§4. Algebraic study of flatness

PDF erstellt am: **22.09.2024**

The set of " nice " fibers is dense in X, so we cannot remove the z-axis and still get a closed subspace of $\mathbf{C}_3$.

§ 4.   *Algebraic study of flatness*

In the following all rings are commutative, with 1, and all modules are unitary.

Definition 1 : An A-module E is *flat,* if for every exact sequence of A-modules

$$0 \rightarrow F' \rightarrow F \rightarrow F'' \rightarrow 0 \,,$$

the sequence $0 \rightarrow E \otimes F' \rightarrow E \otimes F \rightarrow E \otimes F'' \rightarrow 0$ is also exact. We can also say, because $\otimes$ is right exact, that E is flat, if for every injective homomorphism $F' \rightarrow F$, $E \otimes F' \rightarrow E \otimes F$ is also injective.

Examples of modules which are not flat :

(1) if $A = \mathbf{Z}$, $E = \mathbf{Z}_2 = \mathbf{Z}/2\,\mathbf{Z}$, $F = F' = \mathbf{Z}$; then the sequence $0 \rightarrow \mathbf{Z} \xrightarrow{2I} \mathbf{Z}$ $(2I : x' \rightarrow 2x)$ is exact. But now $\mathbf{Z}_2 \otimes \mathbf{Z} = \mathbf{Z}_2$, and the homomorphism $\mathbf{Z}_2 \xrightarrow{2I} \mathbf{Z}_2$ is the zero homomorphism, which is not injective. So $\mathbf{Z}_2$ is not a flat $\mathbf{Z}$ module.

(2) If $A = \mathbf{C}\{x\}$, $E = \mathbf{C} = \mathbf{C}\{x\}/(x)$, $F = F' = \mathbf{C}\{x\}$, then the sequence $0 \rightarrow F \xrightarrow{xI} F'$ $(xI : p(x) \rightarrow xp(x))$ is exact. But the homomorphism $E \xrightarrow{xI} E$ is not injective.

Proposition 1 : If A is an integral domain and E a flat A-module, then E is torsion-free.

Proof : Let $a \in A$, $a \neq 0$. Because A is an integral domain, the sequence $0 \rightarrow A A \xrightarrow{aI}$ $(aI : x \rightarrow ax)$ is exact. Since E is flat, the sequence $0 \rightarrow E \xrightarrow{aI} E$ is also exact. In other words E has no torsion elements.

Proposition 2 : If A is a principal-ideal domain, then E is flat if and only if E is torsionfree.

Proof : See corollary of prop. 6.

Examples of flat modules :

(1) The inductive limit of flat modules is flat, because the inductive limit preserves exactness, and it commutes with the tensor product.

(2) Every free module is flat. In fact, if E is free and finite type, then $E = A^n$ and $E \otimes F = F^n$. If $F' \to F$ is injective, so is $F'^n \to F^n$ too.

If E is an arbitrary free module, then it is an inductive limit of free modules of finite type, and the flatness of E follows from (1).

(3) Let S be a multiplicative system in A. Then the ring of fractions $S^{-1}A$ is a flat A-module. In fact the ring $S^{-1}A$ can be identified with an inductive limit of free modules, so it is flat ((1) (2)). We assume for simplicity that S has only regular elements. We can define in the set S a partial order in the following way:

$$s' \geq s \Leftrightarrow \exists\, t \in A, \quad ts = s' \quad \text{(such a } t \text{ is then unique).}$$

Let $E_s = A$ for every $s \in S$, and if $s' \geq s$ (i.e. $s' = ts$) then let $f_s^{s'}$ be the homomorphism $t . I_A : E_s \to E_{s'}$. The family $(E_s)_{s \in S}$ with the homomorphisms $(f_s^{s'})$ is an inductive system.

Let $E = \lim\limits_{\to} E_s$ be the inductive limit of this system, and φ_s the canonical homomorphism $E_s \to E$. We shall define an isomorphism $\psi : E \to S^{-1}A$.

We first define for every s a homomorphism $\psi_s : E_s = A \to S^{-1}A$; $x \to x/s$. Now if $s' \geq s$, then

$$(\psi_{s'} \circ f_s^{s'})(x) = \psi_{s'}(tx) = \frac{tx}{s'} = \frac{tx}{ts} = \frac{x}{s} = \psi_s(x) .$$

Therefore there exists a homomorphism $\psi : E \to S^{-1}A$, satisfying $\psi_s = \psi \circ \varphi_s$ for every $s \in S$.

Because every element of $S^{-1}A$ has the form a/s, ψ is surjective. On the other hand if $\psi(\phi_s(x)) = 0$, then $\psi_s(x) = x/s = 0$. Thus $x = 0$, and ψ is also injective.

The above proof can be extended to the general case, not assuming that the elements of S are regular. The extended proof involves the notion of inductive limit of an inductive system indexed by a category instead of an ordered set.

From (1) and (2) above, any module which is the inductive limit of free modules, is flat. Conversely:

Theorem 1 : (Daniel, Lazard)

Any flat module is a inductive limit of free modules.
For the proof: See *C.R. Acad. Sci. Paris,* 258 (1964), pp. 6313-6316.

Some elementary properties of flat modules :

(1) If E and F are flat A-modules, then $E \otimes F$ is also flat. In fact, if $G' \to G$ (over A) is injective, then $F \otimes G' \to F \otimes G$ is injective, and also $E \otimes (F \otimes G') \to$ $\to E \otimes (F \otimes G)$ is injective. The result follows from the assosiativity of the tensor product.

(2) Let $\phi : A \to B$ be a ring homomorphism, and E a flat A-module. The module $B \underset{A}{\otimes} E$ is a flat B-module.

If F is a B-module, then $F \underset{B}{\otimes} (B \underset{A}{\otimes} E) = (F \underset{B}{\otimes} B) \underset{A}{\otimes} E = F \underset{A}{\otimes} E$ further if F' and F are B-modules, and $F' \to F$ an injective homomorphism of B-modules, we can consider this homomorphism as an injective homomorphism of A-modules. Because E is A-flat,

$$F' \otimes_A E \to F \otimes_A E \text{ is injective.}$$

(3) Let $\phi : A \to B$ be a ring homomorphism, such that B is a flat A-module. If F is a flat B-module, then F is a flat A-module. In fact : if $E' \to E$ is injective, then $E' \otimes B \to E \otimes B$ (over A, B) is injective, and also $(E \otimes B) \otimes F' \to (E \otimes B) \otimes F$ (over A, B, A, B) is injective. But $(E' \underset{A}{\otimes} B)_B \otimes F' = E' \underset{A}{\otimes} F$; $(E \underset{A}{\otimes} B) \underset{B}{\otimes} F = E \underset{A}{\otimes} F$.

If an A-module E is not flat, we want to measure how far it is from being flat. For this purpose we introduce the functor Tor.

Definition 2 : A free resolution of E is an exact sequence : $\dots \to L_n \to L_{n-1} \to$ $\to \dots \to L_1 \to L_0 \to E \to 0$, where all L_i are free A-modules.

The complex of the resolution is the sequence

$$(L.) \dots \to L_n \to L_{n-1} \to \dots \to L_1 \to L_0 \to 0 .$$

Every module has a free resolution. Two resolutions are algebraically homotopy-equivalent. Forming the tensorproducts $L_i \otimes F$, we get

$$(L. \otimes F) \dots \to L_n \otimes F \to L_{n-1} \otimes F \to \dots \to L_1 \otimes F \to L_0 \otimes F \to 0 .$$

Definition 3 :

$$\text{Tor}_n^A (E, F) = H_n (L. \otimes F) = \frac{\text{Ker}(L_n \otimes F \to L_{n-1} \otimes F)}{\text{Im}(L_{n+1} \otimes F \to L_n \otimes F)}$$

if $n \geq 1$, and $\text{Tor}_0^A (E, F) = \text{Coker}(L_1 \otimes F \to L_0 \otimes F) = E \otimes F.$

Basic properties of Tor :

(1) $\text{Tor}_n (E, F)$ is independent of the choice of the resolution (up to a canonical isomorphism).

(2) If we take a free resolution of F, we get $\mathrm{Tor}_n\,(F, E) = \mathrm{Tor}_n\,(E, F)$ (Symmetry of the Tor). We can also define $\mathrm{Tor}_n\,(E, F)$ by taking two free resolutions, one of E and one of F.

(3) If $0 \to E' \to E \to E'' \to 0$ is a short exact sequence, then we get a long exact sequence:

$$\mathrm{Tor}_n\,(E', F) \to \mathrm{Tor}_n\,(E, F) \to \mathrm{Tor}_n\,(E'', F) \qquad \to$$
$$\to \mathrm{Tor}_{n-1}\,(E', F) \to \mathrm{Tor}_{n-1}\,(E, F) \to \mathrm{Tor}_{n-1}\,(E'', F) \to$$
$$\to \text{—} \text{—} \text{—} \text{—} \text{—} \text{—} \text{—} \text{—} \text{—} \to$$
$$\to \mathrm{Tor}_1\,(E', F) \to \mathrm{Tor}_1\,(E, F) \to \mathrm{Tor}_1\,(E'', F) \qquad \to$$
$$\to E' \otimes F \to E \otimes F \to E'' \otimes F \to 0 \ .$$

(4) Tor is compatible with inductive limit, i.e. if $E = \lim_{\to} (E_i)$, then

$$\mathrm{Tor}_n\,(\lim_{\to} E_i, F) = \lim_{\to} (\mathrm{Tor}_n\,(E_i, F)).$$

(5) We can define $\mathrm{Tor}_n\,(E, F)$ by taking a flat resolution of E.

Proposition 3: Let E be an A-module. Then the following conditions are equivalent:

(a) E is flat.

(b) For all A-modules F, and for all $n \geqq 1$, $\mathrm{Tor}_n\,(E, F) = 0$.

(c) For all A-modules F, $\mathrm{Tor}_1\,(E, F) = 0$.

Proof: $(a) \Rightarrow (b)$. If $\ldots \to L_n \to L_{n-1} \to \ldots \to L_1 \to L_0 \to F \to 0$ is a free resolution of F, then the sequence

$$\ldots \to E \otimes L_n \to E \otimes L_{n-1} \to \ldots \to E \otimes L_1 \to E \otimes L_0 \to E \otimes F \to 0$$

is exact, thus $\mathrm{Tor}_n\,(E, F) = 0$ for all $n \geqq 1$.

$(b) \Rightarrow (c)$ clear. $(c) \Rightarrow (a)$: If the sequence $0 \to F' \to F \to F'' \to 0$ is exact, so is also (by (3) above) $\mathrm{Tor}_1\,(E, F'') \to E \otimes F' \to E \otimes F \to E \otimes F'' \to 0$. Now $\mathrm{Tor}_1\,(E, F'') = 0$, thus E is flat.

Proposition 4: If I and J are two ideals in A, then $\mathrm{Tor}_1^A\,(A/I, A/J) = I \cap J/I \, . \, J$.

Proof: From the exact sequence $0 \to I \to A \to A/I \to 0$, we get the exact sequence:

$$\mathrm{Tor}_1\,(A, A/J) \to \mathrm{Tor}_1\,(A/I, A/J) \to I \otimes A/J \to A \otimes A/J \to A/I \otimes A/J \to 0 \ .$$

But now $\mathrm{Tor}_1\,(A, A/J) = 0$ (A beeing A-free), and $I \otimes A/J = I/I \, . \, J$; $A \otimes A/J = A/J$. Therefore the sequence $0 \to \mathrm{Tor}_1\,(A/I, A/J) \to I/I \, . \, J \to A/J$ is exact, and $\mathrm{Tor}_1\,(A/I, A/J) = \mathrm{Ker}\,(I/I \, . \, J \to A/J) = I \cap J/I \, . \, J$.

Example : Let U be an open set in $\mathbf{C}^n$, and $x \in U$. Further let $X, Y \subset U$ be two hypersurfaces, defined by $I = (f)$ and $J = (g)$. Supposing that f and g do not have common factors: $I_x \cap J_x = I_x J_x$, and

$$\mathrm{Tor}_1 \left(\mathscr{O}_{X,x}, \mathscr{O}_{Y,x} \right) = \mathrm{Tor}_1 \left(\mathscr{O}_{U,x}/I_x, \quad \mathscr{O}_{U,x}/J_x \right) = \frac{I_x \cap J_x}{I_x \cdot J_x} = 0 \, .$$

Heuristic remark : The formula $\mathrm{Tor}_1 \left(\mathscr{O}_{X,x}, \mathscr{O}_{Y,x} \right) = 0$ expresses the fact that X and Y are " in general position ". If for example X and Y are two linears subspaces in $\mathbf{C}^n$ of dimensions p and q, we have $\mathrm{Tor}_1 \left(\mathscr{O}_{X,x}, \mathscr{O}_{Y,x} \right) = 0$ if $\dim (X \cap Y) = p+q-n$, and $\mathrm{Tor} \left(\mathscr{O}_{X,x}, \mathscr{O}_{Y,x} \right) \neq 0$ otherwise.

Next we shall prove an elementary flatness criterion.

Proposition 5 : Let E be an A-module. The following conditions are equivalent:

(a) E is flat.

(b) For all finitely generated ideals I of A, $\mathrm{Tor}_1 (E, A/I) = 0$.

(c) For all monogenous A-modules F, $\mathrm{Tor}_1 (E, F) = 0$.

Proof : $(a) \Rightarrow (b)$, by prop. 3.

$(b) \Rightarrow (c)$: Because Tor is compatible with inductive limit, we can suppose, that $\mathrm{Tor}_1 (E, A/I) = 0$ for an arbitrary ideal I of A. But every monogenous A-module F can be represented by A/I.

$(c) \Rightarrow (a)$. By prop. 3 it is sufficient to prove that $\mathrm{Tor}_1 (E, F) = 0$ for any A-module F.

First consider the case, where F is finitely generated. We use induction, supposing that $\mathrm{Tor}_1 (E, F) = 0$, when F has n generators. Let F have $(n+1)$ generators $x_1, \ldots, x_n, x_{n+1}$. If F' is the submodule generated by $\{x_1, \ldots, x_n\}$, then $F' \subset F$ and $F'' = F/F'$ is monogenous. The exact sequence $0 \to F' \to F \to F'' \to 0$ gives the exact sequence $\mathrm{Tor}_1 (E, F') \to \mathrm{Tor}_1 (E, F) \to \mathrm{Tor}_1 (E, F'')$. Now $\mathrm{Tor}_1 (E, F') = \mathrm{Tor}_1 (E, F'') = 0$, thus $\mathrm{Tor}_1 (E, F) = 0$. In the general case, F can be considered as an inductive limit of finitely generated modules, and because Tor is compatible with inductive limits, $\mathrm{Tor}_1 (E, F) = 0$.

Proposition 6 : Let A be an integral domain, and E an A-module. Then E is torsionfree if and only if $\mathrm{Tor}_1 (E, A/(a)) = 0$, for any element $a \in A$.

Proof : If E is A-module, $a \in A$, then the exact sequence $0 \to A \xrightarrow{aI} A \to A/(a) \to 0$ gives the exact sequence $0 \to \mathrm{Tor}_1 (E, A/(a)) \to E \xrightarrow{aI} E$. In other words $\mathrm{Tor}_1 (E, A/(a)) = \{x \in E \mid ax = 0\}$, from which the result follows.

Corollary : Let A be a principal ideal domain. E is flat if and only if E is torsionfree.

Proof : We have already proved that, if E is flat, then it is torsion free. The converse follows from prop. 6 and prop. 5.

The first flatness criterion for noetherian local rings is the following:

Theorem 2 : Let A be a noetherian local ring with maximal ideal m; $k = A/m$, and E a finitely generated A-module. The following conditions are equivalent:

(a) E is free.

(b) E is flat.

(c) $\mathrm{Tor}_1^A (E, k) = 0.$

Proof : We have already proved $(a) \Rightarrow (b) \Rightarrow (c)$.

$(c) \Rightarrow (a)$: We recall first Nakayma's lemma. If A is a local ring with maximal ideal m; $k = A/m$, and E is a finitely generated A-module, such that $k \underset{A}{\otimes} E = E/mE = 0$, then $E = 0$.

The module $\bar{E} = k \underset{A}{\otimes} E = E/mE$ is a finitely generated vector space over k. Let $\{\bar{x}_1. ..., \bar{x}_r\}$ be a base of $\bar{E}$ (over k), and $\{x_1, ..., x_r\}$ E representatives of $\bar{x}_i : s$. Consider the homomorphism $\phi : A^r \to E$, $\phi (a_1, ..., a_r) = \sum a_i x_i$. Denoting by R and Q the kernel and the cokernel of ϕ, we get an exact sequence:

(*) $0 \to R \to A^r \to E \to Q \to 0$

and R, Q are finitely generated A-modules. From (*) we get the exact sequence

$$A^r \underset{A}{\otimes} k \to E \underset{A}{\otimes} k \to Q \underset{A}{\otimes} k \to 0 .$$

But $\bar{E} = E \underset{A}{\otimes} k \simeq k^r = A^r \underset{A}{\otimes} k$, so $Q \underset{A}{\otimes} k = 0$, and by Nakayama's lemma $Q = 0$.

Therefore *ge* have an exact sequence

$$0 \to R \to A^r \to E \to 0 .$$

From this we get: $\mathrm{Tor}_1 (E, k) \to k \underset{A}{\otimes} R \to k^r \to \bar{E} \to 0$ (exact). Now: $\bar{E} \simeq k^r$, $\mathrm{Tor}_1 (E, k) = 0$ (by assumption). Therefore $k \underset{A}{\otimes} R = 0$, and once more by Nakayama's lemma $R = 0$, thus $E \simeq A^r$, i.e. E is free.

Proposition 7: Let $\phi : A \to B$ be a ring homomorphism, and let B be A-flat. If I is an ideal of A, we write $\bar{A} = A/I$, $\bar{B} = B/IB = \bar{A} \underset{A}{\otimes} B$. Let F be a B-module, then: $\operatorname{Tor}_i^A(\bar{A}, F) = \operatorname{Tor}_i^B(\bar{B}, F)$ $(i \geqq 0)$.

Proof: We choose first a B-free resolution of F

$$\to L_{n+1} \to L_n \to \ldots \to L_1 \to L_0 \to F \to 0 .$$

If $L.$ is the respective complex of resolution, then

$$\bar{B} \underset{B}{\otimes} L. = B/IB \underset{B}{\otimes} L. = \bar{A} \underset{A}{\otimes} (B \underset{B}{\otimes} L.) = \bar{A} \underset{A}{\otimes} L.$$

Because every L_i is B-free, and B is A-flat, every L_i is A-flat (Property 3 after Th. 1). Thus $L.$ is a flat A-resolution, and

$$\operatorname{Tor}_i^A(\bar{A}, F) = H_i(\bar{A} \underset{A}{\otimes} L.) = H_i(\bar{B} \underset{B}{\otimes} L.) = \operatorname{Tor}_i^B(\bar{B}, F) .$$

We shall next state the second flatness criterion for noetherian local rings.

Theorem 3: Let A and B be two noetherian local rings, with maximal ideals $\underline{m}, \underline{n}$; $k = A/\underline{m}$. If $\phi : A \to B$ is a local homomorphism (i.e. $\phi(\underline{m}) \subset \underline{n}$), and F finitely generated B module then

$$F \text{ is A-flat} \Leftrightarrow \operatorname{Tor}_1^A(k, F) = 0 .$$

The proof of this theorem is much more difficult than that of th. 20 see for example:

Bourbaki: *Algèbre commutative*, Chapter III § 5, th1, $(i) \Leftrightarrow (iii)$, p. 98.

The conditions in Bourbaki's theorem are here fullfilled:

1° A finitely generated module F over a noetherian local ring B is idealwise separated for $\underline{n}$. (*Ibid.*, § 5. 1. Ex. 1, p. 97.)

2° If $\phi : A \to B$ is a local homomorphism, F is also idealwise separated for $\underline{m}$. (*Ibid.*, § 5, prop. 2, p. 101.)

3° Also the flatness condition is fulfilled, because k is a field.

Remark: The main interest of the theorem lies in the fact, that it is true without any assumption of finitness on B.

Corollary: If the assumptions are the same as in the theorem 3, and if moreover B is A-flat, then

$$F \text{ is A-flat} \Leftrightarrow \operatorname{Tor}_1^B(\bar{B}, F) = 0 ,$$

where $\bar{B} = B/mB$.

Proof: $\mathrm{Tor}_1^A (k, F) = \mathrm{Tor}_1^B (\bar{B}, F)$, by prop. 7.

§ 5. Geometric applications of the flatness criterions

A) Flatness for finite morphisms

Proposition 1: Let $\pi: X \to S$ be a finite morphism (i.e. proper with finite fibres) of analytic spaces. Then $\pi_* (\mathcal{O}_X)$ is a coherent analytic sheaf over S. The following conditions are equivalent:

(a) π is flat (i.e. for every $x \in X$, $\mathcal{O}_{X,x}$ is a flat $\mathcal{O}_{S,s}$-module, $s = \pi(x)$).

(b) For every s, $(\pi_* \mathcal{O}_X)_s$ is a flat $\mathcal{O}_{S,s}$-module.

(c) $\pi_* \mathcal{O}_X$ is a locally free sheaf.

Proof: Because π is finite $\pi_* (\mathcal{O}_X)_s = \bigoplus_{x \varepsilon \pi^{-1}(s)} \mathcal{O}_{X,x}$, thus the only point to prove is $(b) \Rightarrow (c)$.

Now if $\mathcal{O}_{X,x}$ is a flat $\mathcal{O}_{S,s}$-module, then (by theorem 2) $\mathcal{O}_{X,x}$ is free, and a coherent sheaf whose fibers are free is a locally free sheaf.

Proposition 2: Let S be a reduced analytic space and $\mathcal{E}$ a coherent $\mathcal{O}_s$-module. Let $E(s)$ be the finite dimensional vector space (over $\mathbf{C}$) $\mathcal{E}_s \otimes_{\mathcal{O}_{S,s}} \mathbf{C}_s$. $\mathcal{E}$ is a locally free $\mathcal{O}_{S,s}$-module if an only if $\dim_{\mathbf{C}} E(s)$ is locally constant.

Proof: If $\mathcal{E}$ is locally free, then $\dim_{\mathbf{C}} E(s)$ is locally constant. Suppose now that $\dim_{\mathbf{C}} E(s)$ is locally constant in an open set $U \subset S$, and that $\mathcal{O}_U^p \xrightarrow{d} \mathcal{O}_U^q \to \mathcal{E}_U \to 0$ is exact. d is determined by a $p \times q$ matrix of analytic functions on U, so it gives a morphism $\mathbf{C}_U^p \xrightarrow{d} \mathbf{C}_U^q$ of trivial vector bundles over U.

From the exact sequence $\mathcal{O}_s^p \xrightarrow{d_s} \mathcal{O}_s^q \to \mathcal{E}_s \to 0$, we get (by making tensor-products with $\mathbf{C}_s$) the exact sequence:

$$\mathbf{C}_s^p \xrightarrow{d(s)} \mathbf{C}_s^q \to E(s) \to 0,$$

which shows that d has constant rank in U. Thus Ker d and Im d are vector bundles, and we can write

$$\mathbf{C}_U^p = F_1 \oplus G_1, \quad \mathbf{C}_U^q = F_0 \oplus G_0,$$

$$d: \begin{cases} F_1 \to 0 \\ G_1 \simeq F_0 . \end{cases}$$