ON THE ALGEBRA OF THE FOUR-COLOR PROBLEM [1])

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1. A regular " map " T on a surface M^2 is a 2-dimensional cell complex whose 1-skeleton is a graph of order three and whose underlying topological space is M^2. It is a well known unsolved problem, whether or not every regular " map " T on the 2-sphere S^2 allows an *admissible 4-coloring* of its " countries ", i.e. an assignment of colors g_1, g_2, g_3, g_4 to the individual 2-cells of T, such that adjacent cells get different colors.

It was observed by Tait [5] that for a given " map " T an admissible 4-coloring of its " countries " exists if an only if there exists an admissible 3-coloring of its edges, i.e. an assignment of colors g_1, g_2, g_3 to the individual 1-cells of T, such that cells with a common endpoint get different colors. Later, Heawood [2] showed that this in turn is the case if and only if a certain system Σ of diophantine equations related to T has a solution.

The proof of these theorems (see Ringel [4] where one may also find an extensive bibliography) in terms of graph theory is rather constructive and requires the distinction of several cases. Because of the ambiguities involved in the construction, it is not clear how the different colorings are related to one another and to the different solutions of the system Σ. Thus one might expect that a treatment of the subject in terms of elementary algebraic topology, as done by Tutte [6] for a related question, should lead to more precise results.

2. An m-coloring of the r-cells of T is nothing but a function from the set T^r of these cells to an arbitrary set M of m elements. This naturally suggests considering r-chains on T over an abelian

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group G of order m. Second, it is clear that two colorings which differ only by a permutation of the colors among themselves should be regarded as the same. Therefore the main objects of our investigation will be equivalence classes of r-chains modulo the symmetric group acting on G or, equivalently, the *m-partitions* of the sets T^r.

In fact, we shall prove that the admissible 4-partitions of T^2 (the equivalence classes of admissible 4-colorings of the " countries " of T) are in one-one correspondence with the admissible 3-partitions of T^1 (theorem 2); and these in turn are in one-one correspondence with the admissible 2-partitions of T^0 (defined in 4.2) or, equivalently, with the solutions of the system Σ in a certain projective space (theorems 3 and 4).

The essential point, however, is that these correspondences of the admissible partitions are induced by the boundary operator ∂, acting on the representing r-chains over suitably chosen groups G. Consequently our main tools will be the well known homology properties of the sphere S^2 and a reduction theorem (theorem 1) on the compatibility of the respective equivalence relations with ∂. Here it is important that for $m = 4$ and $m = 3$ (and $m = 2$; see the remark 5.9) the symmetric group on m letters has a normal subgroup of order m; since this is no longer true for $m > 4$, our methods do not lead to an immediate generalization.

3. Sections I-III of the paper establish the necessary algebraical apparatus. The main results are contained in the diagrams (4.6), (5.19) and in theorem 4 of sections IV and V; in VI we finally give an example.

The following notations will be used without reference:

(I) If X and M are sets, Map (X, M) denotes the set of all functions $f: X \to M$. If G is a group, Aut (G) denotes the group of automorphisms of G.

(II) Otherwise permutation groups are denoted by script letters, the individual permutations by small greek letters. If M is a set, $\mathcal{S}(M)$ denotes the full symmetric group on M; if G is a group, $\mathcal{L}(G)$ denotes the group of left translations λ_g, $g \in G$, on G.

(III) If F is a field, F^+ denotes the additive group of F, F^* the multiplicative group of F, $\mathsf{V}_n(F)$ the vector space of n-tuples over F and $\mathsf{P}_{n-1}(F)$ the associated $(n-1)$-dimensional projective space.

I

1.1. If X is a set, an *m-partition* of X, $m \geqq 1$, is a collection of at most m nonempty disjoint subsets of X whose union is X. The set of different m-partitions of X is denoted by $P_m(X)$.

Let M be a set of $m \geqq 1$ elements. Then any function $f \in \mathrm{Map}(X, M)$ *induces* in a natural way an m-partition of X by virtue of the equivalence relation

$$x \sim y \Leftrightarrow f(x) = f(y) \quad (x, y \in X),$$

and all m-partitions of X can be obtained in this manner. Since different functions $f, h \in \mathrm{Map}(X, M)$ may very well induce the same partition of X, we have to take additional measures to get a unique representation of $P_m(X)$ by means of such functions.

1.2. If $\mathscr{P}$ is a permutation group on M, $\mathscr{P}$ induces an equivalence relation on $\mathrm{Map}(X, M)$ in virtue of

$$h \sim f(\mathscr{P}) \Leftrightarrow h = \pi \circ f, \ \pi \in \mathscr{P}. \tag{1.1}$$

The equivalence class represented by f is denoted by $\mathscr{P} \circ f$; hence

$$\mathscr{P} \circ f = \{\pi \circ f \mid \pi \in \mathscr{P}\}.$$

Correspondingly, for any collection $A \subset \mathrm{Map}(X, M)$ the quotient set under the relation (1.1) is denoted by $A/\mathscr{P}$.

Proposition 1. Let X be a set and M a set with $m \geqq 1$ elements. Then there is a natural identification

$$P_m(X) \approx \mathrm{Map}(X, M)/\mathscr{S}(M). \tag{1.2}$$

In other words: Two functions $f, h \in \mathrm{Map}(X, M)$ induce the same m-partition of X if and only if they are equivalent in the sense of (1.1) with $\mathscr{P} = \mathscr{S}(M)$.

1.3. If certain partitions of X are distinguished as *admissible*, we shall also call admissible the functions inducing such partitions and refer to admissibility by bold face type: $\mathbf{P}_m(X)$, **Map** (X, M) denote respectively the set of different admissible m-partitions of X and the set of admissible functions $f: X \to M$. Clearly (1.2) remains true if restricted to admissible partitions:

$$\mathbf{P}_m(X) \approx \mathbf{Map}(X, M)/\mathscr{S}(M). \tag{1.3}$$

1.4. If G is a group and $\mathscr{P}$ a permutation group on G, we denote by $\mathscr{P}'$ the subgroup of $\mathscr{P}$ containing those $\pi \in \mathscr{P}$ which leave the neutral element of G fixed.

Proposition 2. (cf. [1], p. 86 ff.). Let G be a group and $\mathscr{P}$ a permutation group on G which contains $\mathscr{L}(G)$ as normal subgroup. Then $\mathscr{P}$ is the semidirect product of $\mathscr{L}(G)$ and $\mathscr{P}'$; i.e. each $\pi \in \mathscr{P}$ is a unique product

$$\pi = \lambda_g \circ \pi', \quad g \in G, \pi' \in \mathscr{P}'.$$

Furthermore one has

$$\mathscr{P}' = \mathscr{P} \cap \mathrm{Aut}(G).$$

II

2.1. Let V denote the finite field of four elements $0, p, p^2, p^3 = e$.

Proposition 3. $\mathscr{L}(V^+)$ is a normal subgroup of $\mathscr{S}(V)$.

Proof. Any $\lambda_g \in \mathscr{L}(V^+)$ different from the identity is of order 2 and leaves no element fixed. Thus its cyclic representation must be of the form $(a, b)(c, d)$; whence $\mathscr{L}(V^+)$ consists of the identity and the three permutations $(0, p)(p^2, e)$, $(0, p^2)(p, e)$, $(0, e)(p, p^2)$. It is well known (see e.g. [7], p. 36) that these form a normal subgroup of the symmetric group on four letters.

Proposition 4. If $x_1, x_2, x_3 \in V^*$, then

$$x_1 + x_2 + x_3 = 0 \Leftrightarrow \{x_1, x_2, x_3\} = V^*. \tag{2.1}$$

2.2. Let Z_3 denote the field of integers modulo 3. Clearly

$$\log\colon p^m \leftrightarrows m \quad (m \in Z_3) \tag{2.2}$$

is an isomorphism of the multiplicative group V^* of V with $Z_3{}^+$ (we shall only need that log is one-one).

Proposition 5. $\mathscr{L}\,(Z_3{}^+)$ is a normal subgroup of $\mathscr{S}\,(Z_3)$.

Proof. $\mathscr{L}\,(Z_3{}^+)$ is of index 2 in $\mathscr{S}\,(Z_3)$.

Proposition 6. If $x_1, x_2, x_3 \in Z_3{}^*$, then

$$x_1 + x_2 + x_3 = 0 \Leftrightarrow x_1 = x_2 = x_3 . \qquad (2.3)$$

III

3.1. Let K be an oriented 2-dimensional cell complex whose underlying topological space is the 2-sphere S^2 and let K^r be the set of the " positive " r-cells of K:

$$K^r = \{\sigma_i^r \mid 1 \leqq i \leqq \alpha_r\} \quad (r = 0, 1, 2) .$$

If G is an abelian group and $C^r\,(G)$ the group of r-chains over G, then there is a natural identification

$$C^r\,(G) = \operatorname{Map}\,(K^r, G) \qquad (3.1)$$

which to the chain

$$f^r = \sum_i g_i\,\sigma_i^r\,, \quad g_i \in G$$

lets correspond the function given by

$$f^r\,(\sigma_i^r) = g_i \quad (\text{all } i)\,. \qquad (3.2)$$

Let

$$\partial \cdot \sigma_i^r = \sum_j \eta_{ij}^{r-1} \cdot \sigma_j^{r-1}, \quad \eta_{ij}^{r-1} \in Z$$

be the incidence relations of the complex K. Then, in the notation of (3.2), the boundary of the chain f^r becomes

$$\partial f^r\,(\sigma_j^{r-1}) = \sum_i \eta_{ij}^{r-1} f^r\,(\sigma_i^r) \quad (\text{all } j)\,. \qquad (3.3)$$

3.2. Since S^2 is an orientable manifold, we may assume that the 2-cells of K are coherently oriented. If σ_j^1 is a given 1-cell of K, we therefore obtain from (3.3) the relation

$$\partial f^2\,(\sigma_j^1) = f^2\,(\sigma_{i_1}^2) - f^2\,(\sigma_{i_2}^2)\,, \qquad (3.4)$$

where $\sigma^2_{i_1}, \sigma^2_{i_2}$ (i_1, i_2 depending on j) are the two 2-cells " adjacent " to σ^1_j.
Similarly we get for a given vertex σ^0_k of K:

$$\partial f^1 (\sigma^0_k) = \sum_\kappa \eta^0_{j_\kappa k} f^1 (\sigma^1_{j_\kappa}), \quad \eta^0_{j_\kappa k} = \pm 1, \qquad (3.5)$$

where the $\sigma^1_{j_\kappa}$ (the j_κ depending on k) are the 1-cells in the star $St\ \sigma^0_k$.

3.3. Let $Z^r(G)$ denote the group of r-cycles, i.e. of r-chains f^r with $\partial f^r = 0$. Then we have

Proposition 7. The group $Z^2(G)$ consists of the " constant " 2-chains given by

$$f^2 (\sigma^2_i) = g, \quad g \in G \quad (\text{all } i).$$

Proposition 8. A 1-chain f^1 is a boundary ∂f^2 if and only if it is a cycle or, in the notation of (3.5), if and only if

$$\sum_\kappa \eta^0_{j_\kappa k} f^1 (\sigma^1_j) = 0 \quad (\text{all } k).$$

Propositions 7 and 8 are well known consequences of the fact that S^2 can be regarded as the boundary of a 3-cell (see e.g. [3], p. 112, (21.4) and (22.4)).

3.4. *Remark.* If G is the additive group F^+ of a field F with unit element e, then the operators η^r_{ij} may be identified with the elements $\eta^r_{ij}\, e \in F$ and considered as scalar factors.

3.5. *Remark.* If G is of characteristic 2 and f^r an arbitrary r-chain over G, one has

$$f^r(-\sigma^r_i) = -f^r(\sigma^r_i) = f^r(\sigma^r_i),$$

which shows that f^r can even be considered as a function on the non-oriented cells $\bar{\sigma}^r_i$. Furthermore it is clear that in this case the η^r_{ij} need only be taken mod 2.

3.6. Now let $\mathcal{P}$ be a permutation group on G. Then the $\pi \in \mathcal{P}$ operate also on Map (K^r, G) as described in 1.2 and thus in virtue of the identification (3.1) on the chains over G.

Proposition 9. If $\pi \in \mathrm{Aut}\,(G)$, then π commutes with ∂, i.e.

$$\partial\,(\pi \circ f^r) = \pi \circ \partial f^r \quad (f^r \in \mathrm{Map}\,(K^r, G))\,.$$

Proof. The relations (3.3) immediately give

$$\partial\,(\pi \circ f^r)\,(\sigma_j^{r-1}) = \sum_i \eta_{ij}^{r-1}\,\pi\,(f^r\,(\sigma_i^r))\,. \tag{3.6}$$

Since π commutes with the operators $\eta_{ij}^{r-1} \in Z$, we may replace the right hand side of (3.6) by

$$\pi\left(\sum_i \eta_{ij}^{r-1} f^r\,(\sigma_i^r)\right) = (\pi \circ \partial f^r)\,(\sigma_j^{r-1})\,;$$

here we have used (3.3) again.

3.7. Theorem 1. Let $A^2 \subset C^2\,(G)$ be a collection of 2-chains on K and let $\mathscr{P}$ be a permutation group on G which contains $\mathscr{L} = \mathscr{L}\,(G)$ as a normal subgroup. Then the boundary operator ∂ induces a one-one correspondence

$$\Delta : A^2/\mathscr{P} \leftrightarrows \partial\,(A^2)/\mathscr{P}'\,,$$

such that the following diagram commutes:

$$\begin{array}{ccc} A^2 & \xrightarrow{\partial} & \partial\,(A^2) \\ \downarrow & & \downarrow \\ A^2/\mathscr{P} & \underset{\Delta}{\leftrightarrows} & \partial\,(A^2)/\mathscr{P}'. \end{array} \tag{3.7}$$

Proof. The theorem is an immediate consequence of the relation

$$h^2 \sim f^2\,(\mathscr{P}) \Leftrightarrow \partial h^2 \sim \partial f^2\,(\mathscr{P}')\,, \tag{3.8}$$

for then Δ is well defined by

$$\Delta\,(\mathscr{P} \circ f^2) = \mathscr{P}' \circ \partial f^2 \quad (f^2 \in A^2)$$

and has the required properties.

In order to prove (3.8), we first observe that

$$h^2 = \pi \circ f^2\,, \quad \pi \in \mathscr{P}\,,$$

is by proposition 2 equivalent to

$$h^2 = \lambda_g \circ \pi' \circ f^2 , \quad g \in G , \pi' \in \mathscr{P}' ;$$

and this in turn is equivalent to

$$\mathscr{L} \circ h^2 = \mathscr{L} \circ \pi' \circ f^2 . \tag{3.9}$$

Second, by the definition of $\mathscr{L} \circ h^2$ and proposition 7 one has

$$\mathscr{L} \circ h^2 = h^2 + Z^2 ,$$

and similarly for the right hand side of (3.9). Thus (3.9) is equivalent to

$$h^2 + Z^2 = \pi' \circ f^2 + Z^2 ,$$

and this in turn is equivalent to

$$\partial h^2 = \partial (\pi' \circ f^2) . \tag{3.10}$$

Now it follows from propositions 2 and 9 that (3.10) is the same as

$$\partial h^2 = \pi' \circ \partial f^2 , \quad \pi' \in \mathscr{P}' ;$$

whence (3.8) is proven.

IV

4.1. Let T be a cell complex homeomorphic to S^2, as considered in section III, and assume that the 1-skeleton of T is a graph of order three; thus T is a regular " map " in the sense of the introduction. It follows that each star $St\ \sigma_k^0$ contains exactly three 1-cells and three 2-cells which we shall denote by $\sigma_{j\kappa}^1$ and $\sigma_{i\kappa}^2$ ($\kappa = 1, 2, 3$; the i_κ, j_κ depending on k), respectively.

4.2. Using the incidences of the complex T, we distinguish certain partitions of the sets $T^r = \{ \sigma_i^r \mid 1 \leqq i \leqq \alpha_r \}$ $(r = 0, 1, 2)$ of r-cells as *admissible*:

(I) A 4-partition of T^2 is admissible, if for each 1-cell $\sigma_j^1 \in T^1$ the two " adjacent " 2-cells are in different subsets of T^2.

(II) A 3-partition of T^1 is admissible, if for each vertex $\sigma_k^0 \in T^0$ the three 1-cells $\sigma_{j\kappa}^1$ of $St\ \sigma_k^0$ are respectively in the three subsets of T^1.

(III) A 2-partition of T^0 is admissible, if for each 2-cell $\sigma_i^2 \in T^2$ the vertices of σ_i^2 are partitioned in such a way that their numbers in the two subsets of T^0 differ by a multiple of 3.

In view of (1.3), we may write for the partitions under (I) and (II) above:

$$\mathbf{P}_4 (T^2) \approx \mathbf{Map} (T^2, V)/\mathscr{S} (V) , \qquad (4.1)$$
$$\mathbf{P}_3 (T^1) \approx \mathbf{Map} (T^1, V^*)/\mathscr{S} (V^*) ,$$

where V is again the finite field of four elements considered in 2.1. Now by (3.1) the maps appearing in (4.1) can be regarded at the same time as chains over V, and we may apply the results of 3.3 and 3.7.

4.3. Lemma 1. (Tutte [6]) In the sense of (3.1) one has

$$\partial (\mathbf{Map} (T^2, V)) = \mathbf{Map} (T^1, V^*) ; \qquad (4.2)$$

that is to say: $f^1 \in \mathrm{Map} (T^1, V^*)$ is admissible on T^1 if and only if $f^1 = \partial f^2$ and $f^2 \in \mathrm{Map}\ (T^2, V)$ is admissible on T^2.

Remark. Since V is of characteristic 2, remark 3.5 applies. In particular it follows that the partition of T^1 induced by ∂f^2 is independent of the orientation adopted on the individual 1-cells of T.

Proof of Lemma 1. If f^2 is admissible, then by 4.2 (I) and (3.4) $\partial f^2 = f^1$ is different from 0 on all 1-cells of T, i.e. $f^1 \in \mathrm{Map}\ (T^1, V^*)$.
Let

$$f^1 (\bar{\sigma}_j^1) = x_j \in V^* \quad (\text{all}\, j) \qquad (4.3)$$

and consider a vertex σ_k^0 of T. Then by proposition 8 and (3.5) we have

$$\sum_{\kappa} f^1 (\bar{\sigma}_{j\kappa}^1) = x_{j_1} + x_{j_2} + x_{j_3} = 0 , \qquad (4.4)$$

and thus by (2.1):

$$\{x_{j_1}, x_{j_2}, x_{j_3}\} = V^* . \qquad (4.5)$$

Together with 4.2 (II) this proves that f^1 is admissible.

Conversely, let $f^1 \in \mathbf{Map}\ (\mathrm{T}^1, V^*)$ be given by (4.3). By assumption, for each vertex σ_k^0 one has (4.5); thus (4.4) follows from (2.1). Using proposition 8 again, we get

$$f^1 = \partial f^2,\ f^2 \in C^2(V) = \text{Map}(T^2, V).$$

Finally, f^2 is admissible because $\partial f^2 = f^1$ is nonzero on all 1-cells of T.

4.4. Theorem 2. The boundary operator ∂ induces a one-one correspondence Δ of the different admissible 4-partitions of T^2 and the different admissible 3-partitions of T^1, as given by (4.1), such that the following diagram commutes:

$$\begin{array}{ccc} \mathbf{Map}\,(T^2, V) & \xrightarrow{\partial} & \mathbf{Map}(T^1, V^*) \\ \downarrow & & \downarrow \\ \mathbf{P}_4\,(T^2) & \underset{\Delta}{\rightleftarrows} & \mathbf{P}_3\,(T^1)\,. \end{array} \tag{4.6}$$

Proof. Since $\mathcal{L}(V^+)$ is a normal subgroup of $\mathcal{S}(V)$ (proposition 3), we may apply theorem 1, resp. diagram (3.7), with $A^2 = \mathbf{Map}\,(T^2, V)$ and $\mathcal{P} = \mathcal{S}(V)$. Because of (4.1) and (4.2) the diagram (3.7) goes over into (4.6) but for the lower right entry, where we at first get

$$\partial\,(A^2)/\mathcal{P}' = \mathbf{Map}\,(T^1, V^*)/\mathcal{S}'(V).$$

4.5. Now the permutations in $\mathcal{S}'(V)$, if restricted to V^*, are just the permutations of V^*; i.e.

$$\mathcal{S}'(V)\,|_{V^*} = \mathcal{S}(V^*).$$

Together with (4.1) this shows

$$\mathbf{Map}\,(T^1, V^*)/\mathcal{S}'(V) = \mathbf{Map}\,(T^1, V^*)\,/\mathcal{S}(V^*) \approx \mathbf{P}_3\,(T^1).$$

V

5.1. In order to establish our next main result we shall need the $(\alpha_2 \times \alpha_0)$-matrix $[a_{ik}]$ given by

$$a_{ik} = \begin{cases} 1 & (\sigma_i^2 \text{ has } \sigma_k^0 \text{ as a vertex}) \\ 0 & (\text{otherwise}) \end{cases}$$

(it is clear that $[a_{ik}]$ could also be defined in terms of the matrices $[\eta^r_{ij}]$). Since every σ^0_k is a vertex of exactly three 2-cells σ^2_i, the matrix $[a_{ik}]$ has exactly three entries 1 in each column and the rest zeros.

5.2. Now we modify the complex T and obtain a new complex $\hat{T}$ as follows:

(I) As a point set $\hat{T}$ coincides with T, i.e. with S^2.

(II) In the interior of each 2-cell σ^2_i choose a point τ^0_i $(1 \leqq i \leqq \alpha_2)$. The vertices of $\hat{T}$ are given by the union of the vertices σ^0_k $(1 \leqq k \leqq \alpha_0)$ of T and the τ^0_i $(1 \leqq i \leqq \alpha_2)$.

(III) The 1-cells of $\hat{T}$ are designed by τ^1_{ik} (all i, k) and defined as follows: (*a*) Each 2-cell σ^2_i is triangulated in an obvious way by joining the point τ^0_i with each vertex σ^0_k of σ^2_i. Thus a 1-cell τ^1_{ik} is generated whenever σ^2_i has σ^0_k as a vertex, i.e. whenever $a_{ik} = 1$. We orient the τ^1_{ik} in such a way that

$$\partial \cdot \tau^1_{ik} = \cdot\, \tau^0_i - \cdot\, \sigma^0_k\,. \tag{5.1}$$

(*b*) If σ^0_k is not a vertex of σ^2_i, i.e. if $a_{ik} = 0$, τ^1_{ik} is defined to be the zero 1-chain.

(IV) If we now delete the 1-cells σ^1_j of T, then for each σ^1_j the two " adjacent " triangles pair off to a rhombuslike quadrangle which we shall denote by τ^2_j $(1 \leqq j \leqq \alpha_1)$. $\hat{T}^2$ is defined to be the set of these τ^2_j; we may again assume that they are coherently oriented.

This completes the construction of $\hat{T}$. Note that now the star $St\ \sigma^0_k$ consists of the three 1-cells $\tau^1_{i_\kappa k}$ and the three 2-cells $\tau^2_{j_\kappa}$, where the i_κ, j_κ are the same as in 4.1. Furthermore it follows from (3.5) and (5.1) that for any 1-chain $\hat{f}^1 \in \hat{C}^1(G)$ we have

$$\partial f^1 (\sigma^0_k) = -\sum_\kappa f^1 (\tau^1_{i_\kappa k}) \quad (\text{all } k)\,. \tag{5.2}$$

Similarly at the vertices τ^0_i:

$$\partial f^1 (\tau^0_i) = \sum_\lambda f^1 (\tau^1_{ik_\lambda}) \quad (\text{all } i)\,, \tag{5.3}$$

the k_λ depending on i and determined by the condition $a_{ik_\lambda} = 1$.

5.3. The one-one correspondence

$$\sigma_j^1 \leftrightarrows \tau_j^2 \quad (1 \leqq j \leqq \alpha_1) \tag{5.4}$$

of T^1 and $\hat{T}^2$ induces a trivial one-one correspondence

$$\Lambda\colon P_m(T^1) \leftrightarrows P_m(\hat{T}^2) \quad (m \geqq 1) \tag{5.5}$$

of the respective partitions. It is natural to call a 3-partition of $\hat{T}^2$ *admissible*, if the corresponding 3-partition of T^1 is admissible. A comparison with 4.2 (II) shows that this is the case if and only if the three 2-cells $\tau_{j\kappa}^2$ of each star $St\,\sigma_k^0$ are respectively in the three subsets of $\hat{T}^2$. Since these $\tau_{j\kappa}^2$ are " separated " by the $\tau_{i\kappa k}^1$, this in turn is equivalent to the following:

($\hat{\text{I}}$I) A 3-partition of $\hat{T}^2$ is admissible, if for each nonzero 1-cell τ_{ik}^1 the two " adjacent " 2-cells $\tau_{j_1}^2$, $\tau_{j_2}^2$ (j_1, j_2 depending on i, k) are in different subsets of $\hat{T}^2$.

5.4. In $\hat{T}$ we consider chain groups

$$\hat{C}^r(Z_3) = \mathrm{Map}\,(\hat{T}^r, Z_3) \tag{5.6}$$

over the field Z_3; the boundary operator is again denoted by ∂. We remark that (2.2) and (5.4) induce a natural one-one correspondence

$$\lambda : \mathrm{Map}\,(T^1, V^*) \leftrightarrows \mathrm{Map}\,(\hat{T}^2, Z_3),\ \lambda(f^1) = \hat{f}^2,$$

from the functions on the (nonoriented) 1-cells of T with values in V^* to the functions (5.6) on the 2-cells of $\hat{T}$ via

$$\hat{f}^2(\tau_j^2) = \log(f^1(\bar{\sigma}_j^1)) \quad (1 \leqq j \leqq \alpha_1).$$

Since λ induces the trivial correspondence (5.5) between the partitions of T^1 and $\hat{T}^2$, it follows in particular from (4.1) and

$$\mathbf{P}_3(\hat{T}^2) \approx \mathbf{Map}(\hat{T}^2, Z_3)/\mathscr{S}(Z_3) \tag{5.7}$$

that the diagram

$$\begin{array}{ccc} \mathbf{Map}(T^1, V^*) & \xrightarrow{\ \lambda\ } & \mathbf{Map}(\hat{T}^2, Z_3) \\ \downarrow & & \downarrow \\ \mathbf{P}_3(T^1) & \underset{\Lambda}{\rightleftarrows} & \mathbf{P}_3(\hat{T}^2) \end{array} \tag{5.8}$$

commutes.

5.5. Lemma 2. The set

$$\hat{A}^1 = \partial\,(\mathbf{Map}\,(\hat{T}^2, Z_3)) \tag{5.9}$$

of boundaries of admissible functions $\hat{f}^2$ consists exactly of the functions $\hat{f}^1$ of the form

$$\hat{f}^1\,(\tau^1_{ik}) = x_k \neq 0 \quad (\tau^1_{ik} \neq 0)\,, \tag{5.10}$$

where the x_k satisfy the system of equations

$$\sum_{k=1}^{\alpha_0} a_{ik}\, x_k = 0 \pmod 3 \quad (1 \leqq i \leqq \alpha_2)\,. \tag{5.11}$$

Remark. Remark 3.4 applies also to the a_{ik}; thus (5.11) can be considered as a system of linear equations over the field Z_3.

Proof of Lemma 2. If $\hat{f}^2$ is admissible, then by 5.3 $(\hat{\text{II}})$ and (3.4) $\partial \hat{f}^2 = \hat{f}^1$ is different from zero on all nonzero 1-cells of $\hat{T}$:

$$\hat{f}^1\,(\tau^1_{ik}) = x_{ik} \neq 0 \quad (\tau^1_{ik} \neq 0)\,. \tag{5.12}$$

Consider now a vertex σ^0_k of $\hat{T}$. By (5.2) and proposition 8 we have

$$-\sum_{\kappa} \hat{f}^1\,(\tau^1_{i_\kappa k}) = -\,(x_{i_1 k} + x_{i_2 k} + x_{i_3 k}) = 0, \;\; x_{i_\kappa k} \neq 0\,.$$

From (2.3) it follows that the $x_{i_\kappa k}$ are equal, and if we denote their common value by x_k, (5.12) goes over into (5.10).

Second, for a fixed vertex τ^0_i, (5.3) and proposition 8 together with (5.10) yield

$$\sum_{\lambda} \hat{f}^1\,(\tau^1_{ik_\lambda}) = \sum_{\lambda} x_{k_\lambda} = 0\,, \tag{5.13}$$

which by definition of the k_λ is nothing else but (5.11).

Conversely, let $\hat{f}^1$ have properties (5.10) and (5.11). Then we have for a fixed k:

$$-\sum_{\kappa} \hat{f}^1\,(\tau^1_{i_\kappa k}) = -\,(x_k + x_k + x_k) = 0\,. \tag{5.14}$$

Second, for a fixed i, (5.11) implies (5.13) by definition of the k_λ.

But (5.14) and (5. 13), together with (5.2), (5.3) and proposition 8 imply

$$\hat{f}^1 = \partial \hat{f}^2 , \ \hat{f}^2 \in \hat{C}^2 (Z_3) = \mathrm{Map}\,(T^2, Z_3) .$$

It is clear that $\hat{f}^2$ is admissible, because $\partial \hat{f}^2 = \hat{f}^1$ is different from zero on all nonzero 1-cells of $\hat{T}$.

5.6. Lemma 2 suggests considering the $\hat{f}^1 \in \hat{A}^1$ not as functions on the τ^1_{ik} but as functions on the σ^0_k, the vertices of the original complex T. Thus we introduce the injection

$$\vartheta : \hat{A}^1 \to \mathrm{Map}\,(T^0, Z_3), \ \ \vartheta\,(\hat{f}^1) = f^0,$$

defined by

$$f^0\,(\sigma^0_k) = x_k \ \ (1 \leqq k \leqq \alpha_0) , \tag{5.15}$$

where the x_k are given by lemma 2. It is obvious that ϑ induces an injection Θ of the equivalence classes mod $\mathcal{S}'\ (Z_3)$, such that the diagram

$$\begin{array}{ccc} \hat{A}^1 & \xrightarrow{\ \vartheta\ } & \mathrm{Map}\,(T^0, Z_3) \\ \downarrow & & \downarrow \\ \hat{A}^1 / \mathcal{S}'(Z_3) & \xrightarrow[\Theta]{} & \mathrm{Map}(T^0, Z_3)/\mathcal{S}'(Z_3) \end{array} \tag{5.16}$$

commutes.

Lemma 3.

$$\vartheta\,(\hat{A}^1) = \mathbf{Map}\,(T^0, Z_3^*) .$$

Proof. Let $\hat{f}^1 \in \hat{A}^1$. First it is clear that $\vartheta\,(\hat{f}^1) = f^0$ has values in Z_3^* only. Second, for a fixed i, $1 \leqq i \leqq \alpha_2$, consider the equation (5.11) and write it in the form

$$x_{k_1} + x_{k_2} + \ldots + x_{k_n} = 0\ (\mathrm{mod}\ 3) , \ x_{k_\lambda} \neq 0 , \tag{5.17}$$

where the k_λ are again determined by the condition $a_{ik_\lambda} = 1$, i.e. that $\sigma^0_{k_\lambda}$ should be a vertex of σ^2_i. It follows that the numbers of positive and of negative terms in (5.17) differ by a multiple of 3, which shows that the vertices of σ^0_i are partitioned as required in 4.2 (III). Thus f^0 is admissible.

Conversely, let $f^0 \in \mathbf{Map}\,(T^0, Z_3^*)$ be given by (5.15). Since f^0 is admissible, for each i equation (5.17) is fulfilled and thus, by definition of the k_λ, so is equation (5.11). This shows that $f^0 = \vartheta(\hat{f}^1)$, where $\hat{f}^1$ is in $\hat{A}^1$ and is given by (5.10).

5.7. *Remark.* We repeat that an f^0 given by (5.15) is admissible if and only if (*a*) f^0 sums to 0 on the vertices of each 2-cell σ_i^2 and (*b*) $x_k \neq 0$ (all k). Furthermore we note that as a consequence of (1.3) one has

$$\mathbf{P}_2\,(T^0) \approx \mathbf{Map}\,(T^0, Z_3^*)/\mathscr{S}\,(Z_3^*)\,. \qquad (5.18)$$

5.8. We are now ready to state

Theorem 3. The boundary operator ∂ on $\hat{T}$, composed with ϑ, induces a one-one correspondence $\Theta \circ \Delta$ of the different admissible 3-partitions of $\hat{T}^2$, as given by (5.7), with the different admissible 2-partitions of T^0, as given by (5.18), such that the following diagram commutes:

$$\begin{array}{ccccc} \mathbf{Map}\,(T^2, Z_3) & \xrightarrow{\partial} & \hat{A}^1 & \xleftrightarrow{\theta} & \mathbf{Map}\,(T^0, Z_3^*) \\ \downarrow & (a) & \downarrow & (b) & \downarrow \\ \mathbf{P}_3\,(\hat{T}^2) & \xleftrightarrow[\Delta]{} & \hat{A}^1/\mathscr{S}'(Z_3) & \xleftrightarrow[\Theta]{} & \mathbf{P}_2\,(T^0)\,. \end{array} \qquad (5.19)$$

Proof. First we remark that $\mathscr{L}\,(Z_3^+)$ is a normal subgroup of $\mathscr{S}\,(Z_3)$ (proposition 5). If we apply theorem 1, resp. diagram (3.7), with $A^2 = \mathbf{Map}\,(\hat{T}^2, Z_3)$ and $\mathscr{P} = \mathscr{S}\,(Z_3)$, the square (*a*) follows from (5.7) and (5.9).

As for the square (*b*), (5.16) and Lemma 3 immediately give

$$\begin{array}{ccc} \hat{A}^1 & \xleftrightarrow{\theta} & \mathbf{Map}\,(T^0, Z_3^*) \\ \downarrow & & \downarrow \\ \hat{A}^1/\mathscr{S}'(Z_3) & \xleftrightarrow[\Theta]{} & \mathbf{Map}\,(T^0, Z_3^*)/\mathscr{S}'(Z_3)\,. \end{array}$$

But the same argument as in 4.5 shows that here the lower right entry may be replaced by $\mathbf{Map}\,(T^0, Z_3^*)/\mathscr{S}\,(Z_3^*)$, which by (5.18) is nothing but $\mathbf{P}_2\,(T^0)$.

5.9. *Remark.* It is fairly obvious that a third reduction step, using $Z_2^+ \approx Z_3^*$ as coefficient group and the coboundary operator d instead of ∂, would finally establish a one-one correspondence of $\mathbf{P}^2(T^0)$ [$\approx$ **Map** $(T^0, Z_2)/\mathcal{S}(Z_2)$] with a certain collection of 1-chains mod 2 on T, because $\mathcal{S}(Z_2) = \mathcal{L}(Z_2^+)$. We shall not carry this out in detail, since the transfer of the system (5.11) will not lead to a simplification.

5.10. The contents of theorems 2 and 3 may be summarized in

Theorem 4. The admissible 4-partitions of T^2 are in one-one correspondence with the solutions of the system

$$(a) \sum_{k=1}^{\alpha_0} a_{ik} x_k = 0 \; (1 \leqq i \leqq \alpha_2) \quad (b) \; x_k \neq 0 \quad (\text{all } k) \qquad (5.20)$$

in $\mathsf{P}_{\alpha_0 - 1}(Z_3)$.

Proof. Diagrams (4.6), (5.8) and the first part of (5.19) show immediately that $\mathbf{P}_4(T^2)$ is in one-one correspondence with $\hat{A}^1/\mathcal{S}'(Z_3)$.

Second it is clear from Lemma 2 and the subsequent remark that $\hat{A}^1$ is in natural one-one correspondence with the solutions of (5.11), resp. (5.20 a), in $\mathsf{V}_{\alpha_0}(Z_3)$. Note that in $\hat{A}^1$ the $\pi \in \mathcal{S}'(Z_3)$ operate simultaneously on all x_k, and this is still true after the injection of $\hat{A}^1$ into $\mathsf{V}_{\alpha_0}(Z_3)$.

Therefore, by definition of $\mathsf{P}_{\alpha_0 - 1}(Z_3)$, one only has to verify that $\mathcal{S}'(Z_3)$ coincides with $\mathbf{Z}_3^*$, acting on Z_3.

VI

6.1. As an example we consider the complex formed by the 2-skeleton of an n-sided prism S, $n \geqq 3$. Let x_k, y_k $(1 \leqq k \leqq n)$ be the values of an admissible f^0 (cf. the remark 5.7) on the vertices of the top and of the bottom face of S, respectively. Then it is easily seen that (5.20) goes over into the following system of equations and constraints:

$$x_1 + x_2 + \ldots + x_n = 0 \,, \; x_k \neq 0 \quad (\text{all } k) \,, \qquad (6.1)$$

$$y_1 + y_2 + \ldots + y_n = 0 \,, \; y_k \neq 0 \quad (\text{all } k) \,, \qquad (6.2)$$

$$y_k + y_{k+1} = - x_k - x_{k+1} \quad (\text{all } k) \,; \qquad (6.3)$$

here and in the sequel the index k is taken mod n wherever it occurs.

Lemma 4. If the x_k are given and satisfy (6.1), then (6.2), (6.3) admit only the solution

$$y_k = -x_k \quad (\text{all } k); \tag{6.5}$$

except, if n is even and

$$x_k + x_{k+1} = 0 \quad (\text{all } k), \tag{6.6}$$

in which case

$$y_k = x_k \quad (\text{all } k) \tag{6.7}$$

is an additional solution of (6.2), (6.3).

Proof. We only have to show that there are no other solutions. Thus let the y_k satisfy (6.2), (6.3). If one has $y_{k_0} = -x_{k_0}$ for some k_0, then (6.5) follows from (6.3) by induction. If, on the other hand, one has (6.7), then (6.3) implies (6.6). Since (6.6) is the same as $x_{k+1} = -x_k$ (all k), this is possible only for even n.

6.2. Let B_n be the number of solutions of (6.1). Then we immediately obtain the following

Corollary. The number of different solutions of the system (6.1)-(6.3) is B_n (n odd), resp. B_n+2 (n even).

Lemma 5.

$$B_n = \frac{1}{3}(2^n + 2(-1)^n). \tag{6.8}$$

Proof. (6.1) is the same as

$$x_n = -(x_1 + x_2 + \ldots + x_{n-1}), \quad x_k \neq 0 \quad (\text{all } k),$$

which shows that the solutions of (6.1) are in one-one correspondence with the solutions of

$$x_1 + x_2 + \ldots + x_{n-1} \neq 0, \, x_k \neq 0 \quad (1 \leqq k \leqq n-1).$$

But the number of these is easily seen to be $2^{n-1} - B_{n-1}$; whence we obtain the difference equation

$$B_n = 2^{n-1} - B_{n-1} \ (n > 1), \ B_1 = 0 \qquad (6.9)$$

for the B_n. Now (6.8) is just the solution of (6.9).

6.3. Thus Theorem 4 and the corollary to Lemma 4 yield

Theorem 5. There are exactly

$$\frac{1}{3}(2^{n-1} - 1) \ (n \text{ odd}), \quad \text{resp.} \ \frac{1}{3}(2^{n-1} + 4) \ (n \text{ even})$$

different admissible 4-partitions of the faces of an n-sided prism.

6.4. We now return to theorem 4. It has already been observed by Heawood in his original paper [2] that the rank of the matrix $[a_{ik}]$ is $\alpha_2 - 1$ in general and $\alpha_2 - 3$ in the somewhat singular case, where $[a_{ik}]$ has row sums 0 (mod. 2), i.e. where each 2-cell of T has an even number of vertices.

If we consider for the moment $\mathsf{V}_{\alpha_0}(Z_3)$ as a probability space with constant point measure $1/3^{\alpha_0}$, then it is clear that the subspace U defined by the homogeneous system (5.20 a) has probability $\left(\frac{1}{3}\right)^{\alpha_2 - 1}$ (in the general case). If we now assume that the constraints (5.20 b) are independent relative to U (which they certainly aren't; see however (6.8) for the case of only one equation), we obtain immediately the following expectancy for the number of solutions of (5.20) in $\mathsf{P}_{\alpha_0 - 1}(Z_3)$:

$$\frac{1}{2} \cdot 3^{\alpha_0} \cdot \left(\frac{1}{3}\right)^{\alpha_2 - 1} \cdot \left(\frac{2}{3}\right)^{\alpha_0}.$$

Note that in our case $3\alpha_0 = 2\alpha_1$, which together with Euler's formula gives $\alpha_0 = 2\alpha_2 - 4$. Consequently we finally get as a heuristic estimate for the cardinality of $\mathbf{P}_4(T^2)$ the expression

$$E(\alpha_2) = \frac{3}{32}\left(\frac{4}{3}\right)^{\alpha_2}.$$

Note, however, that for arbitrary $\alpha_2 \geqq 4$ there are " maps " with α_2 " countries " on the 2-sphere, for which crd $\mathbf{P}_4(T^2) = 1$.

REFERENCES

1. M. Hall, *The Theory of Groups*, New York 1959.
2. P. J. Heawood, On the Four-Colour Map Theorem. *Quart. J. Math. 29*, 270-285 (1898).
3. S. Lefschetz, Algebraic Topology. *Am. Math. Soc. Coll Publ.*, vol. 27, New York 1942.
4. G. Ringel, Färbungsprobleme auf Flächen und Graphen. *Math. Monogr.*, 2, Berlin 1959.
5. P. G. Tait, Note on a Theorem in Geometry of Position. *Trans. Royal Soc. Edinbourgh 29*, 657-660 (1880).
6. W. T. Tutte, On the Imbedding of Linear Graphs in Surfaces. *Proc. London Math. Soc. (II) 50*, 474-483 (1950).
7. B. L. van der Waerden, *Algebra I*, 4. Aufl., Berlin 1955.

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