REMARKS ON A THEOREM OF A. WINTNER *

by JAMES S. W. WONG

Consider the following general second order differential equation

(1) $$x'' + f(t, x) = 0, \quad t \geqq 0,$$

where $f(t, x)$ is sufficiently smooth to guarantee existence of solutions. We are here concerned with the following result of Wintner [2] concerning the non-existence of L^2 solutions of the linear equation:

(2) $$x'' + a(t)x = 0,$$

where a (t) is a continuous function of t over the non-negative real axis $t \geqq 0$.

THEOREM A (Wintner). *If*

(3) $$\int_0^\infty t^3 \, | a(t) |^2 \, dt < \infty,$$

then (2) *cannot have any non-trivial* L^2 *solutions.*

Slightly modifying Wintner's original proof, Suyemoto and Waltman [1] obtain the following extension:

THEOREM B (Suyemoto and Waltman). *If* (3) *holds, then the differential equation*

(4) $$x'' + a(t)x^p = 0,$$

where p $\geqq$ 1, *cannot have any non-trivial* L^{2p} *solution.*

The purpose of this note is to present an extension of Winter's result for the more general equation (1). Although the basic idea of proof remains essentially the same as that of Wintner, we may by sharpening his argument, widen the applicability of Wintner's Theorem to a larger class of equations.

We assume throughout that $f(t, x)$ in addition satisfies

(5) $$|f(t, x)| \leqq a(t) \varphi(x),$$

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$\varphi(x)$ is a continuous function of x and $a(t)$ satisfies (3). Let $L\Phi$ denote the collection of all functions $x(t)$ on $[0, \infty)$, satisfying

$$\int_0^\infty |\varphi(x(t))|^2 \, dt < \infty . \tag{6}$$

THEOREM. *If the function* $\varphi(u)$ *satisfies*

$$\varphi(u) \neq o \quad \text{if} \quad u \neq 0 , \tag{7}$$

$$\underline{\lim}_{u \to \infty} \varphi(u) \geqq \alpha > 0 , \tag{8}$$

and for each L^2 *solution* $x(t)$ *of* (1), *we have*

$$\overline{\lim_{t \to \infty}} \frac{\int_t^\infty |x(s)|^2 \, ds}{\int_t^\infty |\varphi(x(s))|^2 \, ds} > 0 , \tag{9}$$

then equation (1) *has no non-trivial* $L\Phi$ *solutions.*

LEMMA. *Let* $x(t)$ *be a non-trivial* $L\Phi$ *solution of* (1). *Then* $x(t)$ *and its derivative* $x'(t)$ *tend to zero as* $t \to \infty$.

Proof. Integrating (1), we obtain

$$x'(t) - x'(s) = - \int_s^t f(\tau, x(\tau)) \, d\tau . \tag{10}$$

Using (3) and (5), we may estimate

$$|x'(t) - x'(s)|^2 \leqq \left(\int_s^t |a(s)\varphi(x(s))|^2 \, ds\right)^2 \tag{11}$$

$$\leqq \left(\int_s^t |a(s)|^2 \, ds\right)\left(\int_s^t |\varphi(x(s))|^2 \, ds\right).$$

Under the present hypothesis, the right hand side of (11) tends to zero as $t, s \to \infty$. Thus $\lim_{t \to \infty} x'(t)$ exists, say λ. If $\lambda \neq 0$, then $|x(t)| \to \infty$ as $t \to \infty$, which on account of (8) contradicts the assumption (6). Since $x'(\infty) = 0$, we may now write for any $t_0 \geqq 0$

$$\left(\int_{t_0}^\infty |x'(t)| \, dt\right)^2 \leqq \left(\int_{t_0}^\infty \left| \int_t^\infty a(s)\varphi(x(s)) \, ds \right| dt\right)^2$$

$$\leqq \left(\int_{t_0}^{\infty} |s\,a(s)\,\varphi(x(s))|\,ds\right)^2 \tag{12}$$

$$\leqq \left(\int_{t_0}^{\infty} |s\,a(s)|^2\,ds\right)\left(\int_{t_0}^{\infty} |\varphi(x(s))|^2\,ds\right) < \infty\,,$$

from which it follows that $x(t)$ is absolutely continuous, and in particular $\lim_{t\to\infty} x(t)$ exists, say μ. In case $\mu \neq 0$, we readily see that (7) contradicts the hypothesis (6). The proof is complete.

Proof of the theorem. From the fact that $|x(t)| \leqq \int_t^{\infty} |x'(t)|\,dt$, we may estimate by (12) in the following manner.

$$\int_t^{\infty} |x(t)|^2\,dt \leqq \int_t^{\infty} \left[\int_s^{\infty} |\tau\,a(\tau)\,\varphi(x(\tau))|\,d\tau\right]^2 ds$$

$$\leqq \int_t^{\infty} \left[\left(\int_s^{\infty} |\tau\,a(\tau)|^2\,d\tau\right)\left(\int_s^{\infty} |\varphi(x(\tau))|^2\,d\tau\right)\right] ds \tag{13}$$

$$\leqq \left(\int_t^{\infty} |\varphi(x(\tau))|^2\,d\tau\right)\int_t^{\infty} \tau^3\,|a(\tau)|^2\,d\tau\,.$$

Since $x(t)$ is non-trivial, we may divide $\int_t^{\infty} |\varphi(x(\tau))|^2\,d\tau$ on both sides and take lim sup which yields a contradiction on account of (9).

Corollary 1. If in place of (9), we have for some positive constant k_1

$$|\varphi(u)| \leqq k_1\,|u|^p \qquad p \geqq 1\,, \tag{14}$$

in some neighborhood of zero, say $\{u : |u| < \varepsilon_1, \varepsilon_1 > 0\}$, then the conclusion of the theorem remains valid.

Proof. Let $x(t)$ be a non-trivial $L\Phi$ solution of (1). From the above Lemma, we infer that $x(t) \to 0$ as $t \to \infty$. Thus, for sufficiently large t, we have on account of (14) that

$$|\varphi(x(t))| \leqq k_1\,|x(t)|^p \leqq k_1\,|x(t)|\,. \tag{15}$$

Substituting (15) in (9), we easily arrive at a contradiction.

Corollary 2. Let $\varphi(u)$ satisfy (7), (8), (14) and in addition

$$|\varphi(u)| \leqq k_2\,|u|^p \tag{16}$$

for some positive constant k_2 in some neighborhood of infinity say, $\{u : |u| > \frac{1}{\varepsilon_2}, \varepsilon_2 > 0\}$, where p is the same fixed number as given in (14). Then (1) has no non-trivial L^{2p} solutions.

Proof. Let $x(t)$ be a non-trivial L^{2p} solution. Denote $A = \{t : |x(t)| < \varepsilon_1\}$, $B = \{t : |x(t)| > \frac{1}{\varepsilon_2}\}$ and $C = [0, \infty) - A - B$. Observe from (14) and (16) that

$$\int_0^\infty |\varphi(x(t))|^2 \, dt = \int_A + \int_B + \int_C$$

$$\leqq \int_A k_1 |x(t)|^{2p} \, dt + \frac{1}{\varepsilon_2} \mu(C) + \int_B k_2 |x(t)|^{2p} \, dt$$

$$< \infty ,$$

where $\mu(C)$ denotes the ordinary Lebesgue measure of C. Hence $x(t) \in L\Phi$. Following the proof of the main theorem, we again arrive at a contradiction, proving that (1) has no non-trivial L^{2p} solutions.

REMARK 1. Corollary 1 reduces to the result of Suyemoto and Waltman by taking $\varphi(u) = u^p$, $p \geqq 1$ which reduces to the result of Wintner when $p = 1$.

REMARK 2. Let $\varphi(u) = 2u + \sin u$. It is easy to see that $\varphi(u)$ satisfies all (7), (8), (14) and (16). Hence, from Corollary 2, we may conclude that equation (1) has no square integrable solutions.

REFERENCES

[1] SUYEMOTO, L. and P. WALTMAN, Extension of a theorem of A. Wintner. *Proc. Amer. Math, Soc.*, 14 (1963), 970-971.

[2] WINTNER, A., A criterion for the non-existence of L^2-solutions of a nonoscillatory diffreential equation. *J. London Math, Soc.*, 25 (1950), 347-351.

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University of Alberta
Edmonton, Alberta.

Added in Proof. For a closely related paper, see J. Burlak, "On the non-existence of L^2 solutions of a class of non-linear differential equations", Proc. Edin. Math. Soc., 14(1965), 257-268, which contains several similar results.