I. Introduction

ON ERMAKOF'S CONVERGENCE CRITERIA AND ABEL'S FUNCTIONAL EQUATION. *)

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I. INTRODUCTION

1. We owe to V. Ermakof ([1], [2]) very remarkable criteria for the convergence or divergence of infinite series $\Sigma f(\nu)$ $(f(x) > 0)$ which uses the quotient

$$\frac{f(\Psi(x))\,\Psi'(x)}{f(x)} \tag{1}$$

for continuously differentiable function $\Psi(x)$ with the properties

$$\Psi(x) > x, \; \Psi(x) \to \infty \; (x \to \infty) .$$

As a matter of fact, the first discussion given by Ermakof [1] only established directly the connection with the convergence or the divergence of the integral

$$\int^{\infty} f(x)\, dx \tag{2}$$

so that in order to obtain the results concerning the infinite series we have to assume that $f(x)$ is monotonically decreasing or to make some analogous assumptions to permit the transition from the integral to the infinite series. We discuss some conditions of this kind in the sections 33-38.

2. In his second paper [2] Ermakof developped however a new and very ingenious method of proof using Abel's functional equation

$$\varphi(\Psi(x)) = \varphi(x) + 1 .$$

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This method allows, under suitable regularity conditions on $\Psi(x)$, to connect directly the behavior of (1) with the convergence or divergence of the infinite series $\Sigma f(\nu)$, *without any monotony condition for* $f(x)$.[1])

But Ermakof only sketched his discussion and indicated as the sufficient additional condition to impose on $\Psi(x)$ that, in our notations, $\Psi'(x_0)$ is $\geqq 1$, for a suitable x_0.[2])

It appears however that this additional condition is not sufficient to carry the discussion through. In a paper [5], published 1955, I showed that if beyond Ermakof's condition $\Psi'(x)$ is supposed monotonically increasing, the method can be carried through, indeed. If on the other hand $\Psi'(x)$ is supposed monotonically decreasing the method worked but Ermakof's additional condition was not necessary.

3. In this communication I develop a new method of proof which allows to avoid Abel's functional equation and to obtain the essential results for not necessarily monotonic $f(x)$. This gives a direct and very elementary way of proof as well for monotonically increasing as, (in the case of convergence), for monotonically decreasing $\Psi'(x)$. Beyond that, this method allows also to prove the convergence criteria in the case that $\lim_{x \to \infty} \Psi'(x)$ exists and is finite (Theorems 4-6).

4. As to the divergence criterion, here too, a new result in the case of monotonically decreasing $\Psi'(x)$ can be obtained (Theorem 7), however, with a different method which has more points of contact with Ermakof's second proof — here we have to form a minorant of $f(x)$, which can be interpreted as the derivative of a solution of Abel's functional equation —.

5. In the first sections of this paper we give 3 Theorems concerning the convergence and divergence of the integral (2) generalizing some results given in our first paper [5]. Finally, in the last part of the paper we discuss Pringsheim's treatment

1 Curiously enough, Abel's functional equation was also treated by Korkine in the note [4] where he gave another and direct proof of Ermakof's criteria for monotonic $f(x)$, without using, however, this functional equation.

2) Ermakof says in his paper [2] in a footnote on p. 142: " C'est la seule condition pour que notre démonstration soit juste."

of the problem and prove generalized versions of Pringsheim's results.

This note brings therefore an improvement and simplification of the sections I-V and XI of [5], while I have nothing to add to the sections VI-X of [5].

II. Ermakof's direct Method

6. The form of the expression (1) makes it plausible that we will have to use the integral transformation formula

$$\int_a^b f(\Psi(x))\,\Psi'(x)\,dx = \int_{\Psi(a)}^{\Psi(b)} f(x)\,dx. \tag{3}$$

In order to be able to use (3) we have in any case to assume that $f(x)$ is integrable in the integration interval and $\Psi(x)$ totally continuous between a and b. However, additional conditions are necessary and two such conditions are known either of which ensures the relation (3):

J_1: $|f(x)|$ is uniformly bounded in the integration interval;
J_2: $\Psi(x)$ is monotonically increasing or monotonically decreasing.

7. Theorem 1. *Assume that* $\psi(x)$ *and* $\Psi(x)$ *are totally continuous for* $x \geqq x_0$ *and that we have for a sequence* $b_\nu \geqq x_0$ $(\nu = 1,2, \ldots)$

$$\psi(b_\nu) \leqq \Psi(b_\nu),\ \Psi(b_\nu) \to \infty\ (\nu \to \infty). \tag{4}$$

Let $f(x)$ *be* $\geqq 0$ *on no half-line* $x \geqq \xi$ *almost everywhere* $= 0$, *and measurable in an interval* J *containing all values of* $\psi(x)$ *and* $\Psi(x)$ *for* $x \geqq x_0$. *Assume further that for any finite subinterval of* J *the transformation formula* (3) *holds as well for* $\psi(x)$ *as for* $\Psi(x)$. *Then, if we have for almost all* x *with* $x \geqq x_0$ *and for an* α *with* $0 < \alpha < 1$:

$$f(\Psi(x))\,\Psi'(x) \leqq \alpha f(\psi(x))\,\psi'(x)\ (x \geqq x_0),\ 0 < \alpha < 1, \tag{5}$$

the integral (2) *is convergent and we have for all* $x \geqq x_0$:

$$\Psi(x) > \psi(x) \quad (x \geqq x_0). \tag{6}$$