Section 2. Topological properties

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After applying Fubini's Theorem we see that the last expression is equal to

$$\frac{q}{r}\Big(\int_0^\infty [yf(y)]^q y^{-r-1}\,dy\Big)^{1/q}.$$

The proof of the second inequality is the same except that r is replaced by $-r$.

$$\int_E |f(x)g(x)|\,dm(x) \leqq \int_0^{m(E)} f^*(t)g^*(t)\,dt. \tag{1.9}$$

Proof. We may assume f and g are non-negative simple functions. We then write $f = \Sigma f_j$ and $g = \Sigma g_k$ as in (1.5). (1.9) is clearly true for the functions $f_j\, g_k$ and the result follows.

Finally, let us note

$$\frac{1}{y}\int_0^y g(t)\,dt \leqq \frac{1}{x}\int_a^x g(t)\,dt \quad \text{for } 0 < x \leqq y, \tag{1.10}$$

where g (t) *is non-negative and non-increasing on* t > 0.

(1.10) is geometrically obvious.

Section 2. Topological properties

(1.6) implies that $f + g \in L(p, q)$ if $f, g \in L(p, q)$. Since $\|\cdot\|^*_{pq}$ is positive homogeneous we see that $L(p, q)$ is a linear space. $\|\cdot\|^*_{pq}$ leads to a topology on $L(p, q)$ such that $L(p, q)$ is a topological vector space. $f_n \to f \in L(p, q)$ in this topology if and only if $\|f - f_n\|^*_{pq} \to 0$. We shall see that this space is metrizable.

For p, q fixed we define two analogues of f^*. Choose r such that $0 < r \leqq 1$, $r \leqq q$ and $r < p$. Let

$$f^{**}(t) = f^{**}(t, r) = \begin{cases} \sup\limits_{m(E)\geqq t} \Big(\dfrac{1}{m(E)}\displaystyle\int_E |f(x)|^r\,dm(x)\Big)^{1/r}, & t \leqq m(M) \\[2ex] \Big(\dfrac{1}{t}\displaystyle\int_M |f(x)|^r\,dm(x)\Big)^{1/r}, & t > m(M). \end{cases}$$

Consider $(f^*)^{**}(t)$. Since any g^{**} is non-negative and non-increasing we can use (1.9) and (1.10) to see that

$$(f^*)^{**}(t) = (\frac{1}{t}\int_0^t [f^*(y)]^r dy)^{1/r}.$$

f^{**} leads easily to a metric on $L(p, q)$, avoiding technical difficulties which might occur when the measure m is atomic. f^{**} is also useful because for some purposes it is more closely related to f than is f^*. $(f^*)^{**}$ is especially suited for applications of Hardy's inequality.

(2.1) $$f^* \leqq f^{**}(t) \leqq (f^*)^{**}(t).$$

The first inequality in (2.1) follows from the fact that if $E = \{x \in M : |f(x)| \geqq f^*(t)\}$ then $m(E) \geqq t$. The second inequality follows from (1.9) and (1.10).

Let $\|f\|_{pq} = \|f^{**}\|^*_{pq}$.

f^*, f^{**} and $(f^*)^{**}$ are further related by

(2.2) $$\|f\|^*_{pq} \leqq \|f\|_{pq} \leqq \|f^*\|_{pq} \leqq (p/(p-r))^{1/r} \|f\|^*_{pq}.$$

(2.2) follows immediately from (2.1) and Hardy's inequality.

It is clear that

$$[(f+g)^{**}(t)]^r \leqq [f^{**}(t)]^r + [g^{**}(t)]^r,$$

so that $\rho(f, g) = \|f-g\|^r_{pq}$ is a metric on $L(p, q)$. (2.2) implies the topology of $L(p, q)$ given by $\|.\|^*_{pq}$ is equivalent to the metric topology given by $\|.\|^r_{pq}$.

(2.3) L $(p$, q) *is complete with respect to the metric* ρ (f, g) $= \|$ f$-$g $\|^r_{pq}$.

Proof. Suppose $\rho(f_m, f_n) \to 0$ as $m, n \to \infty$, where $f_n \in L(p, q)$, $n \geqq 1$. We have $\|f\|^*_{p\infty} \leqq \|f\|^*_{pq} \leqq \|f\|_{pq}$. It follows from (1.7) that the sequence $\{f_n\}$ is fundamental in measure and, hence, the exists a subsequence $\{f_{n_k}\}$ which converges almost uniformly to a function f. (See [7, p. 93].)

Fix L such that $\rho(f_n, f_L) < \varepsilon$ for $n \geqq N(\varepsilon)$. Let $\varphi_k = f_{n_k} - f_L$ and $\varphi = f - f_L$. Then φ_k converges almost uniformly to φ and by Fatou's lemma, $\varphi^{**}(t) \leqq \liminf_{k \to \infty} \varphi_k^{**}(t)$, and $\|\varphi\|^r_{pq} \leqq \liminf \|\varphi_k\|^r_{pq}$. That is, $\rho(f, f_L) < \varepsilon$. Hence, $f \in L(p, q)$ and $\rho(f, f_L) \to 0$ as $L \to \infty$.

(2.4) *Simple functions are dense in* L (p, q), $q \neq \infty$.

Proof. Suppose $f \in L(p, q)$, $p \neq \infty$. We may assume that $f \geqq 0$. We show that given any $\varepsilon, \delta > 0$ there exists a simple function f_n such that $0 \leqq f_n \leqq f$ and $(f-f_n)^*(t) \leqq \varepsilon$ for all $t \geqq \delta$. Note that $f^*(t) \to 0$ as $t \to \infty$. It follows that $m(E_\varepsilon[f]) < \infty$. Hence, we can find a simple function $f_n \geqq 0$

such that $f_n(x) = 0$ for $x \notin E_\varepsilon[f]$ and $0 \leqq f(x) - f_n(x) < \varepsilon$ for all $x \in E_\varepsilon[f]$ except for a set of measure less than δ. Then $m(\{x \in M : |f(x) - f_n(x)| > \varepsilon\}) < \delta$, so $(f-f_n)^*(t) \leqq \varepsilon$ for $t \geqq \delta$. We obtain a sequence of simple functions f_n such that $(f-f_n)^*(t) \to 0$ as $n \to \infty$ and $f_n^*(t) \leqq f^*(t)$, for each $t > 0$. We have $(f-f_n)^*(t) \leqq f^*(t/2) + f^*(t/2) \leqq 2f^*(t/2)$ and Lebesgue's Theorem on dominated convergence implies $\|f-f_n\|_{pq}^* \to 0$ as $n \to \alpha$ for $q \neq \infty$.

It is well known that a linear mapping of one Frechet space into another is continuous if and only if it maps bounded sets into bounded sets. (See [6, p. 54].) Since $\|f\|_{pq}^*$ is positive homogeneous, a linear operator T which maps $L(p, q)$ into $L(p', q')$ is continuous if and only if there exists a positive number c such that $\|Tf\|_{p'q'}^* \leqq c\|f\|_{pq}^*$, where c is independent of $f \in L(p, q)$.

Let us note the following interesting and useful result:

(2.5) *Suppose* T *is a linear operator which maps characteristic functions* χ_E, $m(E) < \infty$, *into a Banach space* B *and* $\|T\chi_E\| \leqq c\|\chi_E\|_{p1}^*$, *where* c *is independent of* χ_E. *Then there exists a unique linear extension of* T *to a continuous mapping of* L (p, 1) *into* B.

Proof. Suppose $f \geqq 0$ is a simple function. According to (1.5) we write $f = \Sigma f_n$, where $f_n = c_n \chi_{F_n}$ and $f^* = \Sigma f_n^*$. Then

$$\|Tf\| = \|T(\Sigma f_n)\| \leqq \Sigma \|Tf_n\| \leqq c\Sigma \|f_n\|_{p1}^* = c\|f\|_{p1}^* .$$

Then $\|Tf\| \leqq c'\|f\|_{p1}^*$ for any complex-valued simple function f. Since the simple functions are dense in $L(p, 1)$ we can then extend T uniquely to a bounded operator of $L(p, 1)$ into B.

It is of interest to know which of the $L(p, q)$ spaces may be considered to be Banach spaces.

(2.6) L (1, 1) *and* L (p, q), $1 < p \leqq \infty$, $1 \leqq q \leqq \infty$, *are Banach spaces for any measure space* (M, m). *For any other* p, q *there are measure spaces such that* L (p, q) *cannot be considered to be a Banach space in such a way that the topology corresponding to the norm is comparable to the metric topology.*

Proof. It is immediate that $\|.\|_{pq}^r$, with $r = 1$ is a norm. This norm is applicable to the spaces $L(p, q)$, $1 < p \leqq \infty, 1 \leqq q \leqq \infty$. $\|.\|_{11}^*$ is already a norm for $L(1, 1)$. Also, note that $\|.\|_{11}^* = \|.\|_{1\infty}^{x}$.

Let $M = (0\ \infty)$ and m be real Lebesgue measure. Since $L(p, q)$ is a Frechet space, (2.6) follows from the fact that none of the remaining spaces contain a bounded convex open set. (See [16].) This is easily seen from the following constructions:

In case $0 < q < 1$ let

$$f_k(t) = \begin{cases} 2^k & 0 < t < 2^{-kp} \\ 0 & t \geqq 2^{-kp}, \quad k \geqq 1 . \end{cases}$$

Then $\|f_k\|^*_{pq} = 1$, but $\|\frac{1}{n} \sum_{k=1}^{n} f_k\|^*_{pq} \to \infty$ as $n \to \infty$.

In case $0 < p < 1$ choose ε such that $1 < \varepsilon < \frac{1}{p}$ and let

$$f_k(t) = \begin{cases} k^{-(1/p)+\varepsilon} & k < t \leqq k+1 \\ 0 & \text{otherwise}, \ k \geqq 1 . \end{cases}$$

Then $\|f_k\|^*_{pq} \leqq 1$, but $\|\frac{1}{n} \sum_{k=1}^{n} f_k\|^*_{pq} \to \infty$ as $n \to \infty$.

In the cases where $p = 1$ divide $(0, \infty)$ into pairs of intervals

$$I_{k0}, I_{k1}\text{, where } I_{00} = (0, 1], \quad I_{01} = (1,2]$$

$$I_{k0} = (2^{k-1}(3) + (k-1)\,2^{k-1}, \quad 2^k(3) + (k-1)\,2^{k-1}] \quad \text{and}$$

$$I_{k1} = (2^{k-1}(3) + (k-1)\,2^{k-1}, \quad 2^k(3) + (k-1)\,2^k], \quad k \geqq 1. \quad \text{Let}$$

$$J_{k0} = \left(\bigcup_{i=0}^{k} I_{i0}\right) \cup \left(\bigcup_{i=0}^{k-1} I_{i1}\right) \quad \text{and } J_{k1} = I_{k1}. \text{ Note that } |J_{k0}| = |J_{k1}| .$$

If f_{k0} is zero on J_{k1} define f_{k1} by

$$f_{k1}(t) = \begin{cases} 0 & t \in J_{k0} \\ f_{k0}(t - |J_{k0}|) & t \in J_{k1} \\ f_{k0} & \text{otherwise} . \end{cases}$$

In case $q = \infty$ let

$$f_{00} = \begin{cases} 2^{k-1} & t \in I_{k0} \\ 0 & t \in I_{k1}, \quad k \geqq 0 . \end{cases}$$

In case $1 < q < \infty$ choose $\frac{1}{q} < \alpha < 1$ and let

$$f_{00}(t) = \begin{cases} 2^{-k} \cdot k^{-\alpha} & t \in I_{k0} \\ 0 & t \in I_{k1}, \quad k \geqq 0 . \end{cases}$$

The result is then seen by considering sums of the form $f_{k+1,0} = (f_{k0}+f_{k1})/2$, $k = 0, 1, \ldots$.

For the remainder of this section let us consider continuous linear functionals l on $L(p, q)$. We have $| l(f) | \leqq B \| f \|^*_{pq}$ for all $f \in L(p, q)$.

Consider $L(p, 1)$, $1 \leqq p < \infty$. Define $\mu(E) = l(\chi_E)$. $\mu(E)$ is a measure and $| \mu(E) | \leqq B \| \chi_E \|^*_{p1} = B [m(E)]^{1/p}$. Hence, μ is absolutely continuous with respect to m. The Radon-Nikadyn Theorem (see [7, p. 138]) then gives a function $g(x)$ such that $\mu(E) = l(\chi_E) = \int_M \chi_E(x)\, g(x)\, dm(x)$. This leads to $l(f) = \int_M f(x)\, g(x)\, dm(x)$ and hence $| \int_M f(x)\, g(x)\, dm(x) | \leqq B \| f \|^*_{p1}$ for all $f \in L(p, 1)$. Setting $f(x) = [\exp(-i \arg g(x))] . \chi_E(x)$ we obtain $\int_E | g(x) |\, dm(x) \leqq B [m(E)]^{1/p}$. Therefore,

$$\frac{1}{m(E)} \int_E | g(x) |\, dm(x) \leqq B [m(E)]^{-1/p'} \leqq B t^{-1/p'}$$

for $t \leqq m(E)$, where $1/p + 1/p' = 1$. It follows that $g^{**}(t) \leqq B t^{-1/p'}$, so $g \in L(p', \infty)$ and $\| g \|^*_{p'\infty} \leqq B$. (It is interesting to note how naturally g^{**} appeared in the above discussion.) Conversely, for any $g \in L(p', \infty)$, $l(f) = \int_M g(x)\, f(x)\, dm(x)$ defines a continuous linear functional on $L(p, 1)$. Since

$$| \int_M g(x) f(x)\, dm(x) | \leqq \int_0^\infty g^*(t) f^*(t)\, dt \leqq \| g \|^*_{p'\infty} \int_0^\infty t^{-1/p'} f^*(t)\, dt$$

$$= p \| g \|^*_{p'\infty} \| f \|^*_{p1} .$$

This proves that $L(p', \infty)$ is the conjugate space of $L(p, 1)$. For the same reasons that L^1 is not the conjugate space of L^∞ we cannot expect $L(p, 1)$ to be the conjugate space of $L(p', \infty)$.

Suppose now that l is a continuous linear functional on $L(p, q)$, $1<p<\infty$, $1<q<\infty$. Since $\|f\|^*_{pq} \leqq \|f\|^*_{p1}$, l is also a continuous linear function on $L(p, 1)$. Hence, there exists a function $g \in L(p', \infty)$ such that

$$(*) \qquad l(f) = \int_M f(x)\, g(x)\, dm(x) \quad \text{for all} \quad f \in L(p, 1).$$

In particular, (*) holds for all simple functions. Using (*) and $|l(f)| \leqq B\|f\|^*_{pq}$ it can be shown that $g \in L(p', q')$, $\frac{1}{p} + \frac{1}{p'} = 1$, $\frac{1}{q} + \frac{1}{q'} = 1$, and (*) holds for all $f \in L(p, q)$. Conversely, for any $g \in L(p', q')$, (*) defines a continuous linear functional on $L(p, q)$. We have obtained

(2.7) *The conjugate space of* L (p, 1) *is* L (p', ∞), *where* $\frac{1}{p} + \frac{1}{p'} = 1$.
The conjugate space of L (p, q), $1 < p < \infty$, $1 < q < \infty$, *is* L (p', q'), *where* $\frac{1}{p} + \frac{1}{p'} = 1$, $\frac{1}{q} + \frac{1}{q'} = 1$, *and hence, these spaces are reflexive.*

According to (2.5) any continuous linear functional on $L(p, q)$, $1 \leqq p < \infty$, $q < 1$, can be extended to a continuous linear functional on $L(p, 1)$.

Suppose l is a continuous linear functional on $L(p, 1)$, $0 < p < 1$. Let us assume that $m(M) < \infty$. Since (M, m) is σ-finite this will result in no loss of generality in the following argument. We have

$$|l(\chi_E)| \leqq B \|\chi_E\|^*_{p1} = B[m(E)]^{1/p} \leqq B[m(M)]^{(1/p)-1} \|\chi_E\|^*_{11}.$$

Hence, by (2.5), l can be extended to a continuous linear functional on $L(1, 1) = L^1$. Then there exists a function $g \in L^\infty$ such that $l(f) = \int_M f(x)\, g(x)\, dm(x)$ for all $f \in L^1$. Also, $\left|\int_M g(x) f(x)\, dm(x)\right| \leqq B\|f\|^*_{p1}$.
As before, we have

$$\frac{1}{m(E)} \int_E |g(x)|\, dm(x) \leqq B[m(E)]^{(1/p)-1}.$$

In case (M, m) is non-atomic this implies that $g(x) = 0$ a.e. and, hence $l \equiv 0$ on $L(p, 1)$. It follows that *the trivial functional* $l \equiv 0$ *is the only continuous linear functional on the spaces* $L(p, q)$, $0 < p < 1$, $0 < q < \infty$.

If l is a continuous linear functional on $L(1, q)$, $1 < q$, then l is a continuous linear functional on $L(1, 1) = L^1$, so there exists a function $g \in L^\infty$

such that $l(f) = \int_M f(x)\, g(x)\, dm(x)$ for all $f \in L(1,1)$ and $| \int_M f(x)\, dm(x) | \leqq B \, \|f\|^*_{1q}$. If (M, m) is non-atomic we can use this to show that $g = 0$ a.e. and, hence *the trivial functional* $l \equiv 0$ *is the only continuous linear functional on* L (1, q), $1 < q < \infty$.

Section 3. INTERPOLATION THEOREMS

Suppose T is an operator which maps $L(p_i, q_i)$ boundedly into $L(p'_i, q'_i)$, $i = 0, 1$. An interpolation theorem for $L(p, q)$ spaces can then be described as a method which leads to inequalities of the form $\| Tf \|^*_{p'q'} \leqq B \, \|f\|^*_{pq}$, B independent of $f \in L(p, q)$. The intermediate spaces $L(p, q)$ and $L(p', q')$ and the corresponding constant B are determined by the method of interpolation.

Interpolation theorems can generally be classified as either weak type or strong type. The two types of theorems are easily characterized. The weak type theorems are proved by real variable methods which utilize only minimal hypotheses. Since the weak hypotheses are characteristic of the real method of proof, the conclusions are limited. In the case of Lorentz spaces the essential part of the weak type hypothesis is that the range spaces of the given end point conditions are weak L^p spaces. We can then conclude only that an intermediate space $L(p, q)$ is mapped boundedly into an appropriate space $L(p', q')$, where $q' \geqq q$. In order to utilize a stronger hypothesis to arrive at a stronger conclusion, we must go to the complex methods of proof which are characteristic of the strong type theorems. The two methods also differ in the intermediate spaces obtained and in the behavior of the corresponding constants B. In general, we obtain more intermediate spaces by the weak type methods. However, the constants corresponding to the weak type methods are, in some sense, not as satisfactory. This is seen in the prototypes of the weak and strong type theorems, the interpolation theorem of Marcinkiewicz and the Riesz-Thorin convexity theorem.

An operator T mapping functions on a measure space into functions on another measure space is called *quasi-linear* if $T(f+g)$ is defined whenever Tf and Tg are defined and if $| T(f+g) | \leqq K(| Tf | + | Tg |)$ a.e., where K is independent of f and g. An argument similar to that which led to (1.6) gives

$$(3.1) \qquad (T(f+g))^*(t) \leqq K((Tf)^*(t/2) + (Tg)^*(t/2)).$$