MEROMORPHIC MAPPINGS

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MEROMORPHIC MAPPINGS

by K. STEIN

INTRODUCTION

We study meromorphic mappings of complex spaces. The notion of meromorphic mapping we use was introduced by Remmert [9], [11][1]. Some part of the material dealt with in these lectures is contained in [15] and we shall therefore not give proofs for all statements.

The first sections are preliminary. The concept of correspondence is discussed and used to define meromorphic mappings (these are not mappings in the usual sense). Extension problems are studied in Section 4. Essential use is made of the extension theorem for analytic sets first proved by Thullen [21] in a special case and later generalized by Remmert and Stein [13]. The final section deals with maximal meromorphic mappings.

1. CORRESPONDENCES

Let X and Y be sets. A *correspondence*, denoted $f : X \underset{k}{\rightarrow} Y$, assigns to each $x \in X$ a subset $f(x) \subset Y$, which may be empty. $f : X \underset{k}{\rightarrow} Y$ is called empty if $f(x) = \varnothing$ for all $x \in X$. For $A \subset X$ we set $f(A) = \bigcup_{x \in A} f(x)$. A mapping $\varphi : X \rightarrow Y$ is looked upon as a special correspondence (we do not distinguish between a set consisting of one element and the element).

Each correspondence $f : X \underset{k}{\rightarrow} Y$ can be characterized by its *graph* $G_f = \{ (x, y) \mid x \in X, y \in f(x) \} \subset X \times Y$. The projection maps of G_f into X and Y are denoted by $\check{f} : G_f \rightarrow X$ and $\hat{f} : G_f \rightarrow Y$. Then, we have $f(x) = \hat{f}\left(\check{f}^{-1}(x)\right)$. If $f : X \underset{k}{\rightarrow} Y$, $f' : X \underset{k}{\rightarrow} Y$ are correspondences, we say that f is contained in f' if $G_f \subset G_{f'}$. For a subset $A \subset X$ we define the *restriction*

1) Another notion of meromorphic mapping and related concepts were defined by W. Stoll [16], [17].

$f|A : A \underset{k}{\rightarrow} Y$ by setting $G_{f|A} = G_f \cap (A \times Y)$. To every correspondence $f : X \underset{k}{\rightarrow} Y$ there is associated the *inverse correspondence* $f^{-1} : Y \underset{k}{\rightarrow} X$ whose graph is $G_{f^{-1}} = \{(y, x) \mid (x, y) \in G_f\}$. We have the rule $(f^{-1})^{-1} = f$. The *Cartesian product* $f \times f_1 : X \times X_1 \underset{k}{\rightarrow} Y \times Y_1$ of two correspondences $f : X \underset{k}{\rightarrow} Y$ and $f_1 : X_1 \underset{k}{\rightarrow} Y_1$ assigns, by definition, to $(x, x_1) \in X \times X_1$ the set $f(x) \times f_1(x_1)$. If $X = X_1$, we define the *junction* $(f, f_1) : X \underset{k}{\rightarrow} Y \times Y_1$ by $(f, f_1)(x) = f(x) \times f_1(x)$. The *product* $g \circ f : X \underset{k}{\rightarrow} Z$ of the correspondences $f : X \underset{k}{\rightarrow} Y$ and $g : Y \underset{k}{\rightarrow} Z$ is defined by $g \circ f(x) = g(f(x))$. Hence, $(f, f_1) = (f \times f_1) \circ (I_X, I_X)$ where I_X is the identity mapping of X. We have the following rules: $(f \times f_1)^{-1} = f^{-1} \times f_1^{-1}$, $h \circ (g \circ f) = (h \circ g) \circ f$, $(g \circ f)^{-1} = f^{-1} \circ g^{-1}$.

Definition 1. Let X and Y be topological spaces. A correspondence $f : X \underset{k}{\rightarrow} Y$ is *continuous* at $x \in X$ if

1) $f(x)$ is quasicompact, and

2) given a neighborhood V of $f(x)$, there exists a neighborhood U of x such that $f(U) \subset V$.

The correspondence f is *continuous* if it is continuous at every $x \in X$.[1]

Proposition 1. Let $f : X \underset{k}{\rightarrow} Y$ be a correspondence such that $f(x)$ is quasicompact for all $x \in X$. Then f is continuous if and only if f^{-1} is closed (in the sense that the images of closed sets are closed).

Proof. Let f be continuous and let N be a closed set in Y. Assume that $f^{-1}(N)$ is not closed, then there is a point $x \in \overline{f^{-1}(N)} \cap (X - f^{-1}(N))$. We have $f(x) \in Y - N$ since $x \in X - f^{-1}(N)$, hence $Y - N$ is a neighborhood of $f(x)$. Because of the continuity of f there exists a neighborhood U of x such that $f(U) \subset Y - N$. It follows $U \cap f^{-1}(N) = \varnothing$, but this contradicts the assumption that $x \in \overline{f^{-1}(N)}$. Assume now that f^{-1} is closed. Let x be a point of X and V an open neighborhood of $f(x)$. Then $f^{-1}(Y - V)$ is closed and does not contain x, therefore $U = X - f^{-1}(Y - V)$ is a neighborhood of x, and we have $f(U) \subset V$. Hence f is continuous.

Remark. The statement of Proposition 1 becomes false if " closed " is replaced by " open " as can be seen by simple examples.

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1) This definition and some of the following developments are due to K. Wolffhardt [22].

Proposition 2. If $f: X \underset{k}{\rightarrow} Y$ is a continuous correspondence, $f(K)$ is quasicompact for every quasicompact set $K \subset X$.

Proof. Consider a covering of $f(K)$ by open sets V_i. For each $x \in K$ there are finitely many V_i which cover $f(x)$, let $\tilde{V}_x$ be the union of those V_i. f is continuous at x, hence there is a neighborhood U_x of x such that $f(U_x) \subset \tilde{V}_x$. Now finitely many U_x, say $U_{x1}, \ldots, U_{x_n}$, cover K, hence $\tilde{V}_{x1} \cup \ldots \cup \tilde{V}_{x_n} \supset f(K)$, therefore finitely many V_i cover $f(K)$.

Proposition 3. Let $f: X \underset{k}{\rightarrow} Y$, $f_1 : X_1 \underset{k}{\rightarrow} Y_1$, $f_1' : X \underset{k}{\rightarrow} Y_1$, $g : Y \underset{k}{\rightarrow} Z$ be continuous correspondences. Then $f \times f_1$, (f, f_1'), $g \circ f$ are continuous.

Proof. For $g \circ f$, the assertion follows by applying propositions 1 and 2, Furthermore, $f \times f_1 (x, x_1) = f(x) \times f_1 (x_1)$ is quasicompact for all $x \in X$. $x_1 \in X_1$ since $f(x)$ and $f_1(x_1)$ are quasicompact. Let V be a neighborhood of $f(x) \times f_1(x_1)$; V contains a neighborhood $W \times W_1$ of $f(x) \times f_1(x_1)$ where W, W_1 are neighborhoods of $f(x)$ resp. $f_1(x_1)$. There are neighborhoods U, U_1 of x resp. x_1 such that $f(U) \subset W$, $f_1(U_1) \subset W_1$, then $f \times f_1 (U \times U_1) \subset$ $\subset W \times W_1$; hence $f \times f_1$ is continuous. As for (f, f_1') one has $(f, f_1') =$ $(f \times f_1') \circ (I_X, I_X)$, therefore (f, f_1') is continuous because $f \times f_1'$ and (I_X, I_X) are continuous.

Proposition 4. A correspondence $f: X \underset{k}{\rightarrow} Y$ is continuous if and only if $\check{f}^{-1} : X \underset{k}{\rightarrow} G_f$ is continuous.

Proof. Since $f = \hat{f} \circ \check{f}^{-1}$, the continuity of $\check{f}^{-1}$ implies that of f by proposition 3. Let f be continuous and let x be a point of X. Since $\check{f}^{-1}(x)$ is homeomorphic to $f(x)$, it is quasicompact. Let $W \supset \check{f}^{-1}(x)$ be open in G_f. We can cover $\check{f}^{-1}(x)$ by a finite number of sets of the form $(U_i \times V_i) \cap$ $\cap G_f \subset W$, $U_i \ni x$ open in X and V_i open in Y. Then $V = \cup V_i \supset f(x)$ and there exists a neighborhood U' of x such that $f(U') \subset V$. If $U = (\cap U_i) \cap U'$, $\check{f}^{-1}(U) \subset W$.

It follows that a correspondence f is continuous if and only if the projection $\check{f}$ is a proper map, that is, $\check{f}$ is continuous, closed, and $\check{f}^{-1}(x)$ is always quasicompact.

Proposition 5. If $f: X \underset{k}{\to} Y$ is continuous and Y is a Hausdorff space, G_f is closed in $X \times Y$.

Definition 2. A correspondence f is *proper* if f and f^{-1} are continuous.[1]

Proposition 6. If $f: X \underset{k}{\to} Y$, $f_1: X_1 \underset{k}{\to} Y_1$, $g: Y \underset{k}{\to} Z$ are proper, then $f \times f_1$ and $g \circ f$ are proper.

The junction of two proper correspondences need not, however, be proper. The diagonal mapping (I_X, I_X) serves as an example if X is not a Hausdorff space. If X is Hausdorff, the junction (f, f') of proper correspondences $f: X \underset{k}{\to} Y$ and $f': X \underset{k}{\to} Y'$ remains proper.

Proposition 7. Let $f: X \underset{k}{\to} Y$, $f_1: X \underset{k}{\to} Y_1$, $g: Y \underset{k}{\to} Z$ be continuous where all the spaces are locally compact. Then we have:

1) If f is proper, then (f, f_1) and (f_1, f) are proper,

2) If $g \circ f$ is proper and g^{-1} surjective, then f is proper,

3) If $g \circ f$ is proper and f surjective, then g is proper.

2. Holomorphic correspondences

We consider reduced complex spaces (X, θ) where X is assumed Hausdorff and where the structure sheaf θ has no nilpotent elements. For the definition and related concepts we refer to [8]. The structure sheaf is usually omitted in the notation.

Definition 3. Let X and Y be complex spaces. A correspondence $f: X \underset{k}{\to} Y$ is called *holomorphic* if

1) f is continuous,

2) the graph G_f is an analytic set in $X \times Y$.

If only the condition 2) is fulfilled, f is said to be *weakly holomorphic.*

Let $f: X \underset{k}{\to} Y$ be weakly holomorphic. Then f^{-1} is weakly holomorphic; furthermore, if $A \subset X$ is analytic in X, $f \mid A$ is weakly holomorphic. Since $\check{f}^{-1}(x) = G_f \cap (\{x\} \times Y)$, $x \in X$, is analytic in G_f, $f(x) = \hat{f}(\check{f}^{-1}(x))$

1) Compare [3] where another notion of proper correspondence is defined.

is analytic in Y. If f is holomorphic and $A' \subset Y$ analytic in Y, then, since $\hat{f}^{-1}(A')$ is analytic in G_f and $\check{f}$ is proper, $f^{-1}(A') = \check{f}(\hat{f}^{-1}(A'))$ is analytic in X by Remmert's mapping theorem [11] (see also [8], p. 129).

The correspondences $f \times f_1$, (f, f'_1), and $g \circ f$ are holomorphic if the correspondences f, f_1, f'_1, and g are holomorphic.

A weakly holomorphic correspondence $f: X \underset{k}{\rightarrow} Y$ is called *reducible* resp. *irreducible* if G_f is reducible resp. irreductible. G_f is always a union of irreducible components $G^{(i)}$; let $f_i : X \underset{k}{\rightarrow} Y$ be the (weakly holomorphic) correspondence whose graph is $G^{(i)}$. Then the correspondences f_i are called the irreducible components of f and we write $f = \cup f_i$.

3. Meromorphic mappings

Let $f: X \underset{k}{\rightarrow} Y$ be a correspondence where X is a topological space A point $x \in X$ is called a *distinguished point of f* if there is a neighborhood U of x such that the restriction $f \mid U$ is a mapping (in the usual sense).

Definition 4. A holomorphic correspondence $f: X \underset{k}{\rightarrow} Y$ is called a *meromorphic mapping* if the following holds. If X is irreducible, then

1) f is irreducible,

2) There exists a distinguished point $x_0 \in X$ of f.

In the general case, if $X = \cup X^{(i)}$ is the decomposition of X into irreducible components, then there exist holomorphic correspondences $f_i : X \underset{k}{\rightarrow} Y$ such that

1) $f_i \mid X^{(i)}$ is a meromorphic mapping and $f_i \mid X - X^{(i)}$ is empty,

2) $f = \cup f_i$.

A meromorphic mapping f is *bimeromorphic* if f^{-1} is meromorphic.

We use the notation $f: X \underset{m}{\rightarrow} Y$ for a meromorphic mapping. Note that a meromorphic mapping is in general not a mapping in the strong sense.

An example of a meromorphic mapping is the correspondence f of $\mathbf{C}^2$ onto the extended complex plane $\mathbf{P}_1$ defined by $f(z_1, z_2) = \dfrac{z_1}{z_2}$ if $(z_1, z_2) \neq (0, 0)$, and $f(0, 0) = \mathbf{P}_1$.

Definition 5. A proper holomorphic mapping $\varphi : X' \to X$ is called *a proper modification map* if there exists an open subset $U \subset X$ such that

1) $U \cap X^{(i)} \neq \varnothing$ and $\varphi^{-1}(U) \cap X'^{(j)} \neq \varnothing$ for all irreducible components $X^{(i)} \subset X$ and $X'^{(j)} \subset X'$,

2) $\varphi^{-1} \mid U : U \underset{k}{\to} X'$ is a holomorphic mapping.

It follows that a correspondence f is a meromorphic mapping if an only if $\check{f}$ is a proper modification map.

A proper modification map $\varphi : X' \to X$ is always surjective. The inverse correspondence $\varphi^{-1} : X \underset{k}{\to} X'$ is always a meromorphic mapping.

A normalization $(\tilde{X}, \nu)$ of a complex space X is a normal complex space $\tilde{X}$ ([8], p. 114) and a proper modification map $\nu : \tilde{X} \to X$, such that all fibres $\nu^{-1}(x)$, $x \in X$, are finite. To every complex space X there exists a normalization (see [8]). Let X_1 and X_2 be complex spaces with normalizations $(\tilde{X}_1, \nu_1)$, $(\tilde{X}_2, \nu_2)$ where $\tilde{X}_1 = \tilde{X}_2$. Then it can easily be shown that $\nu_2 \circ \nu_1^{-1}$: $X_1 \underset{k}{\to} X_2$ is a bimeromorphic mapping.

Definition 6. Let f be a meromorphic mapping of X. A point $x_0 \in X$ is called *non-singular with respect to f* if there exists an open neighborhood U of x_0 such that $f \mid U$ is a holomorphic mapping. Otherwise x_0 is called *singular*. The set of singular points of f is denoted by $S(f)$.

The meromorphic mapping in the example on *p.* 5 has the origin as a singular point.

Proposition 8. Let f be a meromorphic mapping of X. Then

1) $S(f)$ is a nowhere dense analytic set in X,

2) If X is locally irreducible at x, $f(x)$ is connected,

3) If X is normal at x, then x is singular if and only if $\dim f(x) > 0$.

For the proof we refer to [15].

The set of singular points is of importance in connection with the compositions of meromorphic mappings. Let f: $X \underset{m}{\to} Y$, f_1: $X_1 \underset{m}{\to} Y_1$, f_1': $X \underset{m}{\to} Y_1$, $g : Y \underset{m}{\to} Z$ be meromorphic mappings where all the spaces are irreducible.[1] Then the correspondence $f \times f_1$ is easily seen to be meromorphic. The junc-

1) This restriction is introduced here for the sake of simplicity.

tion (f, f_1') need not, however, be a meromorphic mapping. Let $f = f_1'$ be the meromorphic mapping in the example on p. 5. Then the graph $G_{(f,f_1')} \subset \mathbf{C}^2 \times (\mathbf{P}_1 \times \mathbf{P}_1)$ is not irreducible. The product $g \circ f$ too, may be reducible; moreover, it may happen that there is no distinguished point of $g \circ f$.

We can always define a " meromorphic junction " in the following way. There are distinguished points of (f, f_1'), for example, all points of $X - (S(f) \cup S(f_1')) \neq \varnothing$. Now it can easily be shown: If a holomorphic correspondence from an irreducible complex space into a complex space has a distinguished point, then the graph of the correspondence has exactly one irreducible component which is the graph of a meromorphic mapping. It follows that there exists a unique meromorphic mapping contained in (f, f_1'); this meromorphic mapping is called the *meromorphic junction* of f and f_1' and denoted by $[f, f_1'] : X \underset{m}{\to} Y \times Y_1$. The meromorphic junction is associative: $[[f_1, f_2], f_3] = [f_1 [f_2, f_3]]$, hence the meromorphic junction $[f_1, ..., f_n] : X \underset{m}{\to} Y_1 \times ... \times Y_n$ of n meromorphic mappings $f_\nu : X \underset{m}{\to} Y_\nu$ is defined in a unique manner.

Furthermore we can define a " meromorphic product " of f and g if there is a distinguished point of $g \circ f$: There is then again a uniquely determined meromorphic mapping contained in $g \circ f$. This is called the *meromorphic product* of f and g and denoted by $g \triangle f : X \underset{m}{\to} Z$. A sufficient condition for the existence of a distinguished point of $g \circ f$ is that $f(X) \not\subset S(g)$. This condition is, in particular, fulfilled if f is surjective or if $S(g)$ is empty (i.e., if g is a holomorphic map; in this case we have $g \triangle f = g \circ f$). Note that the meromorphic product of bimeromorphic mappings always exists. The associative law $h \triangle (g \triangle f) = (h \triangle g) \triangle f$ holds if both sides exist.

As an example we consider the " meromorphic restriction " which is defined as follows. Let A be an irreducible analytic subset of X. Then the correspondence $f | A : A \underset{k}{\to} Y$ need not be irreducible. But if $A \not\subset S(f)$, we can form the meromorphic product $f \triangle I_X^A$ where $I_X^A : A \to X$ is the inclusion map. We set $f \underset{m}{|} A = f \triangle I_X^A : A \underset{m}{\to} Y$ and call $f \underset{m}{|} A$ the *meromorphic restriction* of f to A.

Proposition 9. Let $f : X \underset{m}{\to} Y$ and $g : Y \underset{m}{\to} Z$ be bimeromorphic. Then

1) $f^{-1} \triangle f = I_X$,

2) $g \triangle f$ is bimeromorphic and $(g \triangle f)^{-1} = f^{-1} \triangle g^{-1}$.

Proposition 10. Let $f : X \underset{m}{\to} Y$, $f'_1 : X \underset{m}{\to} Y_1$, $g : Y \underset{m}{\to} Z$ be meromorphic mappings, assume that $g \triangle f$ exists. Then we have:

1) If f is proper, $[f, f'_1]$ is proper,

2) If f and g are proper, $g \triangle f$ is proper,

3) If $g \triangle f$ is proper, f is proper,

4) If $g \triangle f$ is proper and f surjective, g is proper.

4. Extension of meromorphic mappings

We start with some classical results. Let D be a domain in $\mathbf{C}^n$ and $A \neq D$ an irreducible analytic set in D. Let $\varphi : D - A \to \mathbf{C}$ be a holomorphic mapping and $f : D - A \underset{m}{\to} \mathbf{P}_1$ a meromorphic mapping. Then we have (see [2], [8], [14] and the references given there):

1) If codim $A > 1$, then φ and f have extensions over A.

2) Assume codim $A = 1$. Then

a) φ has an extension over A if for some $z_0 \in A$ there is a neighborhood U of z_0 such that φ is bounded in $U - (A \cap U)$,

b) f has an extension over A if for some $z_0 \in A$ f has an extension into a neighborhood of z_0.[1]

We shall see that these statements can be generalized in some respects.[2]

Throughout this section, X and Y are irreducible complex spaces, $A \neq X$ is an irreducible analytic set in X, $f : X - A \underset{m}{\to} Y$ a meromorphic mapping. We shall study conditions under which f has an extension over A, which means that there exists a meromorphic mapping $g : X \underset{m}{\to} Y$ such that $g \mid X - A = f$.

The meromorphic mapping f can always be extended topologically to a correspondence $\bar{f} : X \underset{k}{\to} Y$ by setting $G_{\bar{f}} = \overline{G_f}$ where the closure is with respect to $X \times Y$. On the other hand, if $\tilde{f} : X \underset{m}{\to} Y$ is an extension of f, then

1) The generalization 2a) of Riemann's classical theorem on removable singularities is due to Kistler and Hartogs. 2b) is due to Hartogs and E. E. Levi. 1) follows easily from 2); the statement 1) for holomorphic functions φ is sometimes called " the second Riemann theorem on removable singularities " (2. Riemannscher Hebbarkeitssatz).

2) The extension problem for holomorphic maps is also treated in [1] and [6].

$\tilde{f} = \bar{f}$. We are thus led to study the properties of $\bar{f}$. Of essential use is the following extension theorem for analytic sets.

Theorem 1. Let Z be a complex space and M an irreducible analytic set in Z. Let further N be a pure dimensional (all irreducible components have the same dimension) analytic set in $Z-M$ such that dim N = dim M. Then the closure $\bar{N}$ of N with respect to Z is an analytic set in Z if it is analytic in at least one point of M.

This theorem was proved by Thullen [21] in the case where Z is a domain in $\mathbf{C}^n$ and where dim M = dim $N = n-1$. In [13] the theorem is stated without restriction on the dimension of M but likewise for a domain Z in $\mathbf{C}^n$ (the special case treated by Thullen is used here in the proof). From this one can obtain the theorem in the form above by using imbeddings of open sets of Z into domains of number space.

Corollary 1. If dim $N >$ dim M, then $\bar{N}$ is analytic in Z.

This can be deduced from Theorem 1 by imbedding arguments in an obvious manner. A direct proof is contained in [8].

Corollary 2. Let Z and M be as in the theorem and $\{N_i\}$ a set of mutually different irreducible analytic sets in $Z-M$ for which dim $N_i \geqslant$ dim M, and $\cup N_i$ is analytic in $Z-M$. If every neighborhood of a point $z_0 \in M$ intersects an infinite number of sets N_i, then every point of M has this property.

This is a simple consequence of Theorem 1 and Corollary 1.

Proposition 11. Let D be a domain in $\mathbf{C}^n$, M an irreducible analytic set in D, N a pure dimensional analytic set in $D-M$ such that dim N = dim M. Suppose there exists an analytic plane E_0 through a point $z_0 \in M$ such that the following conditions hold:

1) E_0 is in general position with respect to M, i.e., dim $(E_0 \cap M)$ = dim E_0 + dim M − dim D,

2) There exists a neighborhood U of z_0 such that for every analytic plane E with dim E = dim E_0 which is parallel to E_0 and which intersects U, $\bar{N} \cap E$ is analytic in D ($\bar{N}$ is the closure of N with respect to D).

Then $\bar{N}$ is analytic in z_0 and hence in D by Theorem 1.

As to the proof we refer to [13], p. 301.[1]

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1) The statement actually proved in [13] is a little more special than Proposition 11, but by suitable supplementary arguments one can obtain the proposition in the form above.

We turn now to the study of two problems:

1) When is $\bar{f}$ weakly holomorphic?

2) When is $\bar{f}$ continuous?

If $\bar{f}$ is weakly holomorphic, then $\bar{f}$ is irreducible, because the irreducibility of G_f implies that of $G_{\bar{f}}$. Hence $\bar{f}$ is a meromorphic mapping if it is weakly holomorphic and continuous.

Moreover, if $\bar{f}$ is weakly holomorphic, then the closure $\overline{f^{-1}(y)}$ of $f^{-1}(y)$ with respect to X is analytic in X for every $y \in Y$: $f^{-1}(y)$ is analytic in $X-A$ and $\bar{f}^{-1}(y)$ is analytic in X; since $\overline{f^{-1}(y)} \subset \bar{f}^{-1}(y)$ and $\overline{f^{-1}(y)} \cap (X-A) = \bar{f}^{-1}(y) \cap (X-A) = f^{-1}(y)$, it follows that $\overline{f^{-1}(y)}$ is analytic in X.

We assume now, in the rest of this section, that $\dim X - \dim Y \geqslant \dim A$. We set $Z = X \times Y$, $M = A \times Y$, $N = G_f$. Then $\dim M = \dim A + \dim Y$, $\dim N = \dim G_f = \dim X$ and, by our assumption, $\dim N \geqslant \dim M$. If $\dim X - \dim Y > \dim A$, i.e., if $\dim N > \dim M$, Corollary 1 of Theorem 1 implies that $\bar{f}$ is weakly holomorphic. Furthermore, we have

Proposition 12. Assume $\dim X - \dim Y = \dim A$. Then the correspondence $\bar{f}$ is weakly holomorphic if there exists a non-empty open set $V \subset Y$ such that the closure $\overline{f^{-1}(v)}$ of $f^{-1}(v)$ with respect to X is analytic in X for all $v \in V$.

Proof. The condition $\dim X - \dim Y = \dim A$ implies that $\dim N = \dim M$. Hence, by Theorem 1, $\bar{N} = G_{\bar{f}}$ is analytic in $Z = X \times Y$, i.e., $\bar{f}$ is weakly holomorphic, if there is a point of $M = A \times Y$ in which $\bar{N}$ is analytic. We show that this is the case for points of $A \times V$. Choose a point $(a_0, v_0) \in A \times V$ such that A is irreducible in a_0 and such that v_0 is an ordinary point of Y. There are open neighborhoods $U_1 \subset X$ of a_0 and $U_2 \subset V$ of v_0 with the following properties: $A' = A \cap U_1$ is an irreducible analytic set in U_1; U_1 can be mapped biholomorphically onto an analytic set X' in a domain D_1 of a number space $\mathbf{C}^{n_1}$; U_2 can be mapped biholomorphically onto a domain D_2 of a number space $\mathbf{C}^{n_2}$ ($n_2 = \dim Y$). It is enough to show that the closure $\overline{N'}$ of $N' = G_f \cap (U_1 \times U_2)$ with respect to $U_1 \times U_2$ is analytic in $U_1 \times U_2$. Set $D = D_1 \times D_2$, $M' = A' \times D_2$ and, for $w \in D_2$, $E_w = \mathbf{C}^{n_1} \times \{w\}$. Then we have $\dim (E_w \cap M') = \dim (A' \times \{w\}) = \dim A' = \dim A$, on the other hand $\dim E_w + \dim M' - \dim D = n_1 + (\dim A' + n_2) - (n_1 + n_2) = \dim A$. The hypothesis on the analyticity of $\overline{f^{-1}(v)}$ for all $v \in V$ implies that $\overline{N'} \cap E_w$ is analytic in D for every $w \in D_2$. Hence, by Proposition 11, $\overline{N'}$ is analytic in D; then $\overline{N'}$ is, in particular, analytic in $X' \times D_2 = U_1 \times U_2$.

Concerning the continuity of $\bar{f}$ we have

Proposition 13. The correspondence $\bar{f}$ is continuous if it is continuous at one point $a_0 \in A$.

Proof. We assume first that the topology of Y has a countable base. Then $\bar{f}$ is continuous at $a \in A$ if and only if the following condition holds: If (x_ν) and (y_ν), $\nu = 1, 2, \ldots$, are sequences of points such that $x_\nu \in X - A$, $x_\nu \to a$, $y_\nu \in f(x_\nu)$, then the sequence (y_ν) has a point of accumulation in Y. Suppose that $\bar{f}$ is continuous at a point $a_0 \in A$ and let (x_ν), (y_ν) be sequences as above. Then the fibres $f^{-1}(y_\nu)$ are non-empty analytic sets in $X - A$, and the condition $\dim X - \dim Y \geqslant \dim A$ implies $\dim F_\nu^{(\mu)} \geqslant \dim A$ for every irreducible component $F_\nu^{(\mu)}$ of $f^{-1}(y_\nu)$. Suppose that $L = \cup f^{-1}(y_\nu)$ is not analytic in $X - A$. Then there exists a subsequence (y_{ν_i}) such that one can find points $x_i' \in f^{-1}(y_{\nu_i})$ which converge to a point $x_0' \in X - A$. By continuity at x_0' it follows that (y_{ν_i}) has a point of accumulation on $f(x_0')$. Let now L be analytic in $X - A$. Assume first:

(α) There are infinitely many fibres $f^{-1}(y_{\nu_i})$ which have a common irreducible component N.

In this case we take a point of N and use similarly the continuity of f at this point. Suppose now that (α) is not satisfied. Then we apply Corollary 2 of Theorem 1 to the set of irreducible components $F_\nu^{(\mu)}$ of the fibres $f^{-1}(y_\nu)$. Since every neighborhood of a intersects infinitely many components $F_\nu^{(\mu)}$ (this implies, in particular, that the closure $\bar{L}$ of L with respect to X is not analytic in a), the same holds with respect to a_0. The y_ν have then a point of accumulation on $\bar{f}(a_0)$ because $\bar{f}$ is continuous at a_0.

Now we drop the assumption that Y has countable topology. We remark first: To show that $\bar{f}$ is continuous at $a \in A$ we may replace X by any irreducible open subspace which contains the points a and a_0. Therefore we may assume that X has countable topology. Secondly: All points of Y used in the proof above belong to the topological subspace $f(X-A) \cup \bar{f}(a_0) \subset Y$ which has countable topology since X has. If we now restrict Y to an irreducible open subspace with countable topology containing $f(X-A) \cup \bar{f}(a_0)$, the proof given above applies.

Corollary. If $\dim X - \dim Y > \dim A$, then $\bar{f}$ is always continuous.

In this case the hypothesis on the continuity of $\bar{f}$ at a point $a_0 \in A$ is not needed in the proof of Proposition 13: We have now $\dim F_\nu^{(\mu)} > \dim A$. If L is analytic in $X - A$, Corollary 1 of Theorem 1 implies that $\bar{L}$ is analytic in every point of A, and the condition (α) is necessarily satisfied.

Combining the preceding statements we have the following result.

Theorem 2. Let $f: X-A \underset{m}{\rightarrow} Y$ be a meromorphic mapping and $\dim X - \dim Y \geqslant \dim A$. Then $\bar{f}$ is a meromorphic mapping if and only if

1) there exists a non-empty open set $V \subset Y$ such that $\overline{f^{-1}(v)}$ is analytic in X for all $v \in V$, and

2) $\bar{f}$ is continuous at a point $a_0 \in A$.

If $\dim X - \dim Y > \dim A$, then $\bar{f}$ is always a meromorphic mapping.

Corollary. Assume there is an open subset $U \subset X$ and a compact set $K \subset Y$ different from Y such that $U \cap A \neq \varnothing$ and $f(U-(U \cap A)) \subset K$. Then $\bar{f}$ is a meromorphic mapping.

To conclude this from Theorem 2 we remark first that the set $V = Y-K$ satisfies the above condition 1): If $v \in V$, then $f^{-1}(v)$ does not intersect U, hence $\overline{f^{-1}(v)}$ is analytic in every point of $U \cap A$ and therefore, by Theorem 1, analytic in X. On the other hand, $\bar{f}$ is continuous at every point $a_0 \in U \cap A$. For $\bar{f}(a_0)$ is compact since it is a closed subset of K. Moreover, let V_0 be a neighborhood of $\bar{f}(a_0)$; we assert that there is a neighborhood U_0 of a_0 such that $\bar{f}(U_0) \subset V_0$. If this were false, then there would exist points x in $U-(U \cap A)$ arbitrarily near a_0 such that $f(x) \cap (K-(K \cap V)) \neq \varnothing$. But then it follows that $f(a_0) \cap (K-(K \cap V_0)) \neq \varnothing$, which is a contradiction.

As to the extension of holomorphic maps we state:

Theorem 3. Let X be, in addition to the earlier assumptions, a complex manifold and $f: X-A \rightarrow Y$ a holomorphic map. Then

1) If $\dim X - \dim Y > \dim A + 1$, $\bar{f}$ is a holomorphic map,

2) If $\dim X - \dim Y = \dim A + 1$, then $\bar{f}$ is either a holomorphic map or $\bar{f}$ is a meromorphic mapping and $\bar{f}(a) = Y$ for all $a \in A$.

Proof. Assume $\dim X - \dim Y \geqslant \dim A + 1$. Then, by Theorem 2, $\bar{f}$ is a meromorphic mapping; if $S = S(f) = \varnothing$, $\bar{f}$ is even a holomorphic map. Suppose $S \neq \varnothing$, set $T = \check{f}^{-1}(S)$ and let T_0 be an irreducible component of T. Set $S_0 = \check{f}(T_0)$. By Remmert's mapping theorem S_0 is an irreducible analytic set in X. We have

$$\dim T_0 = \dim S_0 + \inf_{z \in D_0} \dim_z (g^{-1}(g(z)) \text{ where } g = \check{f} \mid T_0 ,$$

furthermore $\dim S_0 \leqslant \dim S \leqslant \dim A$ because $S \subset S_0 \subset A$. Every fibre

$g^{-1}(g(z)), z \in T_0$, is mapped injectively into Y by $\hat{f}$, hence $\dim(g^{-1}(g(z)) \leqslant \dim Y$. Thus we obtain the inequalities

$$(*) \quad \dim T_0 \leqslant \dim A + \dim Y \leqslant \dim X - 1 .$$

Now we shall see that $\dim T_0 = \dim X - 1$. Therefore we have equality in (*), hence $\dim X - \dim Y = \dim A + 1$. We obtain also $\dim S_0 = \dim S = \dim A$, hence $S_0 = S = A$, since A is irreducible; moreover, $\dim(g^{-1}(a)) = \dim Y$ for every $a \in A$, consequently $\bar{f}(a) = \hat{f}(g^{-1}(a)) = Y$.

In order to show that $\dim T_0 = \dim X - 1$, we use the following theorem due to Grauert and Remmert [5] (a proof was also given by Kerner [7]):

Let X be a complex manifold, Z a normal complex space, K an analytic set in Z with $\operatorname{codim} K \geqslant 2$, $\tau : Z \to X$ a holomorphic map such that $\tau \mid Z - K$ is locally biholomorphic. Then τ is locally biholomorphic.

Now assume first that $G_{\bar{f}}$ is a normal complex subspace of $X \times Y$. The holomorphic map $\check{f}: G_{\bar{f}} \to X$ is locally biholomorphic in a point $\zeta \in G_{\bar{f}}$ if and only if $\zeta \in T = \check{f}^{-1}(S)$. Hence, by the theorem of Grauert and Remmert, T is puredimensional and $\dim T = \dim X - 1$. If $G_{\bar{f}}$ is not normal, we take a normalization $(\tilde{G}, \nu)$ of $G_{\bar{f}}$ and look at $\check{f} \circ \nu : \tilde{G} \to X$ and $\tilde{T} = (\check{f} \circ \nu)^{-1}(S)$ instead of $\bar{f}$ and T. We see then that $\tilde{T}$ is puredimensional with $\dim \tilde{T} = \dim X - 1$, but then it follows that $\nu(\tilde{T}) = T$ has the same properties.

Remark. If Y is not compact, then $\bar{f}$ is always a holomorphic map under the hypothesis of Theorem 3 since $\bar{f}(a)$ is compact for $a \in A$. If the assumption that X be a complex manifold is dropped, then both assertions of Theorem 3 become false as can be shown by examples.

5. Maximal meromorphic mappings

All complex spaces in this section are irreducible. Before we state the problem we give the necessary definitions.

Let $f : X \underset{k}{\to} Y$ be weakly holomorphic and not empty. The *rank* $\operatorname{rk} f$ of f is by definition the global rank of the holomorphic mapping $\hat{f} : G_f \to Y$, i.e.,

$$\operatorname{rk} f = \sup_{z \in G_f} \operatorname{codim}_z \hat{f}^{-1}(\hat{f}(z)).$$

For two meromorphic mappings $f : X \underset{m}{\rightarrow} Y$ and $f_0 : X \underset{m}{\rightarrow} Y_0$ we always have rk $[f, f_0] \geqslant \max\{\operatorname{rk} f, \operatorname{rk} f_0\}$. We say that f_0 *depends on* f, if $\operatorname{rk} f =$ *rk* $[f, f_0]$. If f_0 depends on f and f depends on f_0, we say that f_0 is *related* to f. Then clearly $\operatorname{rk} f = \operatorname{rk} f_0$.

Let $f : X \underset{m}{\rightarrow} Y$ and $f_0 : X \underset{m}{\rightarrow} Y_0$ be given. Suppose that there exists a meromorphic mapping α: $Y \underset{m}{\rightarrow} Y_0$ such that the meromorphic product $\alpha \triangle f$ is defined and $f_0 = \alpha \triangle f$. Then we say that f *majorizes* f_0. If this is the case, f_0 depends on f ([15]).

If $f : X \underset{m}{\rightarrow} Y$ is surjective and if f majorizes every meromorphic mapping g dependent on f, f is called meromorphically maximal or *m*—*maximal.*

Let us now consider the following problem:

Given $f_0 : X \underset{m}{\rightarrow} Y_0$, is it possible to find a meromorphic mapping $f_s : X \underset{m}{\rightarrow} Y_s$ such that f_s is related to f_0 and f_s is *m*-maximal ? If possible, the pair (f_s, Y_s) is called a meromorphic base or an *m*—*base* with respect to f_0.

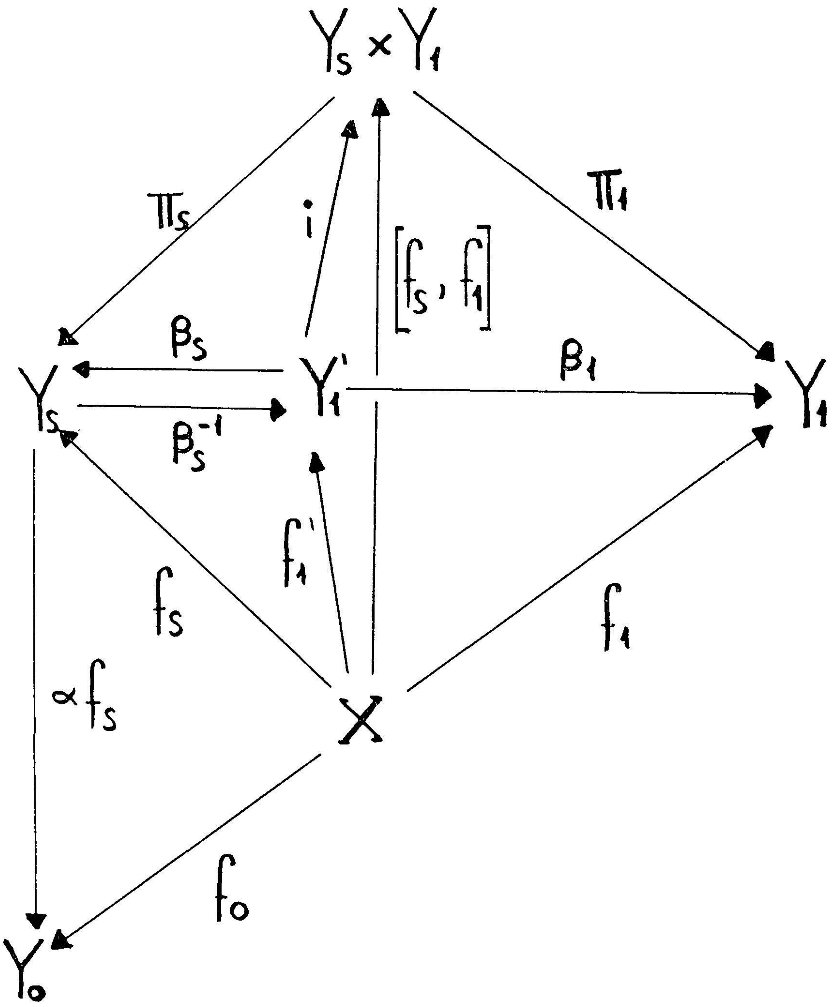
Proposition 14. If $f_0 : X \underset{m}{\rightarrow} Y_0$ is proper, then an *m*-base with respect to f_0 exists.

We give a sketch of the proof (compare [15]).

Since f_0 is proper, $f_0(X) = Y_0'$ is an irreducible $\operatorname{rk} f_0$ — dimensional analytic set in Y_0; there is a surjective meromorphic mapping $f_0' : X \underset{m}{\rightarrow} Y_0'$ such that $f_0 = I^{Y_0'}_{Y_0} \circ f_0' \left(I^{Y_0'}_{Y_0} \text{ is the inclusion map } Y_0' \rightarrow Y_0 \right)$. f_0' is proper by Proposition 10, moreover it is surjective and related to f_0. Now, a complex *m*-base with respect to f'_0 is also a complex *m*-base with respect to f_0. Therefore we can suppose that f_0 is surjective.

We consider the class $\mathfrak{F}$ of those surjective meromorphic mappings of X which are dependent on f_0 and majorize f_0. If $(f : X \underset{m}{\rightarrow} Y) \in \mathfrak{F}$, there exists a unique surjective meromorphic mapping $\alpha_f : Y \underset{m}{\rightarrow} Y_0$ such that $f_0 = \alpha_f \triangle f$. This implies that f is related to f_0 and, by Proposition 10, that f and α_f are proper. We have $\operatorname{rk} f = \dim Y$, $\operatorname{rk} \alpha_f = \dim Y_0 = \operatorname{rk} f_0$, $\operatorname{rk} f = \operatorname{rk} f_0$, hence $\dim Y = \dim Y_0 = \operatorname{rk} \alpha_f$. Thus (Y, α_f, Y_0) is a " meromorphic covering " of Y_0 with a well defined number $n(f)$ of sheets. The $n(f)$, $f \in \mathfrak{F}$, have a finite upper bound: If not, one can show that there exists a point $y_0 \in Y_0$ such that $f_0^{-1}(y_0)$ has infinitely many connected components, but this is impossible since f_0 is proper.

Let $(f_s : X \underset{m}{\to} Y_s) \in \mathfrak{F}$ be such that $n(f_s)$ is maximal. We claim that (f_s, Y_s) is an m-base with respect to f_0. Suppose that $f_1 : X \underset{m}{\to} Y_1$ depends on f_s, we have to show that f_s majorizes f_1. The meromorphic junction $[f_s, f_1] : X \underset{m}{\to} Y_s \times Y_1$ is proper (Proposition 10) and rk $[f_s, f_1] =$ rk $f_s =$ rk f_0, therefore $[f_s, f_1](X) = Y_1'$ is a rk f_0 – dimensional analytic subset of $Y_s \times Y_1$. There is a meromorphic mapping $f_1' : X \underset{m}{\to} Y_1'$ such that $[f_s, f_1] = \mathrm{i} \circ \mathrm{f}_1'$ where $i: Y_1' \to Y_s \times Y_1$; f_1' is surjective, proper and related



to f_0. Let π_s and π_1 be the projections from $Y_s \times Y_1$ onto Y_s and Y_1, set $\beta_s = \pi_s \circ i$, $\beta_1 = \pi_1 \circ i$, respectively. We have $f_s = \beta_s \circ f_1'$, hence f_1' majorizes f_s. The holomorphic mapping $\pi_s \circ i = \beta_s$ is surjective and, by Proposition 10, proper. The meromorphic product $\alpha_{f_s} \triangle \beta_s$ is defined since β_s is surjective; we have $f_0 = (\alpha_{f_s} \triangle \beta_s) \triangle f_1'$, hence f_1' majorizes f_0 and, consequently, $f_1' \in \mathfrak{F}$. Then $n(f_1') \geqq n(f_s)$ since f_1' majorizes f_s, thus $n(f_1') = n(f_s)$ since $n(f_s)$ is maximal. It follows that the number of sheets of the covering (Y_1', β_s, Y_s) equals 1, and this implies that β_s is a bimeromorphic mapping. Now $f_1 = \beta_1 \circ f_1' = \beta_1 \circ (\beta_s^{-1} \triangle f_s) = (\beta_1 \circ \beta_s^{-1}) \triangle f_s$. Hence f_s majorizes f_1.

We give, without proof (see [15]) a more general result in this direction.

Theorem 4. Let $f_0 : X \underset{m}{\to} Y_0$ be a meromorphic mapping and A an irreducible analytic set in X such that the holomorphic correspondence

$$a_0 = f_0 \mid A : A \underset{k}{\to} Y_0$$

has at least one irreducible component $a_0': A \underset{k}{\to} Y_0$ which is proper and satisfies $\operatorname{rk} a_0' = \operatorname{rk} f_0$. Then there exists $f_s : X \underset{m}{\to} Y_s$ such that (f_s, Y_s) is an m-base with respect to f_0.

By definition, for $f : X \underset{m}{\to} Y$ a point $x_0 \in X$ is *a point of indeterminacy of degree k*, if $\dim f(x_0) = k$, and a point of indeterminacy of *maximal degree*, if $\dim f(x_0) = \operatorname{rk} f$.

Let now the set A in Theorem 4 consist of one point x_0. Then $a_0 = f_0 \mid \{ x_0 \}: \{ x_0 \} \underset{k}{\to} Y_0$ is a proper holomorphic correspondence and $\operatorname{rk} f_0 \mid \{ x_0 \} = \operatorname{rk} a_0 = \dim f(x_0) \leqslant \operatorname{rk} f_0$. The hypothesis of the theorem means, in this case, that $\dim f_0(x_0) = \operatorname{rk} f_0$; this implies ([15]) that $f_0(x_0) = f_0(x)$. We obtain the following *specialization* of Theorem 4:

Let $f_0 : X \underset{m}{\to} Y_0$ be a meromorphic mapping with a point of indeterminacy of maximal degree. Then there exists an m-base with respect to f_c.

Finally we give applications of Proposition 14 and Theorem 4. We consider *meromorphic functions* defined on the complex space X. These are meromorphic mappings $\varphi: X \underset{m}{\to} \mathbf{P}_1$ such that $\varphi(X)$ does not reduce to the point ∞ of $\mathbf{P}_1$. The set of all meromorphic functions on X form a field $\mathfrak{M}(X)$. Let $\varphi_1, \dots \varphi_k$ be elements of $\mathfrak{M}(X)$. We say that $\varphi_1, \dots \varphi_k$ is a *system of independent meromorphic functions* if for the meromorphic mapping $\Phi = [\varphi_1, \dots, \varphi_k] : X \underset{m}{\to} \mathbf{P}_1 \times \dots \times \mathbf{P}_1 = \mathbf{P}_1^k$ we have $\operatorname{rk} \Phi = k$. There are always maximal systems of independent meromorphic functions on X; the length k of such a system is uniquely determined with $k \leq \dim X$.

Let now X be a *compact* complex space. As a first application we obtain the theorem of *Chow-Thimm* [4], [20] (see also [10]):

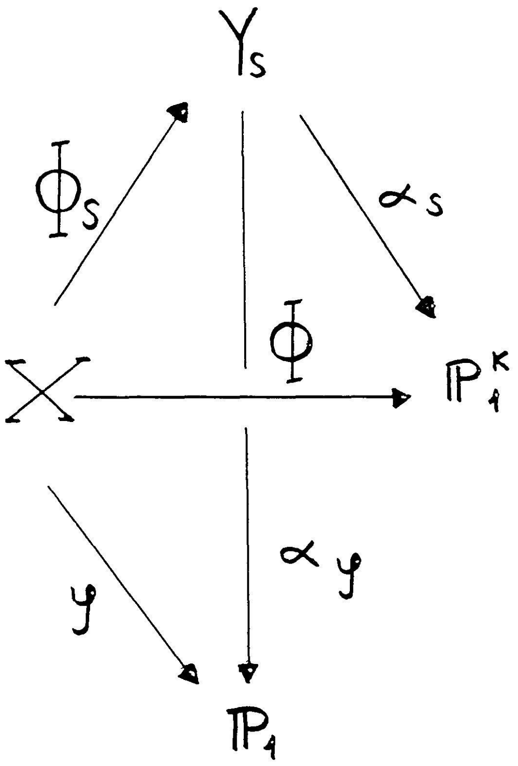
The field $\mathfrak{M}(X)$ of meromorphic functions on an irreducible compact complex space X is isomorphic to a finite algebraic extension of a field of rational functions.

Proof. Choose a maximal system $\varphi_1, \dots, \varphi_k$ of independent meromorphic functions on X and let Φ be defined as above. Φ is proper since X

is compact, thus we can apply Proposition 14. Hence there exists an m-base (Φ_s, Y_s) with respect to Φ and there is a meromorphic mapping α_s: $Y_s \underset{m}{\to} \mathbf{P}_1^k$ such that $\Phi = \alpha_s \triangle \Phi_s$. If $\varphi \in \mathfrak{M}$ (X), we have rk Φ = rk $[\Phi, \varphi]$ since the system $\varphi_1, \ldots, \varphi_k$ is maximal, therefore φ depends on Φ. So Φ_s majorizes every meromorphic function φ on X, i.e., there is a meromorphic function $\alpha_\varphi : Y_s \underset{m}{\to} \mathbf{P}_1$ such that $\varphi = \alpha_\varphi \triangle \Phi_s$. It is easily seen that the assignment $\varphi \mapsto \alpha_\varphi$ gives an isomorphism from $\mathfrak{M}(X)$ onto $\mathfrak{M}(Y_s)$. Now $(Y_s, \alpha_s, \mathbf{P}_1^k)$ is a meromorphic covering of $\mathbf{P}_1^k$; if n is its number of sheets, then every meromorphic function α on Y_s satisfies an equation

$$\alpha^n + (b_1 \triangle \alpha_s) \,.\, \alpha^{n-1} + \ldots + (b_n \triangle \alpha_s) = 0\,,$$

where $b_\nu \in \mathfrak{M}(\mathbf{P}_1^k)$ $(\nu = 1, \ldots, n)$. This implies that $\mathfrak{M}(Y_s)$ is isomorphic to a finite algebraic extension of $\mathfrak{M}(\mathbf{P}_1^k)$. But $\mathfrak{M}(\mathbf{P}_1^k)$ is isomorphic to the field $\mathbf{C}(z_1, \ldots z_k)$ of he rational functions of k complex variables. Hence we obtain an isomorphism of $\mathfrak{M}(X)$ with the desired properties.



As another application we sketch a proof of the following statement: Let $\Phi : X \underset{m}{\to} Y$ be a meromorphic mapping with a point of indeterminacy x_0 of maximal degree. Then the field $\mathfrak{M}_\Phi(X)$ of meromorphic functions on X depending on Φ is isomorphic to a finite algebraic extension of a field of rational functions.

By the special case of Theorem 4 there exists an m-base (Φ_s, Y_s) with respect to Φ. The meromorphic mapping $\Phi_s : X \underset{m}{\to} Y_s$ majorizes every $\varphi \in \mathfrak{M}_\Phi(X)$; if $\varphi = \alpha_\varphi \triangle \Phi_s$, then the assignment $\varphi \mapsto \alpha_\varphi$ gives again an isomorphism $\mathfrak{M}_\Phi(X) \cong \mathfrak{M}(Y_s)$. The point x_0 is also a point of indeterminacy of maximal degree for Φ_s since Φ_s depends on Φ_0 (see [15]), hence $\Phi_s(x_0) = \Phi_s(X) = Y_s$ is compact. Now we can apply the theorem of Chow-Thimm, and we obtain the assertion.

Remark. In the case where $Y = \mathbf{P}_1^k$ and Φ is the junction of k meromorphic functions on X, the statement is a known theorem of Thimm [18], [19]. A proof of this theorem was also given by Remmert [12].

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