3. Matrices related to f

$f_c = f(c_1, \ldots, c_n)$. Then

$$(x_1 - c_1 \ldots x_n - c_n) \begin{bmatrix} \dfrac{\partial^2 f}{\partial c_1\, \partial \bar{c}_1} & \cdots & \dfrac{\partial^2 f}{\partial c_1\, \partial \bar{c}_n} & \left(\overline{\dfrac{\partial f}{\partial c_1}}\right) \\ \vdots & & \vdots & \vdots \\ \dfrac{\partial^2 f}{\partial c_n\, \partial \bar{c}_1} & & \dfrac{\partial^2 f}{\partial c_n\, \partial \bar{c}_n} & \left(\overline{\dfrac{\partial f}{\partial c_n}}\right) \\ \dfrac{\partial f}{\partial c_1} & & \dfrac{\partial f}{\partial c_n} & f_c \end{bmatrix} \begin{bmatrix} x_1 - c_1 \\ \vdots \\ x_n - c_n \\ 1 \end{bmatrix} = 0 \tag{2.1}$$

is called the tangent quadric of f at $(c_1, \ldots, c_n)$. We shall study only the cases that at least one of the first or second derivatives is not zero. It is clear that the tangent plane of (2.1) at $(c_1, \ldots, c_n)$ is the same as the tangent plane of $f = 0$ at this point.

Let the matrix of (2.1) be A, $\xi = (x_1 - c_1 \ldots x_n - c_n)$, and $\eta = (0 \ldots 0\ 1)$. Then by section 8 of [1]

$$\xi A \eta^* = 0 \tag{2.2}$$

is the tangent plane of (2.1) at $(c_1, \ldots, c_n)$. Here η^* is the conjugate transpose of η.

We easily see that (2.2) can be written as

$$\sum_{i=1}^{n} \frac{\partial f}{\partial c_i} (x_i - c_i) = 0\,. \tag{2.3}$$

3. Matrices related to f

Besides A there are other matrices of some interest. We denote the matrix of the quadratic form of (2.1) by Q. The projection on the normal and tangent plane are of some interest. We denote the projection on the normal by P, and clearly $I - P$ is the projection on the tangent plane where I is the identity matrix. It is easy to see that $P = (P_{ij})$, where

$$P_{ij} = \frac{\overline{\left(\frac{\partial f}{\partial x_i}\right)} \frac{\partial f}{\partial x_j}}{\sum \left|\frac{\partial f}{\partial x_i}\right|^2} .$$

This is proved by considering the inner product of a vector $\xi = (x_1, \ldots, x_n)$ and a unit vector on

$$\left(\frac{\partial f}{\partial x_1}, \ldots, \frac{\partial f}{\partial x_n}\right).$$

3. Quadric curvature

If (2.1) becomes of the form

$$[\sum a_i (x_i - c_i)] [\sum \bar{a}_i (\bar{x}_i - \bar{c}_i)] = 0 ,$$

then $f(x_1, \ldots, x_n)$ is called doubly flat at $(c_1, \ldots, c_n)$. Suppose (2.1) does not have this form. Then by sec 6 of [1] centers $(x_1, \ldots, x_n)$ of (2.1) may be obtained by

$$\xi Q = -\left(\frac{\partial f}{\partial \xi}\right), \tag{3.1}$$

where the row matrix ξ is:

$$\xi = (x_1 - c_1 \ldots x_n - c_n) , \; and \; \frac{\partial f}{\partial \xi} = \left(\frac{\partial f}{\partial c_1} \cdots \frac{\partial f}{\partial c_n}\right).$$

The equation (3.1) is a system of n linear equations in n unknowns.

The following cases may occur:

I. Let Q be non-singular. Then the quadric has a unique center which is called the center of quadric curvature of $f(x_1, \ldots, x_n)$ at $\gamma = (c_1, \ldots, c_n)$. Let

$$\xi = -\left(\frac{\partial f}{\partial \xi}\right) Q^{-1} .$$

Then the center is the point defined by $\xi - \gamma$.

II. Let the rank of Q be k, and centers exist. Then these centers are solutions of

$$\xi_k = \xi E = -\left(\frac{\partial f}{\partial \xi}\right) E Q^{-1}, \tag{3.2}$$

where Q^{-1} is the reciprocal of Q, see [2]. That is, if E is the projection on the range of Q, then

$$Q^{-1} Q = Q Q^{-1} = E.$$

Here we choose the center of quadric curvatures at a point of (3.2) so that, it is at the shortest distance from γ.

III. When the rank of Q is k and the quadric does not have centers, then we say that f does not have a center of quadric curvature.

4. Direction of quadric curvature

In part I and II of section 3 we respectively call the vectors ξ and ξ_k the directions of quadric curvature of f at $(c_1, \ldots, c_n)$. In III of section 3, we define the direction of quadric curvature to be a vector δ which satisfies

$$\delta = \delta E = -\left(\frac{\partial f}{\partial \xi}\right) E Q^{-1},$$

where E is the projection described in section 3.

5. Vertex points

Let at the point $\gamma = (c_1, \ldots, c_n)$ of f the direction of quadric curvature be the same as the normal to $f = 0$. Then γ is called a vertex point of the function f.

Theorem: A necessary and sufficient condition for a point to be a vertex point of the function f is that at that point

$$PQ = QP,$$

where P and Q are the matrices described in section 3.