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In E. Hille's *Analytic Function Theory*, Vol. 1 (1959), page 227, the following theorem is given:

Let

$$E_p(z) = (1-z)\exp\left\{ z + \frac{z^2}{2} + \ldots + \frac{z^p}{p} \right\}, \quad p = 1, 2, \ldots ;$$

for $|z| \leqq 1$ we have

$$| E_p(z) - 1 | \leqq | z |^{p+1} .$$

There is also the remark: « This proof was communicated to me some forty years ago by my teacher Marcel Riesz. If I remember correctly, he ascribed it to Fejer. The proof does not seem to have been published. »

The proof is based on the fact that the coefficients of the development of

$$E_p(z) = 1 + \sum_{k=1}^{\infty} A_{k,p}\, z^k$$

have the property

$$A_{1,p} = A_{2,p} = \ldots = A_{p,p} = 0 \,, \text{ and } A_{k,p} < 0 \text{ for } k > p \,,$$

which is a consequence of

$$E'_p(z) = -z^p \exp\left\{ z + \frac{z^2}{2} + \ldots + \frac{z^p}{p} \right\}. \tag{1}$$

But starting from (1) the proof can be obtained immediateyl as follows. Indeed, if $|z| \leqq 1$ and $0 \leqq t \leqq 1$ it follows that

$$|E_p'(tz)| \leqq -|z|^p E_p'(t),$$

and hence

$$|E_p(z)-1| = |z\int_0^1 E_p'(tz)\,dt| \leqq -|z|^{p+1}\int_0^1 E_p'(t)\,dt = |z|^{p+1}.$$

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