < 0, \end{cases}$$

so that, whether  $\alpha \geq 0$  or  $\alpha < 0$ ,

$$a_{n+1} + a_{n+2} + \dots + a_m = O_R [l_n^\alpha \varepsilon \theta(l_n)]. \quad (3.5)$$

In (3.4) and (3.5),

$$l_n^\alpha \theta(l_n) = l_n^{\rho'} \quad \text{where} \quad \rho' = \alpha + \frac{r' - \alpha + \gamma}{r' + 1} \quad (> \gamma).$$

Hence, combining (3.4) and (3.5), we get

$$\overline{\lim}_{n \rightarrow \infty} \max_{l_n \leq l_m < l_n + \varepsilon \theta(l_n)} \frac{a_n + a_{n+1} + \dots + a_m}{l_n^{\rho'}} = o_R(1), \quad \varepsilon \rightarrow 0. \quad (3.6)$$

(3.6) and the fact, following from Theorem B, that  $\Sigma a_n l_n^{-s}$  is summable  $(R, l_n, r')$ , enables us to use (2.10) in the proof of Theorem I (A) with  $r, \rho$  replaced by  $r', \rho'$  respectively, so as to infer that  $\Sigma a_n l_n^{-s}$  is summable  $(R, l_n, k)$ ,  $0 \leq k < r'$ , for

$$\sigma \geq \frac{(r' - k) \rho' + k\gamma}{r'} = \frac{(r' - k)(\alpha + 1) + (k + 1)\gamma}{r' + 1}.$$

This yields (3.2) as required when we let  $r' \rightarrow r$  and recall that  $\gamma (> \eta)$  can be taken arbitrarily close to  $\eta$ .

**THEOREM IV.** *In Theorem III, (3.1) alone can be changed to*

$$\left. \begin{aligned} \sum_{v=1}^n (a_v + |a_v|) l_v^p (l_v - l_{v-1})^{1-p} &= O(l_n^{p(\alpha+1)+1}), \quad l_n - l_{n-1} = \\ &= O \left[ l_n \frac{r - \alpha - p^{-1} + \eta}{r + 1 - p^{-1}} \right], \quad p > 1, \quad \alpha + 1 + p^{-1} \geq 0, \end{aligned} \right\} \quad (3.7)$$

with the conclusion changed in consequence to the assertion that  $\Sigma a_n l_n^{-s}$  is summable  $(R, l_n, k)$ ,  $0 \leq k < r$ , for

$$\sigma > \frac{(r - k)(\alpha + 1) + (k + 1 - p^{-1})\eta}{r + 1 - p^{-1}}. \quad (3.8)$$

*Proof.* We observe that Theorem III may be viewed as the limiting case  $p = \infty$  of Theorem IV.

The proof itself is similar to that of Theorem III with the difference that the choice of  $\gamma$  and  $\theta(x)$  in (3.3) is now altered as below:

$$\eta < \gamma < \alpha + 1, \quad \theta(x) \equiv x^{(r' - \alpha - p^{-1} + \gamma)/(r' + 1 - p^{-1})}$$

1) We suppose that  $l_0 = 0$ .

And furthermore the step corresponding to (3.6) is obtained as follows. Writing  $1 - 1/p = 1/p'$ , we get, for  $l_n \leq l_m < l_n + \varepsilon\theta(l_n)$ ,

$$\begin{aligned}
 a_{n+1} + a_{n+2} + \dots + a_m &\leq a_{n+1} + |a_{n+1}| + \dots + a_m + |a_m| \\
 &= \sum_{v=1}^{m-n} (a_{v+n} + |a_{v+n}|) l_{v+n} (l_{v+n} - l_{v+n-1})^{(1-p)/p} \times \\
 &\qquad \qquad \qquad \times \frac{(l_{v+n} - l_{v+n-1})^{1/p'}}{l_{v+n}} \\
 &\leq \left[ \sum_{v=1}^{m-n} (a_{v+n} + |a_{v+n}|)^p l_{v+n}^p (l_{v+n} - l_{v+n-1})^{1-p} \right]^{1/p} \times \\
 &\qquad \qquad \qquad \times \left[ \sum_{v=1}^{m-n} \frac{l_{v+n} - l_{v+n-1}}{l_{v+n}^{p'}} \right]^{1/p'} \\
 &= O \left[ l_m^{\alpha+1+1/p} \frac{(l_m - l_n)^{1/p'}}{l_{n+1}} \right] (n \rightarrow \infty) \\
 &= O \left[ l_n^{\alpha+1+1/p} \frac{\{\varepsilon \theta(l_n)\}^{1/p'}}{l_n} \right] \tag{3.9}
 \end{aligned}$$

where we have used the hypothesis (3.7) in the passage to the step preceding (3.9). Taking  $m = n+1$  in the step preceding (3.9), we get also

$$\begin{aligned}
 a_{n+1} &= O_R \left[ l_n^{\alpha+1+1/p} \frac{(l_{n+1} - l_n)^{1/p'}}{l_{n+1}} \right] (n \rightarrow \infty) \\
 &= O_R \left[ l_{n+1}^{\alpha+1/p} l_{n+1}^{(r-\alpha-p^{-1}+\eta)/(r+1-p^{-1})p'} \right] \\
 &= o_R \left[ l_{n+1}^{\alpha+1/p} \{\theta(l_{n+1})\}^{1/p'} \right]. \tag{3.10}
 \end{aligned}$$

From (3.9) and (3.10) with  $n+1$  changed to  $n$ , we obtain, instead of (3.6) in the proof of Theorem III,

$$\overline{\lim}_{n \rightarrow \infty} \max_{l_n \leq l_m < l_n + \varepsilon\theta(l_n)} \frac{a_n + a_{n+1} + \dots + a_m}{l_n^{\rho'}} = o_R(1), \quad \varepsilon \rightarrow 0,$$

where

$$\rho' = \alpha + \frac{1}{p} + \frac{(r' - \alpha - p^{-1} + \eta)}{(r' + 1 - p^{-1})p'}.$$

After this the proof is completed exactly like that of Theorem III subsequent to (3.6).

It may be observed that the assumption  $\alpha+1+p^{-1} \geq 0$  involves no loss of generality since  $\alpha+1+p^{-1} < 0$  makes successively  $a_n + |a_n| \equiv 0$ ,  $a_n \equiv 0$  and so  $\sigma_r = -\infty$  for all  $r \geq 0$ .

**THEOREM V.** *In Theorem II, let hypothesis (i) be omitted on account of its being implicit (with  $q = 0$ ,  $\rho = \alpha+1$ ) in hypothesis (iii) modified as under. Let hypothesis (ii) be retained with  $\rho$  changed to  $\alpha+1$ , and hypothesis (iii) replaced by*

$$a_n = O [l_n^\alpha (l_n - l_{n-1})]. \quad (3.11)$$

*Then the conclusion is that  $\Sigma a_n l_n^{-s}$  is summable  $(R, l_n, k)$ ,  $0 \leq k < r$ , for  $\sigma$  satisfying (3.2).*

**THEOREM VI.** *If, in Theorem V, (3.11) alone is changed to*

$$\sum_{v=1}^n |a_v|^p l_v^p (l_v - l_{v-1})^{1-p} = O [l_n^{p(\alpha+1)+1}], \quad p > 1, \quad \alpha + 1 + p^{-1} \geq 0,$$

*the conclusion will become the assertion that  $\Sigma a_n l_n^{-s}$  is summable  $(R, l_n, k)$ ,  $0 \leq k < r$ , for  $\sigma$  satisfying (3.8).*

The proofs of Theorems V, VI are omitted, being obvious simplifications of those of Theorems III, IV, involving the use of Theorem I (A) with hypothesis (2.2) (b) instead of (2.2) (a) as formerly. Theorems V and VI, as pointed out by Chandrasekharan and Minakshisundaram, yield Ananda Rau's and Ganapathy Iyer's extensions of the Schnee-Landau theorem when  $\alpha \rightarrow +0$ .

#### § 4. FURTHER APPLICATIONS

Theorem I (A) is a base which, combined with Theorem B, produces Theorem II, and in this sense Theorem I (A) may be said to correspond to Theorem II. There are results corresponding to each of Theorems III-VI in the same sense. For instance, Deduction 1 below corresponds to Theorem III and shows how other deductions corresponding to Theorems IV-VI may be formulated. Deductions 2,3 are further examples of results based on Theorem I.

**DEDUCTION 1.** (A) *In Theorem I (A), suppose that  $\sigma_r < \alpha+1$  and that (2.2) (a) is replaced by*

$$a_n = O_R [l_n^\alpha (l_n - l_{n-1})], \quad l_n - l_{n-1} = O (l_n^{(r-\alpha+\sigma_r)/(r+1)}). \quad (4.1)$$