DIRECTIONAL DEVIATION NORMS AND SURFACE AREA

DIRECTIONAL DEVIATION NORMS AND SURFACE AREA

by L. V. Toralballa

Introduction

One of the earlier attempts at giving a general definition of surface area was made by J. A. Serret[1] in 1868. Following the quite adequate definition of arc length, he defined the area of a surface to be the L.U.B. of the areas of polyhedra inscribed in it. The inadequacy of this definition was made apparent in 1882 by H. A. Schwarz [2] when he showed that by this definition even such a simple and smooth surface as a circular cylinder has no area. This discovery prompted a vigorous search for a definition of surface area that would have adequate generality. In 1902 Henri Lebesgue [3] proposed that surface area be defined as the G.L.B. of the set of the limit inferiors of the sequences of areas of polyhedral surfaces which converge uniformly to the given surface. An enormous literature [see 4, 5, 6, 7] has grown using Lebesgue's definition as a basis.

However, some mathematicians came to feel that while Lebesgue's definition is quite general, it lacks geometric simplicity. They initiated a return to a presentation by means of inscribed triangular polyhedra. The idea is to limit the class of the inscribed triangular polyhedra in such a manner as to preclude the occurrence of the Schwarz phenomenon. Thus, M. W. H. Young [8], for continuously differentiable surfaces, requires that the angles of the triangles on the xy plane have an upper bound less than π. Rademacher [9] for surfaces satisfying the Lipschitz condition, requires that these angles have a positive lower bound. Kempisty [7] limits consideration to right triangles having the ratio of the base to the altitude between 1/2 and 2. In a certain sense these latter definitions are *ad hoc* and thus seem to lack naturalness.

The present note is restricted to continuously differentiable surfaces. However, it makes use of a simple geometric idea which, as far as this writer knows, has not been considered in the literature.

If a polyhedron, inscribed on a continuously differentiable surface is to be thought of, in a good geometric sense, as an approximation to the surface, one might expect that the direction of the normal to each face of the polyhedron should not differ very much from the directions of the normals to the part of the surface which is subtended by the particular face. One sees that the polyhedra constructed by Schwarz do not have this property; that in fact, as the norms of the polyhedra converge to zero, the angle between the normal to each face and the normals to the surface subtended by the particular face approaches $\pi/2$. It is this pleating effect that produces a set of polyhedral areas that is unbounded.

The present paper is an attempt to take into consideration the angular or directional deviation of the faces of the polyhedra.

We shall here confine ourselves to non-paramatric surfaces. Such a surface is the locus in E^3 of an equation $z = f(x, y)$ where the domain is the closure of a bounded, open, and connected set E in E^2, and f is continupus on $\bar{E}$.

THE BASIS

We shall make use of the following properties of E^3.

1) Let U and V be any two vectors in E^3 such that $|\cos(U, V)| < k$, where $0 < k < 1$. Then for every $\epsilon > 0$ there exists $\delta > 0$ such that if U_1 and V_1 are any two vectors such that $|\sin(U_1, U)| < \delta$ and $|\sin(V_1, V)| < \delta$ then $|\sin(U \times V, U_1 \times V_1)| < \epsilon$.

Let E be an open, bounded, and connected set on the xy plane. Let $f(x, y)$ be defined and continuously differentiable on $\bar{E}$. Then

2) The directional derivative of f is uniformly continuous on $\bar{E}$, i.e. for every $\epsilon > 0$ there exists $\delta > 0$ such that if (x_1, y_1) and (x_2, y_2) are in $\bar{E}$ and $0 < \rho((x_1, y_1), (x_2, y_2)) < \delta$ then $|D_{x_1,y_1;x_2,y_2} f(x_1, y_1) - D_{x_1,y_1;x_2,y_2} f(x_2, y_2)| < \epsilon$. Here $\rho((x_1, y_1), (x_2, y_2))$ is the distance between (x_1, y_1) and (x_2, y_2). $D_{x_1,y_1;x_2,y_2}(x_1, y_1)$ is the directional derivative of f at (x_1, y_1) in the direction of the vector from (x_1, y_1) to (x_2, y_2).

The directional derivative is uniformly Lipschitzian over $\bar{E}$.

3) There exist positive numbers k and δ, $k < 1$ such that if P, P_1, and P_2 are any three distinct points of $\bar{E}$ such that

a) $\rho(P, P_1) < \delta$

b) $\rho(P, P_2) < \delta$ and

c) $\cos(\overrightarrow{PP_1}, \overrightarrow{PP_2}) = 0$, then

$|\cos(\overrightarrow{QQ_1}, \overrightarrow{QQ_2})| < k$, where $Q = f(P)$, $Q_1 = f(P_1)$ and $Q_2 = f(P_2)$.

4) Let P_1 and P_2 be any two distinct points on $\bar{E}$, $Q_1 = f(P_1)$, and $Q_2 = f(P_2)$. Let $P_1 P_2$ denote the closed interval determined by P_1 and P_2 and $Q_1 Q_2$ the closed interval determined by Q_1 and Q_2. Let the curve $C = f(P_1 P_2)$. Then there exists a point R on C such that the tangent line to C at R is parallel to $Q_1 Q_2$.

5) With the notation as in 4), let the *deviation* $D(P_1 P_2)$ denote the L.U.B. of the acute angles φ between the surface chord $Q_1 Q_2$ and any tangent line to C. Then for every $\epsilon > 0$ there exists $\delta > 0$ such that if $0 < \rho(P_1, P_2) < \delta$ then $D(P_1 P_2) < \epsilon$.

6) For every $\epsilon > 0$ there exists $\delta > 0$ such that if P_1 and P_2 are any two distinct points of $\bar{E}$ such that $\rho(P_1, P_2) < \delta$ then $\psi < \epsilon$, where ψ is the acute angle between the surface normals at $f(P_1)$ and at $f(P_2)$.

We need to give some preliminary definitions.

Definitions

We shall call a surface $S = f(\bar{E})$ simple when the boundary of $\bar{E}$ is a simple closed polygon. We shall first be concerned only with simple surfaces.

A polyhedron Π is said to be inscribed on S when all the vertices of Π are in S and the orthogonal projection, Proj Π, on the xy plane is $\bar{E}$. By the norm of a polyhedron we shall mean the greatest of the diameters of the faces (triangles) of Π.

Let Π be inscribed on S and let A be a face of Π. By the deviation $D(A)$ of A we shall mean the L.U.B. of the acute angles between the normal to A and the surface normal at a point of the surface *subtended* by A. By the deviation norm of Π we shall mean the greatest of the deviations of its faces.

We shall consider sequences of polyhedra which are inscribed on S. A sequence $\{\Pi_1, \Pi_2, \ldots\}$ of such polyhedra is said to be a *proper* sequence of polyhedra inscribed on S when the corresponding sequence of norms $\{N_1, N_2, \ldots\}$ converges to zero and the corresponding sequence $\{\phi_1, \phi_2, \ldots\}$ of deviation norms also converges to zero.

We now give our basic definition of surface area:

Let E be a bounded set on the xy plane whose boundary is a simple closed polygon. Let $f(x, y)$ be defined and continuously differentiable on $\bar{E}$. If to every proper sequence of polyhedra inscribed on $S = f(\bar{E})$ the corresponding sequence of polyhedral areas $\{A_1, A_2, \ldots\}$ converges, then

then we say that S is *quadrable* anf that the necessarily unique limit of $\{A_1, A_2, \ldots\}$ is the area of the surface S.

THEOREM 1.

Let E be a bounded set on the xy plane whose boundary is a simple closed polygon. Let $f(x, y)$ be defined and continuously differentiable on $\bar{E}$. Then there exist a proper sequence $\{\Pi_1, \Pi_2, \ldots\}$ of polyhedra inscribed on S.

Proof:

For every positive number r there exists a decomposition of $\bar{E}$ as the union of closed right triangles whose diameters are all less than r. The vertices of these right triangles determine a finite set of points in S whose projection is precisely the set of these vertices. This set of points in S determines a triangular polyhedron which is inscribed on S. We shall show that by making the norm of the decomposition of $\bar{E}$ sufficiently small we can make the acute angle between the normal to each polyhedral face and the surface normal at any point of the portion of S which is subtended by the particular face to be arbitrarily small. Let $\varepsilon > 0$ be given.

By property 3) there exist positive real numbers $k < 1$ and δ_1 such that if PP_1P_2 is a right triangle on $\bar{E}$ (P being the right angled vertex) with diameter $< \delta_1$, then $|\cos(\overrightarrow{QQ_1}, \overrightarrow{QQ_2})| < k$. Let the decomposition of $\bar{E}$ by right triangles be of norm less than δ_1.

By property 1) there exists a positive real number θ such that if $|\sin(\overrightarrow{QQ_1}, \overrightarrow{QQ_1'})| < \theta$ and $|\sin(\overrightarrow{QQ_2}, \overrightarrow{QQ_2'})| < \theta$, then the acute angle between $\overrightarrow{QQ_1} \times \overrightarrow{QQ_2}$ and $\overrightarrow{QQ_1'} \times \overrightarrow{QQ_2'}$ is less than $\varepsilon / 3$.

By properties 4) and 5) there exists a positive real number δ_2 such that if PP_1P_2 is a right triangle on $\bar{E}$ with diameter less than δ_2, then the angle between the chord $\overrightarrow{QQ_1}$ and the tangent line at Q to the curve on S subtented by $\overrightarrow{QQ_1}$ is less than θ. Similarly, the angle between the chord $\overrightarrow{QQ_2}$ and the tangent line at Q to the curve on S subtended by $\overrightarrow{QQ_2}$ is less than θ. It follows that the angle between the normal to the polyhedral face QQ_1Q_2 and the surface normal at Q is less than $\varepsilon / 3$.

By property 6) there exists a positive real number δ_3 such that if the diameter of the triangle PP_1P_2 is less than δ_3, then the angle between the surface normals at any two points of the portion of S which is subtended by the polyhedral face QQ_1Q_2 is less than $\varepsilon / 3$.

Let δ be the least of δ_1, δ_2, and δ_3. If D is any decomposition of $\bar{E}$ int closed right triangles of norm less than δ, then if QQ_1Q_2 is any of the polyhedral faces, the L.U.B. of the angles between the normal to $QQ_1\,Q_2$ and the surface normals at any point of the portion of the surface subtended by $QQ_1\,Q_2$ is less than ε.

Thus corresponding to a sequence $\{\varepsilon_1, \varepsilon_2, ...\}$ converging to zero, there exists a sequence of polyhedra with corresponding sequence of norms converging to zero and also with corresponding sequence of deviation norms converging to zero.

THEOREM 2.

Let E be an open set on the xy plane whose boundary is a simple closed polygon. Let $f(x, y)$ be defined and continuously differentiable on $\bar{E}$. Then for every proper sequence of polyhedra inscribed on S the corresponding sequence $\{A_1, A_2, ...\}$ of polyhedral areas converges and moreover it converges to the double integral

$$\int_{\bar{E}} \sqrt{1 + \left(\frac{\partial f}{\partial x}\right)^2 + \left(\frac{\partial f}{\partial y}\right)^2}\, d(x, y)\,.$$

Proof:

For each n, the projection of the faces of Π_n constitute a decomposition D_n of $\bar{E}$ as the union of a finite set of closed triangles. Let the triangle $\Delta_{mn} = QQ_1\,Q_2$ be the face of Π_n and let $\Delta'_{mn} = \text{Proj}\ QQ_1\,Q_2 = PP_1\,P_2$. Let β_{mn} be the acute angle between the normals to Δ_{mn} and to Δ'_{mn}. Let A_{mn} and A'_{mn} denote the areas of Δ_{mn} and Δ'_{mn}, respectively. Then $A_{mn} = A'_{mn}$ sec β_{mn} and the area A_n of Π_n is $\sum\limits_m A'_{mn} \sec \beta_{mn}$.

Let P_{mn} be any point in Δ'_{mn} and let Q_{mn} be the point of S whose projection is P_{mn}. Let θ_{mn} denote the acute angle between the surface normal at Q_{mn} and the z-axis.

Let $\{\Pi_1, \Pi_2, \Pi_3, ..\}$ be any proper sequence of polyhedra inscribed on S. We shall associate to $\{\Pi_1, \Pi_2, \Pi_3, ...\}$ certain related sequences.

$$\Pi_1,\ \Pi_2,\ \Pi_3,\ \ldots$$

$$\phi_1,\ \phi_2,\ \phi_3,\ \ldots$$

$$\Sigma_1,\ \Sigma_2,\ \Sigma_3,\ \ldots$$

$$\Sigma'_1,\ \Sigma'_2,\ \Sigma'_3,\ \ldots$$

The sequence $\{\phi_1, \phi_2, \phi_3, ...\}$ is the corresponding sequence of deviation norms. The sequence $\{\Sigma_1, \Sigma_2, \Sigma_3, ..\}$ is the corresponding sequence of polyhedral areas. $\Sigma_n = \sum\limits_m A'_{mn} \sec \beta_{mn}$. In the fourth sequence $\Sigma'_n = \sum\limits_m A'_{mn}$ $\sec \theta_{mn}$. Here $\sec \theta_{mn}$ is the value of $\sqrt{1 + \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2}$ at some point of A'_{mn}. Thus the sequence $\{\Sigma'_1, \Sigma'_2, \Sigma'_3, \ldots\}$ is a sequence of Riemann sums of the function $\sqrt{1 + \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2}$ on $\bar{E}$ with corresponding sequence of norms converging to zero. Since $\sqrt{1 + \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2}$ is continuous on $\bar{E}$, this converges to the double integral $\oint = \int\limits_{\bar{E}} \sqrt{1 + \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2}\, d(x, y)$.

We will now consider the sequence $\{\Sigma_1, \Sigma_2, \Sigma_3, \ldots\}$.

Let θ denote the acute angle between the surface normal at a point of S and the z-axis. Sec $\theta = \sqrt{1 + \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2}$ is bounded on $\bar{E}$. Thus there exists an acute angle $\theta^* > 0$ such that $\theta < \theta^*$ for all points of $\bar{E}$ (i.e. for all points of S). Since $\sec \theta$ is uniformly continuous on the closed interval $[0, \theta^*]$, for every $\eta > 0$ there exists $\tau > 0$ such that if $0 < \theta_1 < \theta^*, 0 < \theta_2 < \theta^*$, and $|\theta_1 - \theta_2| < \tau$, then $|\sec \theta_1 - \sec \theta_2| < \eta$.

We now compare the corresponding sequences

$$\{\Sigma_1, \Sigma_2, \Sigma_3, \ldots\}$$
$$\{\Sigma'_1, \Sigma'_2, \Sigma'_3, \ldots\}.$$

Let $\varepsilon > 0$ be given. Take $\frac{\varepsilon}{2A}$, where A = area of $\bar{E}$. There exists $\tau > 0$ such that if $|\theta_1 - \theta_2| < \tau$, then $|\sec \theta_1 - \sec \theta_2| < \frac{\varepsilon}{2A}$. Since $\{\phi_1, \phi_2, \phi_3, \ldots\}$ converges to zero, there exists a positive integer N_1 such that if $n > N_1$ then $\phi_n < \tau$. Thus if $n > N_1$, then

$$|\Sigma_n - \Sigma'_n| = |\sum_m A'_{mn} (\sec \beta_{mn} - \sec \theta_{mn})| < \frac{\varepsilon}{2A} \sum_m A'_{mn} = \frac{\varepsilon}{2}.$$

Since $\{\Sigma'_n, \Sigma'_2, \Sigma'_3, \ldots\}$ converges to $\oint$, there exists a positive integer N_2. such that if $n > N_2$, then $|\Sigma'_n - \oint| < \frac{\varepsilon}{2}$. Let N be the larger of N_1 and N_2. If $n > N$ then

$$|\Sigma_n - \oint| = |\Sigma_n - \Sigma'_n + \Sigma'_n - \oint| \leqslant |\Sigma_n - \Sigma'_n| + |\Sigma'_n - \oint| \leqslant \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \epsilon.$$

Thus $\{\Sigma_1, \Sigma_2, \Sigma_3, \ldots\}$ converges to Φ.

Thus far we have defined the concept of area only for surfaces which are not only continuously differentiable but are also simple. We now remove this latter restriction.

Let E be any quadrable (i.e. Jordan measurable) open set on the xy plane having for boundary a simple closed curve. Let f be defined and continuously differentiable on $\bar{E}$. Let P be any subset of $\bar{E}$ whose boundary is a simple closed polygon. The surface $S_p = f(P)$ is quadrable. Denote its area by A_p. Consider now the set of all such areas A_p. Since $\sec\theta$ is bounded on $\bar{E}$, for every polygonal subset P of $\bar{E}$, $A_p \leqq AM$, where A is the area of $\bar{E}$ and M is an upper bound of $|\sec\theta|$ on $\bar{E}$. We now define the area of $S = f(\bar{E})$ as the L.U.B. of the set [all A_p].

THEOREM 3.

Let E be a quadrable open set on the xy plane having for boundary a simple closed curve. Let f be defined and continuously differentiable on $\bar{E}$. Then the area of $S = f(\bar{E})$ is given by

$$\Phi = \int_{\bar{E}} \sqrt{1 + \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2}\, d(x, y)\,.$$

Proof:

Let B denote the L.U.B. of the set [all A_p]. For each P, $A_p \leqslant \Phi$ and hence $B \leqslant \Phi$. Suppose now that $\Phi - B = 2\varepsilon > 0$.

Let $\{D_1, D_2, D_3, \ldots\}$ be any sequence of triangular " decompositions " of $\bar{E}$ with corresponding sequence of norms converging to zero. Here we permit the triangles to abut beyond the boundary of $\bar{E}$. On each D_n form a Riemann sum of $F(x, y) = \sqrt{1 + \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2}$ in the following manner: If a triangle does not abut beyond the boundary of $\bar{E}$, then take for the point P any point of the triangle. However, if a triangle does abut beyond the boundary of $\bar{E}$, let its contribution to the Riemann sum be zero. Now every sequence $\{S_1, S_2, S_3, \ldots\}$ of such Riemann sums converges and moreover, it converges to Φ. Since $\{S_1, S_2, S_3, \ldots\}$ converges to Φ, there exists a positive integer N such that if $n > N$ then $|\Phi - S_n| < \frac{\varepsilon}{2}$.

On D_n, the set of the triangles which do not abut beyond the boundaries of $\bar{E}$ constitutes a polygonal subset of $\bar{E}$. Call it P_n. There exists a triangular decomposition D_n' of P_n such that if S_n' is a Riemann sum of $f(x, y)$ on D_n', then $| A_{P_n} - S_n' | < \frac{\varepsilon}{4}$ and $| \mathfrak{A} - S_n' | < \frac{\varepsilon}{2}$. It follows that $A_{P_n} > B$. This contradiction shows that $B = \mathfrak{A}$.

BIBLIOGRAPHY

[1] Serret, J. A., Cours de Calcul Differentiel et Integral. 1868.

[2] Schwarz, H. A., Sur une définition érronée de l'aire d'une surface courbe. *Ges. Math.*, Abhandl. II, 1882, p. 309.

[3] Lebesgue, Henri, Intégrale, longueur, aire. *Annali di Mathematica pura ed applicata.* 1902.

[4] Tonelli, A., Sulla quadratura delle superficie. *Atti. Accad. Naz. Lincei*, Mem. Cl. Sci. Fis. Mat. Natur., Ser. I, 6e série, *3* (1926).

[5] Rado, M., Sur le calcul des surfaces courbes. *Fundamenta Math.*, 10 (1927).

[6] Zoard de Geocze, Quadrature des surfaces courbes. *Math. Naturwiss. Berichte Ung.*, 26 (1910).

[7] Kempisty, Sur la méthode triangulaire du calcul de l'aire d'une surface courbe. *Bull. Soc. Math. de France*, 1936.

[8] Young, M. W. H., On the triangulation method of defining the area of a surface. *Proc. London Math. Soc.*, 2nd series, *19* (1919).

[9] Rademacher, M., Ueber partielle und totale Differenzierbarkeit, II. *Math. Ann.*, 81 (1920).

(Reçu le 10 octobre 1966)

Dep. of Math.
New York University
New York, N.Y. 10453.