VI. Comments on Pringsheim's discussion of the problem

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37. Assuming that (35) is satisfied we take x as an integer v and obtain, integrating with respect to y from $\gamma v + A - \gamma$ to $\gamma v + A$:

$$\gamma f(x) \geqq K \int_{\gamma v + A - \gamma}^{\gamma v + A} f(y)\, dy\,,$$

and therefore, for a convenient integer n_0,

$$\frac{\gamma}{K} \sum_{v=n_0}^{n} f(v) \geqq \sum_{v=n_0}^{n} \int_{\gamma v + A - \gamma}^{\gamma v + A} f(y)\, dy = \int_{\gamma n_0 + A - \gamma}^{\gamma n + A} f(y)\, dy\,.$$

From this inequality the assertion corresponding to the condition (35) follows immediately. The Theorem 8 is proved.

38. Corollary. *Assume $f(x)$ non-negative, finite and integrable in any finite subinterval of $\langle x_0, \infty)$. If there exists an integer N such that $x^N f(x)$ is from a certain x on either monotonically increasing or monotonically decreasing, the series (18) converges or diverges according as the integral (2) is convergent or divergent.*

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39. Although Ermakof's convergence and divergence criteria and in particular Ermakof's second proof, using Abel's functional equation, are extremely interesting, they remained very little known and it appears that the author's paper [5] was the first in which the problem was taken up in a modern way. The reason for this may lie partly in the very negligent way in which Ermakof's notes were written and partly in some erroneous and misleading statements about this problem which were formulated by Pringsheim in [6], [7] and [8]. Although the essential merit of Ermakof's second paper consists just in the fact that the function $f(x)$ need not be assumed as monotonic — it is true that Ermakof does not even mention this point in [2] — Pringsheim says in [7], pp. 308-309: -"Es ist mir neuerdings gelungen, dieselben [*that is Ermakof's criteria*] von einer ihnen (auch in der von Herrn Ermakoff gegebenen Darstellung)

anhaftenden, sehr wesentlichen Beschränkung, nämlich der *ausschliesslichen* Anwendbarkeit auf Reihen mit *niemals zunehmenden* Gliedern zu befreien, und zwar lassen sie sich auch in dieser *erweiterten* Form mit Hilfe der oben charakterisierten, in meiner Abhandlung durchgeführten Methode ableiten."

Further, on p. 327 of [7] Pringsheim says after having discussed the case that $f(x)$ is never increasing, in a footnote: " Dies ist der von Herrn Ermakoff ausschliesslich betrachtete Fall."

The same is implied in the statement about Ermakof's criteria in [8] on p. 89: " Die letztere habe ich neuerdings in der Weise verallgemeinert, dass $f(x)$ nicht mehr als *monoton* vorausgesetzt zu werden braucht."

It is obvious that the reader of the last statement cannot help believing that while Ermakof did assume the monotony of $f(x)$, Pringsheim in his paper [7] quoted proved that this assumption can be dropped.

On the other hand, what Pringsheim did in [7] with Ermakof's criteria can be reduced to the observation that the transition from (2) to (18) in the Euler-Maclaurin theorem can be achieved if we have

$$\frac{f(\nu+\theta)}{f(\nu)} \to 1 \ (\nu \to \infty\ , \quad 0 \leqq \theta \leqq 1)$$

for natural ν, uniformly in θ.

This is certainly a pretty unfair way to deal with the ingenious proof of Ermakof and the beautiful result given in his paper [2].

40. However, Pringsheim derived the above result which is, of course, a very special case of our Theorem 8, from an elegant " Corollary " to Ermakof's criteria. In this " Corollary " the expression (1) is replaced by:

$$\frac{f([\Psi(x)])\ \Psi'(x)}{f([x])},$$

where, as usual, $[x]$ denotes the greatest integer contained in x.

This expression is of interest since only the values of f for integer arguments enter into it, and in this discussion the assumption about the monotony of $f(x)$ is not necessary. On the other hand, it is pretty difficult to handle if $f(x)$ is given by an analytic expression.

Since Pringsheim's formulation of this " Corollary " is too special we derive in the sections 44-45 a generalized form of it.

41. In the paper [6] Pringsheim gives a very general convergence criterion which is also mentioned in [7] and [8]. This criterion uses, in the notations of sec. 7-12 and assuming that $\psi(x) \leqq \Psi(x)$ $(x \geqq x_1)$, the expression

$$\varphi_h(x) = \int_{\Psi(x)}^{\Psi(x+h)} f(x)\,dx \Bigg/ \int_{\psi(x)}^{\psi(x+h)} f(x)\,dx \tag{37}$$

for a fixed $h > 0$. Pringsheim proves, that if $\lim\limits_{x\to\infty} \varphi_h(x)$ is > 1, the integral (2) diverges, while this integral is convergent if $\lim\limits_{x\to\infty} \varphi_h(x) < 1$.

In quoting this result in [7] and [8] Pringsheim says that Ermakof's result follows from his for $h \to 0$. This is, of course, not correct since in this passage to the limit something like the uniform differentiability of $\Psi(x)$ in the infinite interval (x, ∞) has to be used. As a matter of fact, Pringsheim mentiones this restriction in his first publication [6], while in [7] and [8] any reference to this restriction is omitted.

42. We give in what follows a proof of Pringsheim's criterion in a generalized form, avoiding the assumption that $\lim\limits_{x\to\infty} \varphi_h(x)$ exists. We prove:

If for a positive ε *from an* $x = x_1$ *on we have*

$$\varphi_h(x) \geqq 1 + \varepsilon \quad (x \geqq x_1), \tag{38}$$

the integral (2) diverges, while this integral converges if we have from an $x \geqq x_1$ *on:*

$$\varphi_h(x) \leqq 1 - \varepsilon \quad (x \geqq x_1). \tag{39}$$

Proof. From (38) follows obviously for a natural n:

$$1+\varepsilon \leqq \sum_{v=1}^{n} \int_{\Psi(x_1+(v-1)h)}^{\Psi(x_1+vh)} f(x)\,dx \Bigg/ \sum_{v=1}^{n} \int_{\psi(x_1+(v-1)h)}^{\psi(x_1+vh)} f(x)\,dx$$

$$= \int_{\Psi(x_1)}^{\Psi(x_1+nh)} f(x)\,dx \Bigg/ \int_{\psi(x_1)}^{\psi(x_1+nh)} f(x)\,dx .$$

If we replace in the numerator of the right hand quotient $\Psi(x_1)$ by $\psi(x_1) \leqq \Psi(x_1)$ this quotient is not decreased and we have

$$\int_{\psi(x_1)}^{\Psi(x_1+nh)} f(x)\,dx \Bigg/ \int_{\psi(x_1)}^{\psi(x_1+nh)} f(x)\,dx \geqq 1+\varepsilon.$$

Therefore the integral (2) is divergent, because otherwise the left hand quotient would tend to 1 with $n \to \infty$.

43. Under the condition (39) we have again for a natural n:

$$1-\varepsilon \geqq \int_{\Psi(x_1)}^{\Psi(x_1+nh)} f(x)\,dx \Bigg/ \int_{\psi(x_1)}^{\psi(x_1+nh)} f(x)\,dx$$

$$\geqq \int_{\Psi(x_1)}^{\Psi(x_1+nh)} f(x)\,dx \Bigg/ \int_{\psi(x_1)}^{\Psi(x_1+nh)} f(x)\,dx .$$

Therefore the integral (2) must converge, since otherwise the right hand quotient would tend to 1 with $n \to \infty$. Combining this result with the Theorem 8 we obtain again criteria for the convergence and divergence of (18).

44. Pringsheim derived in his paper [7] the "Corollary" from Ermakof's results, quoted above, in the following way.

If the function $f(x)$ is defined for all integers $\nu \geqq \nu_0$, define the function $\varphi(x)$ by

$$\varphi(x) \equiv f([x]) \quad (x \geqq \nu_0) . \tag{40}$$

Then we have for an integer n which is $\geqq$ than an integer v_0,

$$\int_{v_0}^{n+1} \varphi(x)\, dx = \sum_{v=v_0}^{n} f(v), \tag{41}$$

and conditions for the convergence or divergence of the series (18) are obtained, applying to the integral $\int^{\infty} \varphi(x)\, dx$ Ermakof's criteria. In this way we obtain corresponding criteria without assuming anything about the monotony of $f(x)$.

45. As a matter of fact Pringsheim formulates only the condition

$$\lim_{x \to \infty} \frac{f([\Psi(x)])\, \Psi'(x)}{f([\psi(x)])\, \psi'(x)} < 1$$

for the convergence and

$$\lim_{x \to \infty} \frac{f([\Psi(x)])\, \Psi'(x)}{f([\psi(x)])\, \psi'(x)} > 1$$

for the divergence, where $\Psi(x)$ and $\psi(x)$ are assumed to tend monotonically to ∞ with $x \to \infty$ and to satisfy $\Psi(x) > \psi(x)$. However, it is obvious, e.g. from the corresponding specialisations of our Theorems 1 and 2 that we can use

$$f([\Psi(x)])\, \Psi'(x) \leqq \alpha f([\psi(x)])\, \psi'(x), \quad \alpha < 1 \tag{42}$$

as convergence condition and

$$f([\Psi(x)])\, \Psi'(x) \geqq f([\psi(x)])\, \psi'(x) \tag{43}$$

as that for divergence.

Incidently, it is clear that we have in these cases the same degree of generality if we take $\psi(x) \equiv x$.

46. Applying the same idea directly to the Theorems 1—3 we have the following three Theorems in which we assume that $\psi(x)$ and $\Psi(x)$ are totally continuous for $x \geqq v_0$ and that $f(v)$ is defined and $\geqq 0$ for all integers $v \geqq v_0$.

THEOREM 1. *Assume that we have (4) for a sequence* $b_\nu \geqq \nu_0$ $(\nu = 1,2,...)$. *Then, if we have (42) for almost all* $x \geqq \nu_0$ *and for a positive* $\alpha < 1$ *the series (18) is convergent.*

Further, assuming that $f(\nu)$ is not $= 0$ for all sufficiently great integers ν, we have for all $x \geqq \nu_0$:

THEOREM 2. *Assume that there exists an* $a \geqq \nu_0$ *and an integer* $\nu_1 \geqq \nu_0$ *such that:*

$$\Psi(a) > \nu_1 \geqq \psi(a)\,,\ f(\nu_1) > 0\,, \tag{44}$$

and a sequence $b_\nu \geqq \nu_0$ $(\nu = 1,2,...)$ *such that we have (8). Then, if (43) holds for almost all* $x \geqq \nu_0$, *the series (18) is divergent and we have (10) for all* $x \geqq a$.

THEOREM 3. *Assume that there exists a constant* γ, $0 < \gamma < 1$, *and a sequence* $b_\nu \geqq \nu_0$ *such that (13) holds and further that for a constant c and for all integers* $\nu \geqq \nu_1$ *we have:*

$$\nu f(\nu) \leqq c \quad (\nu \geqq \nu_1)\,.$$

If then (42) holds for a certain $\alpha < 1$ *the series (18) is convergent, and the relation* $\Psi(a) \leqq \psi(a)$ *is for an* $a \geqq \nu_0$ *only possible, if* $f(\nu) = 0$ *for all* $\nu \geqq [\Psi(a)]$.

Observe that in applying the Theorems 1′, 2′ and 3′ to φ the transformation formula can be certainly applied since $|\varphi(x)|$ is uniformly bounded.

BIBLIOGRAPHY

[1] ERMAKOF, V., Caractère de convergence de séries. *Bull. des Sciences mathématiques et astronomiques*, 1871, II, pp. 250-256.

[2] ERMAKOF, V., Extrait d'une lettre adressé à M. Hoüel. *Bull. des Sciences mathématiques et astronomiques*, 1883, (II), VII, pp. 142-144.

[3] KNOPP, K., Theorie und Anwendung der unendlichen Reihen. Berlin 1947, pp. 305-307.

[4] KORKINE, A., Sur un problème d'interpolation. *Bull. des Sciences mathématiques et astronomiques*, 1882, (II), VI, pp. 228-231.

[5] OSTROWSKI, A., Sur les critères de convergence et divergence dus à V. Ermakof. A Trygve Nagell à l'occasion de son 60e anniversaire. *L'Enseignement mathématique*, 2e série, tome I, 1955, pp. 224-257.