COMPACT ANALYTICAL VARIETIES

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CONTENTS

INTRODUCTION

These lectures deal with the vanishing theorem of Kodaira (cf. e.g. [2], p. 344) and some of its consequences, and with Lefschetz' theorem on hyperplane sections (cf. [1]). Only complex manifolds (and not complex spaces) are considered, but most of the results in the first part could be carried over to the more general case (with similar proofs).

1. PRELIMINARIES

We first give some definitions:

Definition 1.1. Let V be a complex manifold and D a relatively compact, open subset of V. Then D is *strongly pseudoconvex* if for every $x_0 \in \partial D$ there exist a neighbourhood U of x_0 and a real-valued C^2-function φ defined in U such that

(1) $$d\varphi(x_0) \neq 0,$$

(2) $$H(\varphi)(x_0) > 0 \text{ for all } \alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbf{C}^n - \{0\}.$$

(Here $H(\varphi)$ is the complex Hessian form

$$\sum_{i,j=1}^{n} \frac{\partial^2 \varphi}{\partial z_i \partial \bar{z}_j} \alpha_i \bar{\alpha}_j$$

with respect to some system of local coordinates),

(3) $$D \cap U = \{ x \in U; \quad \varphi(x) < 0 \} .$$

It can be shown that strong pseudoconvexity of D is equivalent to the following property: For every $x_0 \in \partial D$ there exist a neighbourhood U of x_0 and a biholomorphic mapping $f \colon U \to \Omega \subset \mathbf{C}^n$ such that $f(U \cap D)$ has a strictly convex boundary (in the Euclidean sense).

Definition 1.2. Let V be a complex manifold and A a subset of V. We say that A " can be blown down to a point " if there exist an analytic space X, a point $x_0 \in X$, and a mapping $f \colon V \to X$ such that $f(A) = x_0$ and $f \colon V - A \to X - \{ x_0 \}$ is an analytic isomorphism.

To give an example of sets which can be blown down to a point, we mention the following theorem (for a proof see [2], pp. 338 and 340):

Theorem 1.3. If D is strongly pseudoconvex, then D has a maximal compact analytic subset A whose dimension at any point is > 0 and each component of A can be blown down to a point.

Lemma 1.4. If A can be blown down to a point, then A has a fundamental system of strongly pseudoconvex neighbourhoods.

Proof. Let X, x_0, and f be as in Definition 1.2. The lemma follows from the fact that the inverse image of a strongly pseudoconvex neighbourhood of x_0 is a strongly pseudoconvex neighbourhood of A.

We now introduce the concept of holomorphic line bundle.

Definition 1.5. Suppose X is a complex manifold. A holomorphic line bundle F on X is a complex manifold F together with a mapping π with the following properties:

(i) $\pi \colon F \to X$ is a holomorphic map (called projection) onto X.

(ii) For $x \in X$, $\pi^{-1}(x)$ has the structure of a one-dimensional vector space over the complex numbers.

(iii) For each $x \in X$ there exist a neighbourhood U of x and a holomorphic mapping h of $F \mid U = \pi^{-1}(U)$ onto $U \times \mathbf{C}$ such that h^{-1} is holomorphic and $h \mid \pi^{-1}(a)$ is a $\mathbf{C}$-isomorphism onto $\{ a \} \times \mathbf{C}$ for every $a \in U$.

Let $\{ U_i \}$ be an open covering of X such that for each i we have a mapping h_i of $F \mid U_i$ onto $U_i \times \mathbf{C}$ with the properties in (iii) above. If $U_i \cap U_j \neq \emptyset$, we get a mapping $h_i \circ h_j^{-1} \colon (U_i \cap U_j) \times \mathbf{C} \to (U_i \cap U_j) \times \mathbf{C}$. If $(x, c) \in (U_i \cap U_j) \times \mathbf{C}$, then the image of (x, c) under the mapping

$h_i \circ h_j^{-1}$ can be written $(x, \gamma'(x, c))$ where $\gamma'(x, c) \in \mathbf{C}$. According to the last property in (iii), for fixed $x \in U_i \cap U_j$ the mapping $c \to \gamma'(x, c)$ is a $\mathbf{C}$-isomorphism of $\mathbf{C}$ onto itself. Therefore

$$\gamma'(x, c) = g_{ij}(x) \cdot c \text{ , where } g_{ij}(x) \neq 0 \text{ ,} \tag{1.1}$$

and it is easily seen that g_{ij} is holomorphic in $U_i \cap U_j$.

The functions g_{ij} obviously satisfy the cocycle conditions

$$g_{ij}\, g_{jk}\, g_{ki} = 1 \quad \text{on} \quad U_i \cap U_j \cap U_k \text{ ,} \tag{1.2}$$

$$g_{ij}\, g_{ji} = 1 \quad \text{on} \quad U_i \cap U_j \text{ .} \tag{1.3}$$

The g_{ij} are called transition functions corresponding to the line bundle F.

Conversely, it is easy to prove (cf. [4], p. 135) that given an open covering $\{ U_i \}$ and functions g_{ij} without zeros in $U_i \cap U_j$ which satisfy the cocycle conditions, we can construct a line bundle which has g_{ij} as transition functions.

Now, let F be a line bundle over a complex manifold X, and let π be the corresponding projection. We denote $\pi^{-1}(a)$ by F_a. Let F_a^* be the $\mathbf{C}$-dual of F_a. Then

$$F^* = \bigcup_{a \in X} F_a$$

is in a natural way a holomorphic line bundle over X, which is called the dual bundle of F. If F has transition functions $\{ g_{ij} \}$, then F^* has transition functions $\{ g_{ij}^{-1} \}$.

Definition 1.6. Let F be a holomorphic line bundle over a compact complex manifold. Then F is *negative* if the zero cross section $\mathfrak{o}$ of F can be blown down to a point. F is *positive* if the dual bundle is negative.

In the sequel we let $\underline{F}$ denote the sheaf of germs of analytic sections of a line bundle F.

2. The vanishing theorem of Kodaira

This is the following theorem, which is our first main result:

Theorem 2.1. Let X be a compact connected complex manifold and F a positive line bundle on X and S a coherent analytic sheaf on X. Then there exists an integer $k(S, F)$ such that for $k > k(S, F)$ we have $H^q(X, S \otimes \underline{F^k}) = 0 \; (\forall q \geqslant 1)$.

The proof uses the following finiteness theorem:

Theorem 2.2. Let V be a complex manifold, S a coherent analytic sheaf on V, and $D \subset \subset V$ a strictly pseudoconvex subdomain of V. Then the cohomology groups $H^q(D, S)$ are finite-dimensional **C**-vector spaces if $q \geq 1$.

For a proof of Theorem 2.2 see Section 4.4 of the lectures by Malgrange in these notes.

Proof of Theorem 2.1.

Let E be the dual bundle of F. By hypothesis, E is negative. Thus, by Lemma 1.4, the zero cross section of E has a strictly pseudoconvex neighbourhood D.

By definition, we have a projection $\pi \colon E \to X$. We will now use π to " lift " S to a coherent analytic sheaf $\tilde{S}$ on E. To do this, we first consider the sheaf of abelian groups $\pi^{-1}(S)$ which to any point a of E assigns the stalk $S_{\pi(a)}$. Since $S_{\pi(a)}$ and the ring $\mathcal{O}_a(E)$ of germs of analytic functions at a both are modules over the ring $\mathcal{O}_{\pi(a)}(X)$, we can form the tensor product $\tilde{S}_a = S_a \otimes \mathcal{O}_a(E)$ over $\mathcal{O}_{\pi(a)}(X)$. Then $\tilde{S}_a$ is a module over $\mathcal{O}_a(E)$, and this defines $\tilde{S}$. Since S is coherent, $\tilde{S}$ is also coherent (cf [3], p. 401).

From Theorem 2.2 it now follows that $H^q(D, \tilde{S})$ are finite-dimensional **C**-vector spaces for $q \geq 1$. We complete the proof of Theorem 2.1 by constructing for every N a natural injection

$$\sum_{k=0}^{N} H^q(X, S \otimes \underline{F^k}) \to H^q(D, \tilde{S}),$$

where the sum is the direct sum as vector spaces. In fact, since $\dim \sum_{k=0}^{N} H^q = \sum_{k=0}^{N} \dim H^q$, the existence of such injections would imply the existence of the desired integer $k(S, F)$.

Let a be a point of the zero cross section $\mathfrak{o}$ in the negative bundle E, and let U be a neighbourhood of a such that $E_U \approx U \times \mathbf{C}$. Identifying $a \in \mathfrak{o} \subset E$ with the point $\pi(a) \in X$, we denote by $\mathcal{O}_a(E)$ and $\mathcal{O}_a(X)$ the rings of germs of analytic functions on E at a and on X at a, respectively.

To a germ $f \in \mathcal{O}_a(E)$ corresponds a Taylor series $\sum_{\nu=0}^{\infty} f_\nu(x)\, z^\nu$, converging in some neighbourhood $U' \times D_r$, where $U' \subset U$ and $D_r = \{z;\ |z| < r\}$.

For $x \in U$, let $e'(x) \in E_x$ correspond to $(x, 1)$ under the isomorphism $E_x \approx U \times \mathbf{C}$, and let $e(x) \in F_x$ be defined by $< e(x), e'(x) > = 1$. Then

$e(x)$ is a holomorphic section of F over U, and every germ $p \in \underline{F}_a^k$ is represented by $p(x)\, e(x) \otimes e(x) \otimes \dots \otimes e(x)$, ($k$ factors $e(x)$), where $p(x)$ is holomorphic in a neighbourhood of a. But $p(x)\; e(x) \otimes e(x) \otimes \dots \otimes e(x) \in \underline{F}_x^k$ can be identified with the multilinear functional

$$(z_1, \dots, z_k) \to p(x)\, z_1 \cdot \dots \cdot z_k$$

and therefore also with the polynomial $p(x)\, z^k$.

Hence, for every N we obtain an injection

$$i_N \colon \sum_{k=0}^{N} \underline{F}_a^k \to \mathcal{O}_a(E)$$

by mapping $(p_0, p_1, \dots, p_N) \in \sum_{0}^{N} \underline{F}_a^k$ onto the germ at a of $\sum_{k=0}^{N} f_k(x)\, z^k$, where $f_k(x)$ is holomorphic in a neighbourhood of a and $f_k(x)\, z^k$ corresponds to $p_k \in \underline{F}_a^k$ in the way described above. Further the map $q_N \colon \sum_{0}^{\infty} f_\nu(x) z^\nu \to f_k(x) z^k$ gives rise to a homomorphism $\mathcal{O}_a(E) \to \underline{F}_a^k$ such that $q_N \circ i_N = \mathrm{id}$. It is obvious that this mapping i_N is injective.

From i_N we also obtain a homomorphism

$$j_N : S \otimes_{\mathcal{O}(X)} \sum_{0}^{N} \underline{F}^k \to S \otimes_{\mathcal{O}(X)} \mathcal{O}(E) = \tilde{S},$$

and the corresponding homomorphism

$$j_N^* : H^q(X, S \otimes \sum_{0}^{N} \underline{F}^k) \to H^q(\mathfrak{o}, \tilde{S}).$$

Further, the map q_N defined above gives rise to a homomorphism

$$\tilde{S} \to S \otimes_{\mathcal{O}(X)} \sum_{0}^{N} \underline{F}^k,$$

and hence a map

$$\eta_N \colon H^q(\mathcal{O}, \tilde{S}) \to H^q(X, S \otimes \sum_{0}^{N} \underline{F}^k)$$

such that $\eta_N \circ j_N^* = \mathrm{id}$. Hence j_N^* is injective.

This mapping can be factored as follows

$$H^q(\mathcal{O}, S \otimes \sum_{0}^{N} \underline{F}^k) = \sum_{0}^{N} H^q(S \otimes \underline{F}^k) \xrightarrow{\alpha} H^q(D, \tilde{S}) \xrightarrow{\beta} H^q(\mathfrak{o}, \tilde{S}),$$

and as $\beta \circ \alpha$ is an injection, α also is an injection, which proves the theorem.

3. An imbedding theorem

Lemma 3.1. If X is a compact complex manifold and S a coherent analytic sheaf over X, then $\Gamma(X, S)$ is a finite dimensional vector space (cf. remark concerning Theorem 2.2).

We will now prove an imbedding theorem (cf. [2], p. 343).

Theorem 3.2. If the complex manifold X is compact, connected, and carries a positive (negative) line bundle, then X can be imbedded biholomorphically in a complex projective space $\mathbf{P}^N$.

Proof: Suppose F is a line bundle on a compact complex manifold X with the property that for every $a \in X$ there exists a section $\sigma \in \Gamma(X, \underline{F})$ with $\sigma(a) \neq 0$. Then F defines a holomorphic mapping of X into a projective space $\mathbf{P}^k$ in the following way:

Since X is compact, $\Gamma(X, \underline{F})$ is finite-dimensional according to Lemma 3.1.

Let $\sigma_0, ..., \sigma_k$ be a basis of $\Gamma(X, \underline{F})$. Then the σ_j have no common zeros.

Since F is locally isomorphic to the product of an open subset of X and $\mathbf{C}$, the σ_j are locally given by holomorphic functions without common zeros.

We map X into $\mathbf{P}^k$ by $x \to (\sigma_0(x), ..., \sigma_k(x))$. The point in the projective space is independent of the isomorphism we are using, for if we use another isomorphism we get a point $(g(x)\,\sigma_0(x), ..., g(x)\,\sigma_k(x))$, where $g(x) \neq 0$ (cf. (1.1)).

We are now going to show that if F is positive, then there exists an integer γ such that the sections of $\Gamma(X, \underline{F}^\gamma)$ have no common zeros and such that the corresponding mapping is an imbedding.

For $a \in X$, let I be the sheaf of germs of holomorphic functions vanishing at a. Since I is coherent, we can apply the vanishing theorem of Kodaira. We conclude that there exists an integer $k(a)$ such that $H^1(X, I \otimes \underline{F}^{k \geqq k(a)}) = 0$

Since $\mathcal{O}_a/I_a \approx \mathbf{C}$, we have the following exact sequence

$$0 \to I \to \mathcal{O}(X) \to \mathbf{C}_a \to 0 ,$$

where $\mathbf{C}_a$ is a sheaf with stalk $\mathbf{C}$ at a and zero outside. From this it follows that the sequence

$$0 \to I \otimes \underline{F}^{k(a)} \to \underline{F}^{k(a)} \to \mathbf{C}_a \otimes \underline{F}^{k(a)} \to 0$$

is exact. We have $\mathbf{C}_a \otimes \underline{F}^{k(a)} \approx \tilde{F}_a^{k(a)}$, where $\tilde{F}_a^{k(a)}$ has stalk $F_a^{k(a)}$ at a and zero outside. Using the fact that $H^1(X, I \otimes \underline{F}^{k(a)}) = 0$, the exact cohomology sequence associated to the above sequence of sheaves gives us an exact sequence

$$\Gamma(X, \underline{F}^{k(a)}) \to \Gamma(X, \tilde{F}_a^{k(a)}) \to 0 .$$

This implies that given $e \in F_a^{k(a)}$ there exists $\sigma \in \Gamma(X, \underline{F}^{k(a)})$ such that $\sigma(a) = e$. Thus, for every $a \in X$ we can find an integer $k(a)$ and a neighbourhood V_a of a such that $\Gamma(X, \underline{F}^{k(a)})$ has a section not vanishing on V_a. Since X is compact, there are finitely many such neighbourhoods V_i ($i=1, \ldots, p$) with corresponding sections of F^{k_i} such that $X = \bigcup_{i=1}^{p} V_i$. Letting $k = k_1 \cdot k_2 \cdot \ldots \cdot k_p$, we get p elements of $\Gamma(X, \underline{F}^k)$ without common zeros, for if $\sigma \in \Gamma(X, \underline{F})$ and $\sigma(x) \neq 0$, then $\sigma' = \underbrace{\sigma \otimes \ldots \otimes \sigma}_{l\text{ - times}} \in \Gamma(X, \underline{F}^l)$ and $\sigma'(x) \neq 0$.

Let $\underline{E} = \underline{F}^k$. Now, for $a \in X$, let $G = q_a^2$, where q_a is the ideal of germs of holomorphic functions vanishing at a. Using the above argument with $\underline{E}$ and G instead of $\underline{F}$ and I, we see that there exists an integer $s(a)$ such that the restriction mapping

$$\Gamma(X, \underline{E}^{s(a)}) \to \{ \mathcal{O}_a / q_a^{\,2} \} \otimes \underline{E}_a^{\,s(a)}$$

is surjective. Since the residue classes in $\mathcal{O}_a / q_a^{\,2}$ are sets of germs f of holomorphic functions at a with fixed values of $f(a)$ and $df(a)$, this implies that we can find a neighbourhood U_a of a and sections $\sigma_1, \ldots, \sigma_t \in \Gamma(X, \underline{E}^{s(a)})$ which are nowhere zero in U_a such that the mapping given by $\sigma_1, \ldots, \sigma_t$ is regular and injective in U_a. We observe that for every positive integer I we can find sections $\sigma_1^{(1)}, \ldots, \sigma_t^{(1)} \in \Gamma(X, E^{ls(a)})$ which have the same properties in U_a. In fact, if σ is a section of $\underline{E}^{s(a)}$ which has no zeros on a set $M \subset X$, we set

$$\sigma' = \sigma \otimes \ldots \otimes \sigma, (l-1) \text{ times.}$$

Then $\sigma' \otimes \sigma_1, \ldots, \sigma' \otimes \sigma_t$ are sections of $\underline{E}^{ls(a)}$, and define the same mapping (at least on M) as $\sigma_1, \ldots, \sigma_t$.

We can cover X by finitely many such neighbourhoods $U_1, \ldots, U_r$. If $s' = s_1 \cdot \ldots \cdot s_r$, then there are elements of $\Gamma(X, \underline{E}^{s'})$ which give a regular, injective mapping in each U_i $(1 \leqslant i \leqslant r)$.

We are now going to show that we can separate points in X by sections of a suitable $\underline{E}^{\alpha}$. Let $U = \bigcup_{i=1}^{r} (U_i \times U_i)$. For $(a, b) \in X \times X - U$, let H be the sheaf of germs of holomorphic functions vanishing at a and b. It is easily seen that the sequence

$$0 \to H \to \mathcal{O}(X) \to \mathbf{C}_a \oplus \mathbf{C}_b \to 0$$

is exact. From this we conclude as above that there exists an integer $s\,(a, b)$ such that the sequence

$$\Gamma\,(X, \underline{E}^{s(a,b)}) \to E_a{}^{s(a,b)} \oplus E_b{}^{s(a,b)} \to 0$$

is exact. Therefore there exists a neighbourhood W of (a, b) in $X \times X$ such that if $(a', b') \in W$, then the sections of $\Gamma\,(X, \underline{E}^{s(a,b)})$ separate a' and b'; that is, if $\sigma_0, ..., \sigma_k$ is a basis of $\Gamma\,(X, \underline{E}^{s(a,b)})$, then $(\sigma_0(a'), ..., \sigma_k(a'))$ and $(\sigma_0(b'), ..., \sigma_k(b'))$ are different points in $\mathbf{P}^k$. Let l be a positive integer, let $(a', b') \in W$, and let σ be a section of $\Gamma\,(X, \underline{E}^{s(a,b)})$ such that $\sigma\,(a') \neq 0$ and $\sigma\,(b') \neq 0$. Then $\sigma^{l-1} \otimes \sigma_0, ..., \sigma^{l-1} \otimes \sigma_k$ are sections of $\Gamma\,(X, \underline{E}^{l\,s(a,b)})$ such that $((\sigma^{l-1} \otimes \sigma_0)\,(a'), ..., (\sigma^{l-1} \otimes \sigma_k)\,(a'))$ and $((\sigma^{l-1} \otimes \sigma_0)\,(b'), ..., (\sigma^{l-1} \otimes \sigma_k)\,(b'))$ are different points in $\mathbf{P}^k$.

This means that for every positive integer l the sections of $\Gamma\,(X, E^{ls(a,b)})$ separate all point pairs in W. Thus, covering $X \times X - U$ by finitely many such neighbourhoods and taking s'' to be the product of the corresponding $s\,(a, b)$, we find that the sections of $\Gamma\,(X, \underline{E}^{s''})$ separate all point pairs in $X \times X - U$.

Let $\alpha = s's''$ and let $\sigma_0, ..., \sigma_d$ be a basis of $\Gamma\,(X, \underline{E}^{\alpha})$. We claim that the mapping f from X into $\mathbf{P}^d$ defined by $f\,(x) = (\sigma_0(x), ..., \sigma_d(x))$ is a biholomorphic imbedding of X into $\mathbf{P}^d$. That this mapping is regular follows from the fact that α is a multiple of s'. What remains to be proved is that the mapping is injective.

Suppose $a, b \in X$, $a \neq b$. If $(a, b) \in U$, then $a, b \in U_i$ for some i, and since α is a multiple of s', we have $f\,(a) \neq f\,(b)$. If $(a, b) \in X \times X - U$, then $f\,(a) \neq f\,(b)$ since α is a multiple of s''. This proves the theorem.

4. Line bundle associated to a divisor

Let X be a complex manifold and D an analytic subset of X of pure codimension 1 at every point. Such a set D is called a *divisor* of X. We shall construct a line bundle F on X, associated to D.

To do this, we observe that every point of X has a neighbourhood U in which there is a holomorphic function s such that $U \cap D = \{\, x \in U;\, s(x) = 0\,\}$, and s generates, at every point of U, the ideal of germs of holomorphic functions vanishing on D. Thus we get a covering of X by open sets U_j and corresponding holomorphic functions s_j. The functions $g_{ij} = s_i/s_j$ are then holomorphic and $\neq 0$ on $U_i \cap U_j$ and $g_{ij}\, g_{jk} = g_{ik}$ on $U_i \cap U_j \cap U_k$. The functions g_{ij} therefore define a line bundle F on X with transition functions g_{ij} (see sect. 1). This bundle F is determined by D uniquely up to isomorphism.

If $f \in \Gamma(X, F)$, then the isomorphism $F \mid U_j \simeq U_j \times \mathbf{C}$ gives a holomorphic function f_j on U_j corresponding to f. The functions f_j are related by $f_i = g_{ij} f_j$ on $U_i \cap U_j$. Conversely, if f_j are holomorphic functions on U_j, satisfying this condition, then there is a section f of F on X, which corresponds to f_j on U_j. In particular, the s_j define a section s_D of F on X, and we have $D = \{ x \in X;\ s_D(x) = 0 \}$.

Example. Let $X = \mathbf{P}^n$, and let H be the hyperplane defined in the homogeneous coordinates $z_0, \ldots, z_n$ by $z_0 = 0$. Then the process above associates to H a line bundle F on $\mathbf{P}^n$. As defining functions we can use $s_j(z_0, \ldots, z_n) = z_0/z_j$ on the set U_j where $z_j \neq 0$, $(j=0, \ldots, n)$. We shall prove that F is positive.

Each homogeneous coordinate z_k defines a section $s^{(k)}$ of F, which on each U_j corresponds to the holomorphic function z_k/z_j, for the transition functions are $g_{ij} = s_i/s_j = z_j/z_i$ and we have $z_k/z_i = (z_k/z_j)\, g_{ij}$. Now any section of F can be regarded as a holomorphic function on $E = F^*$, which is linear on the fibres of E. In particular, $s^{(0)}, \ldots, s^{(k)}$ give a holomorphic mapping $\varphi\colon E \to \mathbf{C}^{n+1}$. It is clear that the zero section in E is equal to $\varphi^{-1}(0)$. It is seen by direct verification that φ maps E onto $\mathbf{C}^{n+1}$ and $E - \varphi^{-1}(0)$ biholomorphically onto $\mathbf{C}^{n+1} - \{ 0 \}$. Hence E is negative and F is positive (see sect. 1).

If V is a submanifold of $\mathbf{P}^n$, then the restriction of F to V is a positive line bundle associated to the hyperplane section $D = V \cap H$. In fact, the dual of the restriction is the restriction $E \mid V$ of E to V, and we can use the restriction of φ to $E \mid V$ as " blowing down mapping ".

Let again X be a complex manifold, D a divisor of X, and F the line bundle on X, associated to D. What are the sections of F^k?

If $U \in \Gamma(X, F^k)$, then s is represented in local coordinates on U_j by a holomorphic function f_j. The f_j are connected by $f_i = g_{ij}^k f_j$ on $U_i \cap U_j$, because the functions g_{ij}^k are transition functions for F^k. Now $s_i^k = g_{ij}^k s_j^k$ on $U_i \cap U_j$, the s_i being local equations for the set D as above, and thus $f_i/s_i^k = f_j/s_j^k$ on $U_i \cap U_j$. Hence there exists a meromorphic function f on X such that $f_j = s_j^k f$ on U_j.

This means that f is meromorphic with poles only on D and of order $\leqslant k$. Conversely, if f is such a meromorphic function, then $f_j = s_j^k f$ are holomorphic on U_j and satisfy $f_i = g_{ij}^k f_j$ on $U_i \cap U_j$. Therefore they give a section s of F^k. This correspondence is obtained simply by associating to the section u of F^k, the meromorphic function $u \otimes s_D^{-k}$.

Let us consider again the space $\mathbf{P}^n$ and the bundle F associated to a hyperplane section. Let $(z_0, ..., z_n)$ denote homogeneous coordinates for $\mathbf{P}^n$. If $u \in \Gamma(\mathbf{P}^n, F^k)$, u defines, for $z \in \mathbf{P}^n$, an element of $F_z = (E_z^*)^k$, E being the dual bundle to F, hence a map of E_z into $\mathbf{C}$ which is homogeneous of degree k. Thus, u defines a map $\hat{u}$ of $E \to \mathbf{C}$, homogeneous of degree k on each fibre. If φ denotes the map of E into $\mathbf{C}^{n+1}$ defined above, $\hat{u}: E \to \mathbf{C}$ is holomorphic, and vanishes on $\varphi^{-1}(0)$, and so defines a holomorphic function v on $\mathbf{C}^{n+1}$ which is homogeneous of degree k (v is holomorphic also at 0 since a continuous function holomorphic outside a point in $\mathbf{C}^{n+1}$, $n \geqslant 1$, is holomorphic also at this point). The Taylor expansion of v about 0 shows that v is a homogeneous polynomial of degree k. Thus, *any* $u \in \Gamma(\mathbf{P}^n, F^k)$ *can be identified with a homogeneous polynomial of degree* k *in the homogeneous coordinates* $(z_0, ..., z_n)$ [i.e. the sections $s^{(0)}, ..., s^{(n)}$ of F defined above].

As an application of the vanishing theorem of Kodaira, we now prove the following result due to Chow (cf. [3], p. 170).

Theorem 4.1. Let A be a subvariety of $\mathbf{P}^n$. Then there exist homogeneous polynomials $f_1, ..., f_k$ such that $A = \{ a \in \mathbf{P}_n; f_1(a) = ... = f_k(a) = 0 \}$.

Proof. We first prove that if $b \notin A$, then there exists a homogeneous polynomial f vanishing on A with $f(b) \neq 0$. Let S be the sheaf of germs of holomorphic functions vanishing on A and let I be the sheaf of germs of holomorphic functions vanishing at b. Let F be the line bundle associated to a hyperplane section of A. Then F is positive. We get an exact sequence

$$0 \to I \otimes S \otimes F^m \to S \otimes F^m \to S_b \otimes F_b^m \to 0 .$$

By the vanishing theorem of Kodaira, part of the corresponding cohomology sequence will be

$$H^o(\mathbf{P}^n, S \otimes F^m) \to H^o(\mathbf{P}^n, S_b \otimes F_b^m) \to 0 ,$$

if m is sufficiently large. Thus there exists $f \in H^0(P^n, S \otimes F^m)$ which is not zero at b. Since $S \subset \mathcal{O}$, we may look upon $H^0(S \otimes F^m)$ as a subspace of $H^0(F^m)$. It is then the subspace of those sections of $H^0(F^m)$ which vanish on A. Since $f \in H^0(\mathbf{P}^n, F^m)$, this gives the desired homogeneous polynomial.

To prove the theorem, it now suffices to consider all homogeneous polynomials which vanish on A without being identically zero and apply the Hilbert basis theorem.

5. Meromorphic forms

Let X be a complex manifold. A holomorphic differential form is a form which in local coordinates can be written as a finite sum

$$\omega = \Sigma a_{i_1 \ldots i_k} dz_{i_1} \wedge \ldots \wedge dz_{i_k} \tag{5.1}$$

with holomorphic coefficients $a_{i_1 \ldots i_k}$.

A form is called meromorphic if it has locally the form (5.1) with coefficients that are meromorphic functions. Every meromorphic function can be written locally as $f\omega$ where f is a meromorphic function and ω a holomorphic form. The exterior differentiation d, satisfying $d^2 = 0$, extends naturally to meromorphic forms.

Let D be a divisor of X and let $\Omega^p(k, D) = \Omega^p(X, k, D)$ be the sheaf of germs of meromorphic p-forms on X with poles only on D and of order $\leqslant k$, and let $\Omega^p = \Omega^p(X)$ be the sheaf of germs of holomorphic p-forms on X.

Lemma 5.1. There is a natural isomorphism

$$\Omega^p(k, D) \simeq \Omega^p \otimes \underline{F}^k .$$

Proof. A germ in $\Omega^p(k, D)$ at $a \in X$ is represented by a form $f\omega$, where f is a meromorphic function in a neighbourhood U of a, with poles only on D and of order $\leqslant k$, and ω is a holomorphic form on U. Now to f corresponds biuniquely a section $s \in \Gamma(U, F^k)$ (see Sect. 4), which gives a germ $s_a \in \underline{F}^k_a$. Also ω defines a germ $\omega_a \in \Omega^p_a$.

The desired mapping $\Omega^p(K, D) \to \Omega^p \otimes \underline{F}^k$ is now uniquely defined by

$$f\omega \to \omega_a \otimes s_a .$$

To see that it is an isomorphism, it is sufficient to observe that the inverse mapping of $\Omega^p \otimes \underline{F}^k$ into $\Omega^p(k, D)$ is induced by the bilinear mapping $\Omega^p \oplus \underline{F}^k \to \Omega^p(k, D)$, which is given by

$$(\omega_a, s_a) \to (f\omega)_a , \qquad (a \in X) .$$

where f is the meromorphic function determined by s_a by the procedure described just before Th. 4.1.

Now let X be a compact submanifold of $\mathbf{P}^n$ and consider hyperplanes H_c in $\mathbf{P}^n$, given in homogeneous coordinates $z_0, \ldots, z_n$ by equations

$$\sum_0^n c_j z_j = 0 \text{ where } c = (c_0, \ldots, c_n) \neq 0 .$$

Theorem 5.2. There is an open dense set Ω in $\mathbf{C}^{n+1}$ such that if $c = (c_0, \ldots, c_n) \in \Omega$, the hyperplane section $D_c = H_c \cap X$ is a non-singular analytic subset of X.

The proof is omitted here.

Let $D = H \cap X$ be a non-singular hyperplane section of X. To D is then associated a positive line bundle F on X (see Sect. 4). By Kodaira's vanishing theorem there is a k_0 such that

$$H^q(X, \Omega^p \otimes \underline{F}^k) = 0 , \qquad (\forall q \geq 1, \forall k \geqq k_0) .$$

Using the isomorphism in Lemma 5.1, we have therefore proved.

Lemma 5.3. If D is a non-singular hyperplane section of a compact submanifold X of $\mathbf{P}^n$, then there exists k_0 such that

$$H^q(X, \Omega^p(k, D)) = 0 , \qquad (\forall q \geq 1, \forall k \geq k_0) .$$

6. The Atiyah-Hodge theorem

We first recall two well-known theorems.

Let X be a paracompact C^∞ manifold and let $\mathscr{E}^p$ be the sheaf of germs of C^∞ p-forms on X (p=0, 1, ...).

Then the sequence

$$0 \to \mathbf{C} \xrightarrow{i} \mathscr{E}^0 \xrightarrow{d} \mathscr{E}^1 \xrightarrow{d} \mathscr{E}^2 \xrightarrow{d} \cdots \tag{6.1}$$

is exact (Poincaré's lemma), and

$$H^q(X, \mathscr{E}^p) = 0 , \qquad (\forall q \geq 1, \forall p \geqq 0) , \tag{6.2}$$

because the $\mathscr{E}^p$ are fine sheaves, i.e. they have partitions of unity. From (6.1) we get the sequence

$$0 \to \Gamma(X, \mathscr{E}^0) \to \Gamma(X, \mathscr{E}^1) \to \ldots ,$$

which need not be exact. Put

$$H^p(\mathscr{E}) = \frac{\operatorname{Ker}(\Gamma(X, \mathscr{E}^p) \to \Gamma(X, \mathscr{E}^{p+1}))}{\operatorname{Im}(\Gamma(X, \mathscr{E}^{p-1}) \to \Gamma(X, \mathscr{E}^p))}, \quad (p \geq 0), \tag{6.3}$$

with $\mathscr{E}^{-1} = 0$. Then one has the following theorem of de Rham:

Theorem 6.1. There are natural isomorphisms

$$H^p(X, \mathbf{C}) \simeq H^p(\mathscr{E}), \quad (p \geq 0).$$

If X is a Stein manifold and Ω^p the sheaf of germs of holomorphic p-forms on X ($p=0, 1, \ldots$), then the sequence

$$0 \to \mathbf{C} \xrightarrow{i} \Omega^0 \xrightarrow{d} \Omega^1 \xrightarrow{d} \Omega^2 \to \ldots \tag{6.4}$$

is exact (Gnothendieck's lemma), and

$$H^q(X, \Omega^p) = 0, \quad (\forall q \geq 1, \forall p \geq 0) \tag{6.5}$$

(Cartan's Theorem B). Put

$$H^p(\Omega) = \frac{\operatorname{Ker}(\Gamma(X, \Omega^p) \to \Gamma(X, \Omega^{p+1}))}{\operatorname{Im}(\Gamma(X, \Omega^{p-1}) \to \Gamma(X, \Omega^p))}, \quad (p \geq 0)$$

with $\Omega^{-1} = 0$. Then one has the following theorem

Theorem 6.2. There are natural isomorphisms

$$H^p(X, \mathbf{C}) \simeq H^p(\Omega), \quad (p \geq 0).$$

Theorems 6.1 and 6.2 both follow if one applies the following lemma to the exact sequences (6.1) and (6.4), respectively:

Lemma 6.3. Let X be a paracompact Hausdorff space and

$$0 \to F \xrightarrow{i} F_0 \xrightarrow{d_0} F_1 \xrightarrow{d_1} \cdots \tag{6.6}$$

an exact sequence of sheaves of abelian groups, such that

$$H^q(X, F_p) = 0, \quad (\forall q \geq 1, \forall p \geq 0). \tag{6.7}$$

Then there are natural isomorphisms

$$H^p(X, F) \simeq \operatorname{Ker} d_p^*/\operatorname{Im} d_{p-1}^*, \quad (p \geq 0),$$

where d_p^* is the mapping $\Gamma(X, F_p) \to \Gamma(X, F_{p+1})$ induced by (6.6) (with $F_{-1}=0$).

Proof. Put $Z_p = \operatorname{Ker} d_p \subset F_p$. Then the exactness of (6.6) gives short exact sequences

$$0 \to Z_{p-1} \to F_{p-1} \to Z_p \to 0\,, \qquad (p \geq 1)\,, \tag{6.8}$$

from which we get long exact sequences of cohomology groups, which we write in part:

$$H^q(X, F_{p-1}) \to H^q(X, Z_p) \to H^{q+1}(X, Z_{p-1}) \to H^{q+1}(X, F_{p-1})\,,$$
$$(q \geq 0,\ p \geq 1)\,. \tag{6.9}$$

When $q \geqslant 1$, we get from (6.7) and (6.9)

$$H^q(X, Z_p) \simeq H^{q+1}(X, Z_{p-1})\,, \qquad (p \geq 1)\,.$$

Since F is isomorphic to Z_0, we therefore have

$$H^p(X, F) \simeq H^{p-1}(X, Z_1) \simeq \ldots \simeq H^1(X, Z_{p-1})\,, \quad (p \geq 1)\,. \tag{6.10}$$

When $q = 0$, (6.9) gives an exact sequence

$$\Gamma(X, F_{p-1}) \overset{d_{p-1}^*}{\longrightarrow} \Gamma(X, Z_p) \to H^1(X, Z_{p-1}) \to 0\,,$$

and thus

$$H^1(X, Z_{p-1}) \simeq \Gamma(X, F_{p-1})/\operatorname{Im} d_{p-1}^* = \operatorname{Ker} d_p^*/\operatorname{Im} d_{p-1}^*\,,$$

which together with (6.10) proves the lemma when $p \geqslant 1$.

To prove it for $p = 0$, we observe that the exact sequence

$$0 \to F = Z_0 \to F_0 \to Z_1 \to 0$$

gives an exact sequence

$$0 \to \Gamma(X, F) \to \Gamma(X, F_0) \overset{d_0^*}{\longrightarrow} (X, Z_1)$$

and thus

$$H^0(X, F) = \Gamma(X, F) \simeq \operatorname{Ker} d_0^* = \operatorname{Ker} d_0^*/\operatorname{Im} d_{-1}^*\,.$$

Now let V be a compact submanifold of $\mathbf{P}^n$ and D a non-singular hyperplane section of V. Then $X = V - D$ is imbedded as a closed submanifold of $\mathbf{C}^n$, and in particular it is a Stein manifold.

Let $\Omega^p(D) = \Omega^p(V, D)$ be the sheaf of germs of meromorphic p-forms on V with poles only on D, $p = 0, 1, \ldots$. Then we have a sequence (not necessarily exact)

$$0 \to \mathbf{C} \to \Omega^0(D) \overset{d_0'}{\longrightarrow} \Omega^1(D) \overset{d_1'}{\longrightarrow} \ldots\,.$$

Define

$$\tilde{H}^p = \operatorname{Ker} d'^*_p/\operatorname{Im} d'^*_{p-1}\,, \qquad (p \geq 0)\,,$$

where d'^*_p is the induced mapping $\Gamma(V, \Omega^p(D)) \to \Gamma(V, \Omega^{p+1}(D))$, (with $\Omega^{-1}(D) = 0$). We shall prove the following theorem of Atiyah and Hodge:

Theorem 6.4. There are natural isomorphisms

$$H^p(X, \mathbf{C}) \simeq \tilde{H}^p, \quad (p \geq 0).$$

Proof. Let $\mathscr{E}^p(D) = \mathscr{E}^p(V, D)$ be the sheaf on V, which is defined by the presheaf that to every open subset U of V associates the module of C^∞ p-forms on $U - D$. Then we have a commutative diagram

$$\begin{array}{ccccc} 0 \to \Omega^0(D) & \xrightarrow{d'_0} & \Omega^1(D) & \xrightarrow{d'_1} & \cdots \\ \downarrow & & \downarrow & & \\ 0 \to \mathscr{E}^0(D) & \xrightarrow{d_0} & \mathscr{E}^1(D) & \xrightarrow{d_1} & \cdots, \end{array} \tag{6.11}$$

where the vertical mappings are the inclusions.

For every p, we can regard $\Omega^p(D)$ as the direct limit of $\Omega^p(k, D) = \Omega^p(V, k, D)$ as $k \to \infty$. Now, by Lemma 5.3, there is a k_0 such that $H^q(V, \Omega^p(k, D)) = 0$ for $q \geqslant 1$ and $k \geqslant k_0$. Hence we can conclude that

$$H^q(V, \Omega^p(D)) = 0, \quad (\forall q \geq 1, \forall p \geq 0). \tag{6.12}$$

We also have

$$H^q(V, \mathscr{E}^p(D)) = 0, \quad (\forall q \geq 1, \forall p \geq 0), \tag{6.13}$$

because $\mathscr{E}^p(D)$ are fine sheaves.

From (6.11) we get a diagram

$$\begin{array}{ccccc} 0 \to \Gamma(V, \Omega^0(D)) & \xrightarrow{d'^*_0} & \Gamma(V, \Omega^1(D)) & \xrightarrow{d'^*_1} & \cdots \\ \downarrow & & \downarrow & & \\ 0 \to \Gamma(V, \mathscr{E}^0(D)) & \xrightarrow{d^*_0} & \Gamma(V, \mathscr{E}^1(D)) & \xrightarrow{d^*_1} & \cdots \end{array} \tag{6.14}$$

The cohomology groups of the upper row in (6.14) are $\tilde{H}^p$, $(p=0, 1, \ldots)$, and those of the lower row are the groups $H^p(\mathscr{E})$ in (6.3), because one can obviously identify $\Gamma(V, \mathscr{E}^p(D))$ with $\Gamma(X, \mathscr{E}^p)$. In view of de Rham's theorem, it is therefore sufficient to prove that the vertical mappings in (6.14) induce isomorphisms between the cohomology groups of the rows.

To do this, we will use the following theorem:

Theorem 6.5. Let X be a paracompact Hausdorff space and suppose that two complexes $\mathscr{E}$ and $\mathscr{E}'$ of sheaves over X are given, together with mappings h such that the diagram

$$\begin{array}{ccccccc} 0 \xrightarrow{d'_{-1}} & \mathscr{E}'_0 & \xrightarrow{d'_0} & \mathscr{E}'_1 & \xrightarrow{d'_1} & \mathscr{E}'_2 & \to \cdots \\ & {\scriptstyle h_0}\downarrow & & {\scriptstyle h_1}\downarrow & & {\scriptstyle h_2}\downarrow & \\ 0 \xrightarrow[d_{-1}]{} & \mathscr{E}_0 & \xrightarrow[d_0]{} & \mathscr{E}_1 & \xrightarrow[d_1]{} & \mathscr{E}_2 & \to \cdots \end{array} \tag{6.15}$$

is commutative. (The rows are not supposed to be exact, but we have $dd = 0$ and $d'd' = 0$.)

Suppose further that

$$H^q(X, \mathscr{E}_k) = 0 \text{ and } H^q(X, \mathscr{E}_k') = 0\,, \ (\forall q \geq 1 \ \forall k \geq 0)\,, \tag{6.16}$$

and that for all $k \geqslant 0$, h induces *isomorphisms* of the cohomology sheaves

$$h_k : \operatorname{Ker} d'_k / Im\, d'_{k-1} \to \operatorname{Ker} d'_k / Im\, d'_{k-1}\,. \tag{6.17}$$

Then it follows that h induces *isomorphisms* for all $k \geqslant 0$:

$$h_k^* : \operatorname{Ker} d'^*_k / \operatorname{Im} d'^*_{k-1} \to \operatorname{Ker} d'^*_k / \operatorname{Im} d'^*_k\,, \tag{6.18}$$

where d^* and d'^* are the mappings induced by d and d' between the groups of global sections of the given sheaves:

$$\begin{array}{ccccccc} 0 \to \Gamma(X, \mathscr{E}'_0) & \overset{d'_0{}^*}{\longrightarrow} & \Gamma(X, \mathscr{E}'_1) & \overset{d'_1{}^*}{\longrightarrow} & (X, \mathscr{E}'_2) & \overset{d'_2{}^*}{\longrightarrow} & \cdots \\ \downarrow & & \downarrow & & \downarrow & & \\ 0 \to \Gamma(X, \mathscr{E}_0) & \overset{d_0{}^*}{\longrightarrow} & \Gamma(X, \mathscr{E}_1) & \overset{d_1{}^*}{\longrightarrow} & \Gamma(X, \mathscr{E}_2) & \overset{d_2{}^*}{\longrightarrow} & \cdots \end{array} \tag{6.19}$$

Proof of Theorem 6.5: Taking $F = 0$ in Lemma 6.3, we see that exactness of a sequence

$$0 \to F_0 \overset{\delta_0}{\longrightarrow} F_1 \overset{\delta_1}{\longrightarrow} \cdots \tag{6.20}$$

together with the conditions (6.7) implies exactness of the sequence

$$0 \to \Gamma(X, F_0) \overset{\delta_0{}^*}{\longrightarrow} \Gamma(X, F_1) \overset{\delta_1{}^*}{\longrightarrow} \cdots. \tag{6.21}$$

With the help of the " mapping cylinder " construction we will reduce the proof of Theorem 6.5 to an application of this fact. We define the sheaves and mappings in (6.20) as follows (where we take $\mathscr{E}_{-1} = 0$).

$$F_k = \mathscr{E}'_k \oplus \mathscr{E}_{k-1}\,; \qquad \delta_k(a', a) = \left(d'_k a', d_{k-1} a + (-1)^k h_k a'\right).$$

Since (6.7) follows from (6.16), it is enough to prove that the fact that (6.17) are isomorphisms for all $k \geqslant 0$ implies that (6.20) is exact, and that the exactness of (6.21) implies that (6.18) are isomorphisms. But we see that (6.21) is obtained from (6.19) by the same construction which lead from (6.15) to (6.20). Thus the proof of Theorem 6.5 will be complete if we apply the following lemma in one direction to (6.15) and (6.20) and in the other direction to (6.19) and (6.21).

Lemma 6.6. Let (6.15) be any diagram of the type considered above (with no condition (6.16) supposed) and such that (6.17) are isomorphisms, and let (6.20) be the corresponding sequence given by the above construction. Then (6.18) are isomorphisms if and only if (6.20) is exact.

Proof of Lemma 6.6. By straightforward calculation we see that $\delta\delta = 0$. Clearly h_k^* is injective if and only if

(i) For every $a' \in \mathscr{E}_k'$ and $a \in \mathscr{E}_{k-1}$ with $d'a' = 0$ and $ha' = da$ there exists $b' \in \mathscr{E}_{k-1}$ with $a' = d'b'$.

Similarly, h_{k-1}^* is surjective if and only if

(ii) For every $b \in \mathscr{E}_{k-1}$ with $db = 0$ there exist $f' \in \mathscr{E}'_{k-1}$ and $c \in \mathscr{E}_{k-2}$ with $d'f' = 0$ and $dc = b - hf'$.

Finally we want to express in a similar way the condition that (6.20) is exact at F_k. If $\alpha \in F_k$ and $\delta\alpha = 0$, the condition is that $\alpha = \delta\gamma$ for some $\gamma \in F_{k-1}$. To get rid of the signs we write $\alpha = ((-1)^{k-1} a', a)$ and $\gamma = ((-1)^{k-1} c', c)$. Then the condition may be written:

(iii) For every $a' \in \mathscr{E}_k'$ and $a \in \mathscr{E}_{k-1}$ with $d'a' = 0$ and $ha' = da$, there exist $c' \in \mathscr{E}_{k-1}'$ and $c \in \mathscr{E}_{k-2}$ such that $d'c' = a'$ and $dc = a - hc'$.

Trivially, (iii) $\Rightarrow$ (i). Taking $a' = 0$ and $a =$ b, we see that (iii) $\Rightarrow$ (ii). To complete the proof we will then assume that (i) and (ii) holds and prove (iii).

Let a' and a be as in (iii). From (i) we get b'. Then $d(a - hb') = da - ha' = 0$ by hypothesis. Apply (ii) with $b = a - hb'$ and define $c' = b' - f'$. Then $d'c' = a'$ and $a\text{-}hc' = a - hb' - hf' = dc$, which completes the proof of Theorem 6.5.

Continuation of the proof of Theorem 6.4. It only remains to prove that we may take $\mathscr{E}' = \Omega(V, D)$ and $\mathscr{E} = \mathscr{E}(V, D)$ in Theorem 6.5. In view of (6.12) and (6.13), it suffices to check that the mappings (6.17) are isomorphisms for all $k \geqslant 0$.

At any point of $V - D$, both cohomologies are trivial, and there is nothing to prove. Thus it only remains to consider points in D. Let us choose a neighbourhood U of such a point a and local coordinates $(z_1, \ldots, z_n)$ in U in such a way that U is the polycylinder given by $|z_i| < 1$, $(i = 1, 2, \ldots, n)$, $U \cap D$ is the part of U where $z_1 = 0$, and a is the point where all $z_i = 0$. Now $U - D = (E - \{0\}) \times E^{n-1}$, where E is the open unit disk in $\mathbf{C}$. Since the second factor is contractible, the mapping $(E - \{0\}) \times E^{n-1} \to E - \{0\}$ induces isomorphisms of $\mathscr{H}'^k = \operatorname{Ker} d'_k / \operatorname{Im} d'_{k-1}$. Thus, by de Rham's theorem, $\mathscr{H}'^k = 0$ if $k \geqslant 2$, and $\dfrac{dz_1}{z_1}$ forms a basis for $\mathscr{H}'^1$. We claim that the same is true for $\mathscr{H}^k = \mathrm{Ker} d_k / \mathrm{Im} d'_{k-1}$. Since h is the natural inclusion, this would complete the proof.

All forms considered in the sequel are meromorphic in U and have poles at most on D. If $\gamma = \Sigma a_{i_1 \ldots i_k} dz_{i_1} \wedge \ldots \wedge dz_{i_k}$ is a k-form, we set $\frac{\partial \gamma}{\partial z_\nu} = \sum \frac{\partial a_{i_1 \cdots i_k}}{\partial z_\nu} dz_{i_1} \wedge \ldots \wedge dz_{i_k}$. Then $d\gamma = \sum_{\nu=1}^{n} dz_\nu \wedge \frac{\partial \gamma}{\partial z_\nu}$.

We also introduce the norm $|\gamma| = \sup |a_{i_1 \cdots i_k}|$. If γ does not involve dz_1, we define $\delta\gamma = \sum_{\nu=2}^{n} dz_\nu \wedge \frac{\partial \gamma}{\partial z_\nu}$. We will need the following lemma.

Lemma 6.8. If γ is a k-form $(k \geqslant 1)$ not involving dz_1, and if $\delta\gamma = 0$, then there exists a form γ' not involving dz_1 and such that $\delta\gamma' = \gamma$.

Proof of Lemma 6.8. We first suppose that γ is a holomorphic. Then we have $\gamma = \sum_{\nu \geqq 0} z_1^\nu \beta_\nu$ and $0 = \Sigma z_1^\nu \delta \beta_\nu$ with convergence for $|z_1| < 1$. Thus for any $\varrho > 1$ we have $|\beta_\nu| \leqslant C \varrho^\nu$.

By the ordinary lemma in a polydisk, there exists β_ν' such that $\beta_\nu = \delta\beta_\nu'$. The mapping $\beta_\nu \to \beta_\nu'$ is a mapping onto the Fréchet space of all closed $(k-1)$-forms. Thus, by the open mapping theorem, we see that the equation $\delta\beta_\nu' = \beta_\nu$ has a solution β_ν' with $|\beta_\nu'| \leqslant C' \varrho^\nu$ on any smaller polydisk P, (C' being a constant which may depend on Thus $\gamma' = \sum_{\nu \geqq 0} z_1{}^\nu \beta_\nu'$ is convergent in $|z_1| < \frac{1}{\varrho}$, which proves the lemma in the holomorphic case. In the general case we have $\gamma = \sum_{i=0}^{k} z_1{}^{-i} \gamma_i$ with holomorphic forms γ_i. We apply the first case to the γ_i and get $\gamma' = \sum_{i=0}^{k} z_1{}^{-i} \gamma_i$ which completes the proof of the lemma.

End of proof of Theorem 6.4. Let ω be any k-form. Then we may write $\omega = dz_1 \wedge \alpha + \beta$, where α and β do not involve dz_1. Suppose now that $d\omega = 0$. This condition takes the form

$$dz_1 \wedge \delta\alpha + dz_1 \wedge \frac{\partial \beta}{\partial z_1} + \delta\beta = 0\,, \tag{6.22}$$

which implies that $\delta\beta = 0$. By Lemma 6.8, we have $\beta = \delta\beta'$ for some $(k-1)$-form β'.

Now ω takes the form

$$\omega - d\beta' = dz_1 \wedge \alpha' . \tag{6.23}$$

We distinguish the two cases $k > 1$ and $k = 1$. In the first case we get from (6.23)

$$dz_1 \wedge \delta\alpha' = 0 ,$$

which implies that $\delta\alpha' = 0$. Since α' is a form of type $q-1 \geqslant 1$, we can apply once again Lemma 6.8 and get $\alpha' = \delta\alpha''$. Thus $dz_1 \wedge \alpha' = d(dz_1 \wedge \alpha'')$, and we get $\omega = d(\beta' + dz_1 \wedge \alpha'')$. This proves that the cohomology under consideration is trivial for $k > 1$.

Finally, in the case $k = 1$, α' is a meromorphic function, independent of $z_2, \ldots, z_n$. Thus by (6.23), $\omega = d\gamma$ for some γ if and only if in the Laurent expansion of α' the coefficient of z_1^{-1} is zero. Thus the cohomology in dimension 1 is generated by $z_1^{-1}\, dz_1$, which completes the proof of Theorem 6.4.

7. Lefschetz' theorem on hyperplane sections

The Lefschetz theorem in the slightly more general setting proved by Andreotti and Frankel [1], is the following:

Theorem 7.1. Let V be a submanifold of $\mathbf{P}^n$ of complex dimension d and let D be a hyperplane section of V (not necessarily non-singular). Then there are natural isomorphisms

$$H^q(V, \mathbf{Z}) \simeq H^q(D, \mathbf{Z}) , \qquad (\forall q < d-1) ,$$

and a natural injection

$$H^{d-1}(V, \mathbf{Z}) \to H^{d-1}(D, \mathbf{Z}) .$$

Proof. $X = V - D$ is a Stein manifold, since it is imbedded as a closed submanifold of $\mathbf{C}^n$. Now one knows that

$$H^q(V, D, \mathbf{Z}) \simeq H^q_c(X, \mathbf{Z}) , \tag{7.1}$$

where the c indicates cohomology with compact support. On the other hand, since X is a topological manifold of dimension $2d$, Poincaré duality gives

$$H^q_c(X, \mathbf{Z}) \simeq H_{2d-q}(X, \mathbf{Z}) . \tag{7.2}$$

Now we shall use the following theorem:

Theorem 7.2. Let X be a Stein manifold of dimension d. Then

$$H_r(X,\mathbf{Z}) = 0, \qquad (\forall r > d). \tag{7.3}$$

Suppose this theorem is proved. Then (7.1) – (7.3) gives

$$H^q(V, D, \mathbf{Z}) = 0. \qquad (\forall q < d). \tag{7.4}$$

Now we have the exact sequence

$$\ldots \to H^q(V,D,\mathbf{Z}) \to H^q(V,\mathbf{Z}) \to H^q(D,\mathbf{Z}) \to H^{q+1}(V,D,\mathbf{Z}) \to \ldots,$$

and using (7.4) we conclude that the mapping

$$H^q(V,\mathbf{Z}) \to H^q(D,\mathbf{Z})$$

is an isomorphism onto when $q < d-1$ and an injection when $q = d-1$.

This proves Lefschetz' theorem.

The proof of Theorem 7.2 is based on *Morse theory*. Let X be a C^∞-manifold with countable base. If f is a real-valued C^∞-function on X, then a point $a \in X$ is called *critical* for f if $df(a) = 0$. A critical point a is *non-degenerate*, if in local coordinates $f(x) - f(a) = \Sigma a_{ij}(x_i - a_i)(x_j - a_j) + o(|x-a|^2)$, where the symmetric matrix (a_{ij}) is non-singular. It is non-degenerate of index r if (a_{ij}) has r eigenvalues < 0. The non-degenerate critical points for f are necessarily isolated. We now quote some facts from Morse theory; for proofs, see [6].

Lemma 7.3. Suppose that $f \in C^\infty(X)$, $f \geqslant 0$, $\alpha < \beta$, and that $X_\beta = \{x \in X; f(x) \leqslant \beta\}$ is compact.

(a) If f has no critical points in $\{x \in X\colon \alpha \leqslant f(x) \leqslant \beta\}$, then X_α is a deformation retract of X_β, and hence

$$H_r(X_\beta, X_\alpha, \mathbf{Z}) = 0, \qquad (\forall r \geqq 0).$$

(b) If all critical points of f in $\{x \in X;\ \alpha \leqslant f(x) \leqslant \beta\}$ are non-degenerate of index $\leqslant d$, then

$$H_r(X_\beta, X_\alpha, \mathbf{Z}) = 0, \qquad (\forall r > d).$$

In particular, if all critical points of f in X_β are non-degenerate of index $\leqslant d$, then

$$H_r(X_\beta, \mathbf{Z}) = 0, \qquad (\forall r > d).$$

In the proof of Theorem 7.2 we shall also use the following lemma of Morse:

Lemma 7.4. Let X be a C^∞-manifold with countable base. Then every real function $g \in C^\infty(X)$ can be approximated in the topology of $C^\infty(X)$ by real functions $f \in C^\infty(X)$, whose critical points are all non-degenerate.

The topology of $C^\infty(X)$ is the topology of uniform convergence of all derivatives on compact sets. Therefore the lemma explicitly means the following:

Let $\varepsilon > 0$, an integer $r \geqslant 0$ and a compact set $K \subset X$ be given, and let $K = K_1 \cup \ldots \cup K_k$, where each K_j is compact and contained in an open set U_j, where we have a coordinate system. Then there is a function f of the prescribed type such that

$$\sup_{j} \sup_{|\alpha| \leq r} \sup_{x \in K_j} |D^\alpha f(x) - D^\alpha g(x)| < \varepsilon .$$

(Here D^α means a derivative of order $|\alpha|$ in the coordinates on U_j.)

To prove Lemma 7.4 we shall use a Lemma of Sard (see [8, Ch. I,§3, Th. 4]):

Lemma 7.5. Let Ω be an open subset of $\boldsymbol{R}^n$ and $f: \Omega \to \boldsymbol{R}^n$ a C^1-mapping. Let A be the critical set of f, i.e. the set of $a \in \Omega$ where $\det(\partial f_i(a)/\partial x_j) = 0$. Then $f(A)$ has Lebesgue measure 0 in $\boldsymbol{R}^n$. In particular, $f(A)$ is nowhere dense in $\mathbf{R}^n$.

Proof of Lemma 7.4. Suppose first that X is an open subset Ω of $\mathbf{R}^n$. If $g \in C^\infty(\Omega)$ is realvalued, consider the mapping

$$\varphi : \Omega \ni x \to (\partial g/\partial x_1, \ldots, \partial g/\partial x_n) \in \mathbf{R}^n .$$

The critical set A of φ is the set in Ω where

$$\det(\partial^2 g/\partial x_i\, \partial x_j) = 0 .$$

The lemma of Sard, applied to φ, shows that there are arbitrarily small $\varepsilon_1, \ldots, \varepsilon_n \in \mathbf{R}$ such that $(\varepsilon_1, \ldots, \varepsilon_n) \notin \varphi(A)$. Put

$$f(x) = g(x) - \varepsilon_1 x_1 - \ldots - \varepsilon_n x_n .$$

A point $x \in \Omega$ is a critical point of f if and only if $\partial g/\partial x_j = \varepsilon_j$, $(j=1, \ldots, n)$.

At such points $\varphi(x) = (\varepsilon_1, \ldots, \varepsilon_n) \in \varphi(A)$ and hence $\det(\partial^2 g/\partial x_i\, \partial x_j) \neq 0$. Hence all critical points of f are non-degenerate.

Since $\varepsilon_1, \ldots, \varepsilon_n$ can be chosen arbitrarily small, the lemma is proved in the case $X = \Omega$.

The general case now follows by a category argument. From the special case we conclude that we can cover X by denumerably many relatively compact open subsets U_j of X, such that $\mathcal{U}_j$ is dense in the space of real C^∞-functions, where $\mathcal{U}_j$ denotes the set of real C^∞-functions, whose critical points in $\overline{U}_j$ are all non-degenerate. It is also easy to see that every $\mathcal{U}_j$ is open in the space of real C^∞-functions. Since this space is a real Fréchet space, we can therefore use Baire's theorem to conclude that the set of all real C^∞-functions, whose critical points in X are all non-degenerate, i.e. $\cap\, \mathcal{U}_j$, is dense. This proves the lemma of Morse.

Proof of Theorem 7.2. Let X be a Stein manifold of dimension d, and let K be a compact subset of X such that

$$K = \{x \in X;\ |f(x)| \leq \|f\|_K, \quad \forall f \text{ holomorphic on } X\}.$$

(Since X is a Stein manifold, every compact subset of X is contained in some K of this kind.) Choose an open set U such that $K \subset U \subset\subset X$. For every $a \in \partial U$ we can find a holomorphic function f on X such that $|f(x)| \geqslant 1$ in a neighbourhood of a and $\|f\|_K < 1$. Since ∂U is compact, we can therefore choose holomorphic functions $f_1, \ldots, f_k$ on X such that

$$\max |f_j(a)| \geq 1, \quad (\forall a \in \partial U),$$

and

$$\|f_j\|_K < 1, \quad (\forall j).$$

By replacing each f_j by a sufficiently high power, we can also arrange that the function

$$p(x) = \Sigma\, |f_j(x)|^2$$

satisfies $p(x) < 1$ on K and $p(x) \geqslant 1$ on ∂U. We can also assume that the rank of $(f_1, \ldots, f_k)$ is maximal at all points of U.

Now $p \in C^\infty(X)$, $p \geqslant 0$, and $U_\beta = \{x \in U; p(x) \leqslant \beta\}$ is compact and contains K if $\beta < 1$ is chosen so that $p(x) < \beta$ in K. By calculating the Levi form and using the maximality of the rank of $(f_1, \ldots, f_k)$, we see that p is strongly plurisubharmonic.

Because of Morse's lemma we can also assume that all critical points of p in U_β are non-degenerate. We shall prove that they are all of index $\leqslant d$.

We expand p at a critical point $a \in U_\beta$ in a local coordinate system:

$$\begin{aligned} p(x) &= p(a) + 2\mathrm{Re} \sum \frac{\partial^2 p(a)}{\partial z_i\, \partial z_j}(z_i - a_i)(z_j - a_j) \\ &\quad + \sum \frac{\partial^2 p(a)}{\partial z_i\, \partial \bar{z}_j}(z_i - a_i)(\bar{z}_j - \bar{a}_j) + \ldots \\ &= p(a) + \mathrm{Re}\, Q(z-a) + L(z-a) + \ldots . \end{aligned}$$

Here $L(z-a)$ is the Levi form of p at the point a. Now, since p is strongly plurisubharmonic, we can choose the coordinates so that $L(z-a) = |z-a|^2$. Then we see that if ζ is an eigenvector corresponding to an eigenvalue < 0 of the symmetric matrix of the real quadratic form Re $Q(z) + L(z)$, then $i\zeta$ is an eigenvector corresponding to an eigenvalue > 0. Hence the number of negative eigenvalues is $\leqslant d$, since the real dimension of X is $2d$. Thus the index of the critical point a is $\leqslant d$.

Now using Lemma 7.3 (b), we see that

$$H_r(U_\beta, \mathbf{Z}) = 0, \qquad (\forall r > d).$$

From this it follows that

$$H_r(X, \mathbf{Z}) = 0, \qquad (\forall r > d),$$

because the singular cycles defining the homology groups $H_r(X, \mathbf{Z})$ have compact supports, and any compact subset of X is contained in some compact set K with a corresponding $U_\beta \supset K$.

A refinement of the above argument leads to the stronger (homotopy) statement:

Any Stein manifold of (complex) dimension d has the same homotopy type as a CW complex of (real) dimension $\leqslant d$. (See [6]).

Moreover, the Lefschetz theorem has an analogue in homology and in homotopy [6]. The latter, for example, asserts that, if V, D are as in Th. 7.1, then the relative homotopy groups $\pi_q(V, D) = 0$ for $q < d$.

Th. 7.2 has been generalised in various directions. It has a relative analogue (relative to a Runge domain). Further, Th. 7.2 remains true if X is any Stein space (with singularities) of complex dimension d, but the corresponding cohomology statement is proved only for some other coefficient groups [5, 7]. Note that in view of the use of Poincaré duality, this does not lead to a Lefschetz theorem for algebraic varieties with singularities.

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