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# THE STONE SPACE OF A BOOLEAN RING

by Alexander ABIAN \*1)

This is an expository paper reproducing some of the basic results in [1] and [2].

DEFINITION 1. — *A ring  $B$  is called Boolean, if*

$$x^2 = x, \quad \text{for every } x \in B. \quad (1)$$

In what follows  $B$  shall represent a given Boolean ring.

The following are well known immediate consequences of Definition 1.

$$x + x = 0, \quad (2)$$

$$xy = yx, \quad (3)$$

$$xy(x + y) = 0, \quad (4)$$

for every two elements  $x$  and  $y$  of  $B$ .

NOTATION. — *In what follows, for every non-zero element  $x$  of  $B$ ,  $p(x)$  shall represent a prime ideal of  $B$  not containing  $x$ , i.e.,  $x \notin p(x)$*

*and*

*$P(x)$  shall represent the set of all prime ideals  $p(x)$ , for a given  $x$ .*

LEMMA 1. — *Let  $I$  be an ideal of  $B$  and  $x$  an element of  $B$  such that  $x \notin I$ . Then there exists a prime ideal  $p(x)$  such that  $I \subset p(x)$ .*

PROOF. — By Zorn's Lemma, in view of (1) and (3), there exists a largest ideal  $M$  of  $B$  such that  $I \subset M$  and  $x \notin M$ . It can be easily verified that the ideal  $M$  is prime [3].

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Let us observe that since 0 is an element of every ideal of  $B$ , hence, in view of Lemma 1,

$$P(x) = \phi \quad \text{if and only if } x = 0. \quad (5)$$

Now, we prove that for every two elements  $x$  and  $y$  of  $B$ ,

$$P(xy) = P(x) \cap P(y) \quad (6)$$

To prove (6), let us observe that since  $p(xy)$  is an ideal not containing  $xy$ , hence,  $x \notin p(xy)$  and  $y \notin p(xy)$ . Thus,  $P(xy) \subset (P(x) \cap P(y))$ . Conversely, since  $p(x)$  is a prime ideal not containing  $x$ , hence, if  $y \notin p(x)$  then  $xy \notin p(x)$ . Thus,  $(P(x) \cap P(y)) \subset P(xy)$ .

From (6) it follows that for every two elements  $x$  and  $y$  of  $B$ ,

$$xy = x \quad \text{implies } P(x) \subset P(y) \quad (7)$$

Since  $p(x+y)$  is an ideal not containing  $x+y$ , hence  $x \in p(x+y)$  implies  $y \notin p(x+y)$ . Therefore, for every two elements  $x$  and  $y$  of  $B$ ,

$$P(x+y) \subset (P(x) \cup P(y)) \quad (8)$$

Further, in view of (4), (5) and (6),

$$P(xy) \cap P(x+y) = \phi$$

so that, in view of (8), for every two elements  $x$  and  $y$  of  $B$ ,

$$P(x+y) \subset (P(x) \oplus P(y)), \quad (9)$$

where  $\oplus$  is the usual set-theoretical *symmetric difference* operator. Also, let us observe that since  $p(x)$  is an ideal not containing  $x$ , hence, if  $p(x) \notin P(y)$  then  $p(x) \in P(x+y)$ . Similarly, if  $p(y) \notin P(x)$  then  $p(y) \in P(x+y)$ . Thus,

$$P(x) \oplus P(y) \subset P(x+y)$$

so that in view of (9), for every two elements  $x$  and  $y$  of  $B$ ,

$$P(x+y) = P(x) \oplus P(y). \quad (10)$$

Let us observe that since

$$P(y) - P(x) = (P(y) \oplus P(x)) \cap P(y),$$

hence, in view of (10), (6) and (1), for every two elements  $x$  and  $y$  of  $B$ ,

$$P(y) - P(x) = P(y + xy). \quad (11)$$

Also, in view of (8), for every positive natural number  $n$ ,

$$x = \sum_{i=1}^n c_i \text{ implies } P(x) \subset \bigcup_{i=1}^n P(c_i) \quad (12)$$

where  $c_i$  is an element of  $B$ . Moreover, in view of (1) and (7),

$$P(ca) \subset P(a), \quad (13)$$

where  $c$  and  $a$  are any two elements of  $B$ .

Now, let  $\mathcal{P}$  represent the set of all proper prime ideals of  $B$ .

**THEOREM 1.** — *The Boolean ring  $B$  is isomorphic to a subring of the algebra of all subsets of  $\mathcal{P}$ .*

**PROOF.** — In view of (5), (6) and (10), the mapping  $f$  from  $B$  into the power set of  $\mathcal{P}$ , given by

$$f(x) = P(x)$$

establishes the desired isomorphism.

Next, in view of (6), we introduce a topology  $\mathcal{T}$  in  $\mathcal{P}$  such that, for every  $x \in B$  the subset  $P(x)$  of  $\mathcal{P}$  is a basis element of  $\mathcal{T}$ .

**DEFINITION 2.** — *The topological space  $(\mathcal{P}, \mathcal{T})$  is called the Stone space of  $B$ .*

**LEMMA 2.** — *In the space  $(\mathcal{P}, \mathcal{T})$ , every basis element is closed.*

**PROOF.** — Let  $P(x)$  be a basis element and let  $p(y) \notin P(x)$ . Clearly,  $p(y) \in (P(y) - P(x))$  and hence in view of (11),

$$p(y) \in P(y + xy).$$

Thus, an element  $p(y)$ , in the complement  $\mathcal{P} - P(x)$  of  $P(x)$ , is contained in a basis element  $P(y + xy)$  which is disjoint from  $P(x)$ . Hence  $P(x)$  is closed.

**LEMMA 3.** — *The space  $(\mathcal{P}, \mathcal{T})$  is totally disconnected.*

**PROOF.** — Let  $p(x)$  and  $p(y)$  be two distinct elements of  $\mathcal{P}$ . Thus, there exists  $z \in B$  such that, say,  $z \in p(x)$  and  $z \notin p(y)$ . But

then  $P(yz)$  is a basis element containing  $p(y)$  and not containing  $p(x)$ . Consequently, in view of Lemma 2, every two distinct elements  $p(x)$  and  $p(y)$  of  $\mathcal{P}$  are contained in two mutually disjoint closed sets of  $(\mathcal{P}, \mathcal{T})$  whose union is  $\mathcal{P}$ . Thus,  $(\mathcal{P}, \mathcal{T})$  is totally disconnected (and in particular, Hausdorff).

LEMMA 4. — *The space  $(\mathcal{P}, \mathcal{T})$  is locally compact.*

PROOF. — It is sufficient to prove that every basis element  $P(x)$  of  $(\mathcal{P}, \mathcal{T})$  is compact. Now, let  $A \subset B$  and  $\bigcup_{y \in A} P(y)$  be a covering of  $P(x)$ , i.e.,

$$P(x) \subset \bigcup_{y \in A} P(y) \tag{14}$$

Let  $(A)$  denote the ideal generated by the elements of  $A$ . Claim that  $x \in (A)$ . Assume the contrary that  $x \notin (A)$ . But then, in view of Lemma 1, there exists a prime ideal  $p(x)$  such that  $(A) \subset p(x)$ , and therefore,  $p(x) \not\subset \bigcup_{y \in A} P(y)$ , contradicting (14).

Hence, our assumption is false and indeed,  $x \in (A)$ . Consequently, there exists a natural number  $n$  such that

$$x = \sum_{i=1}^n (m_i + b_i) a_i$$

where  $m_i$  is an integer,  $b_i \in B$  and  $a_i \in A$ . But then, in view of (12) and (13),

$$P(x) \subset \bigcup_{i=1}^n P((m_i + b_i) a_i) \subset \bigcup_{y \in A} P(y)$$

asserting that, in view of (14),  $\bigcup_{i=1}^n P((m_i + b_i) a_i)$  is a finite subcover of an arbitrary cover  $\bigcup_{y \in A} P(y)$  of  $P(x)$ . Thus, indeed,

$P(x)$  is compact and  $(\mathcal{P}, \mathcal{T})$  is locally compact.

Finally, in view of Lemmas 3 and 4, and Theorem 1, we have,

THEOREM 2. — *Every Boolean ring is isomorphic to a subring of the algebra of all subsets of its Stone space which is totally disconnected and locally compact.*

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