## 5. The Theorems

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Let $u$ be the center of $S^{n-1}$. Since $f$ has no fixed point, it is clear that we can choose $d>0$ so small that a closed solid $n$-sphere $H_{d}^{n}$ of radius $d$ with center at $\theta(u)$ is entirely in $\eta^{n}$, and $H_{d}^{n}$ and its image $f\left(H_{d}^{n}\right)$ are contained in different half-spaces into which $R^{n}$ is separated by some ( $n-1$ )-plane.

Now, let $S^{n-1}$ undergo a deformation by uniform radial shrinking toward $u$ till it reaches a position $S_{2}^{n-1}$ whose image $\sigma_{2}^{n-1}$ under $\theta$ is contained in the interior of $H_{d}^{n}$. By means of $\theta$, there results a deformation of $\sigma^{n-1}$ into $\sigma_{2}^{n-1}$ which by means of the mapping $f$ induces a deformation, on the direction sphere, of the ( $n-1$ )-cycle $f^{n-1}$ resulting from $f$ applied to $\sigma^{n-1}$ into the ( $n-1$ )-cycle $f_{2}^{n-1}$ resulting from $f$ applied to $\sigma_{2}^{n-1}$.

Thus the turning index of $\sigma^{n-1}$ under $f$ equals the turning index of $\sigma_{2}^{n-1}$ under $f$, which by Lemma 2 equals zero. Thus Lemma 4 is proved.

## 5. The Theorems

Theorem 1. Let $\eta^{n} \subset \mathrm{R}^{n}$ be a closed $n$-cell and $f$ a continuous mapping of $\eta^{n}$ into $R^{n}$ such that $f$ maps the boundary $\sigma^{n-1}$ of $\eta^{n}$ into $\eta^{n}$. Then $f$ has at least one fixed point.

Proof. Assume no fixed points. Let, as in the case of Lemma 3, $\eta^{n}$ and $\sigma^{n-1}$ be respectively the images (under the homeomorphism $\theta$ ) of the closed solid $n$-sphere $E^{n}$ with boundary $S^{n-1}$, i.e., $\eta^{n}=\theta\left(E^{n}\right)$ and $\sigma^{n-1}=\theta\left(S^{n-1}\right)$.

Let $u$ be the center of $S^{n-1}$. Consider the mapping $f^{\prime}$ of $\sigma^{n-1}$ which maps every point $\sigma \varepsilon \sigma^{n-1}$ into the point $\theta(u)$. Since $f^{\prime}$ is the mapping which appears in the definition of the index of $\theta(u)$ relative to $\sigma^{n-1}$, we see by Lemma 3 that the turning index of $\sigma^{n-1}$ under $f^{\prime}$ is non-zero.

By hypothesis, $f(\sigma) \varepsilon \eta^{n}$ for every $\sigma \varepsilon \sigma^{n-1}$. Hence we may deform $f\left(\sigma^{n-1}\right)$ as follows. As a parameter $p$ varies from 0 to 1,
the point $\sigma^{\prime}$ moves in $\eta^{n}$ along the path $\theta\left[\overline{\theta^{-1} f(\sigma), u}\right]$ starting from $\sigma$ and ending at $\theta(u)$.

For $p=1$, the above deformation yields the mapping $f^{\prime}$. Therefore, the ( $n-1$ )-cycle resulting from $f$ applied to $\sigma^{n-1}$ is homologous on the direction sphere (as a consequence of a deformation) to the ( $n-1$ )-cycle resulting from $f^{\prime}$ applied to
$\sigma^{n-1}$. Consequently, the turning index of $\sigma^{n-1}$ under $f$ equals the turning index of $\sigma^{n-1}$ under $f^{\prime}$, and hence is not zero. But this contradicts Lemma 4. Thus, Theorem 1 is true.

Theorem 2. Let $\eta^{n} \subset R^{n}$ be a closed $n$-cell with boündary $\sigma^{n-1}$ and $f$ a continuous map of $\eta^{n}$ into $R^{n}$ which leaves no point of $\sigma^{n-1}$ fixed. If there exists an inner point $e$ of $\eta^{n}$ and an angle $\alpha$ with $0 \leq \alpha \leq \pi$, such that for no point $\sigma \varepsilon \sigma^{n-1}$ is $\alpha$ an angle from the vector $\overline{\sigma, f(\sigma)}$ to the vector $\overline{e, \sigma}$ then $f$ leaves at least one point fixed.

Proof. Suppose $f$ leaves no point fixed. We shall show that under the hypotheses of Theorem 2, either
i) for no point $\sigma \varepsilon \sigma^{n-1}$ is the direction from $\sigma$ to $f(\sigma)$ opposite to that from $e$ to $\sigma$,
or
ii) for no point $\sigma \varepsilon \sigma^{n-1}$ is the direction from $\sigma$ to $f(\sigma)$ opposite to that from $\sigma$ to $e$.

For, otherwise we would have points $\sigma_{1}$ and $\sigma_{2} \varepsilon \sigma^{n-1}$ such that, as $\sigma$ traverses a path from $\sigma_{1}$ to $\sigma_{2}$ on $\sigma^{n-1}$, the angle between $\overline{\sigma, f(\sigma)}$ and $\overline{\sigma, e}$ would change continuously from 0 to $\pi$, hence assume the value $\alpha$, a contradiction.

If i) holds, we apply Lemma 1 taking the mapping $g$ of Lemma 1 as the mapping $f$, and as the mapping $h$, we take a mapping which makes correspond to each point $\sigma \varepsilon \sigma^{n-1}$ the intersection of the half line starting at the point $e$ and passing through the point $\sigma$, with an ( $n-1$ )-sphere $V^{n-1}$ whose center is $e$ and which is located completely outside of $\sigma^{n-1}$. We infer by Lemma 1 that the turning indices of $\sigma^{n-1}$ under $f$ and $h$ are equal. Since the turning index of $\sigma^{n-1}$ under $h$ clearly equals the turning index of $\sigma^{n-1}$ relative to $V^{n-1}$, we infer from Lemma 3 that the turning index of $\sigma^{n-1}$ under $f$ is non-zero.

If ii) holds, again by Lemmas 1 and 3 the turning index of $\sigma^{n-1}$ under $f$ is non-zero. (Here, for the mapping $g$ of Lemma 1, we again take the mapping $f$, and for the mapping $h$, we take a mapping which makes correspond to each point $\sigma \varepsilon \sigma^{n-1}$ the intersection of the half line starting at the point $e$ and passing through the point $\sigma$, with an ( $n-1$ )-sphere $V^{n-1}$ whose center is $e$ and which is located completely inside of $\sigma^{n-1}$ ).

In short, the turning index of $\sigma^{n-1}$ under the assumption of the absence of fixed points is non-zero, a fact which contradicts Lemma 4. Hence $f$ has at least one fixed point, and Theorem 2 is proved.

Corollary 1. Let $E^{n}$ be a closed solid $n$-sphere and $f$ a continuous mapping of $E^{n}$ into $R^{n}$ such that $f$ maps the boundary $S^{n-1}$ of $E^{n}$ into $E^{n}$. Then $f$ has at least one fixed point.

Proof. If no point of $S^{n-1}$ is fixed, then the hypotheses of Theorem 2 are seen to be satisfied with $e$ at the center of the sphere $E^{n}$ and $\alpha=0$.

Clearly, Corollary 1 also follows immediately from Theorem 1. Proofs of this corollary also appear in the literature ([3], page 115).

Corollary 2. Let $\eta^{n} \subset R^{n}$ be a closed $n$-cell with boundary $\sigma^{n-1}$, and $f$ and $g$ two continuous maps of $\eta^{n}$ into $R^{n}$ such that for no point $\sigma \varepsilon \sigma^{n-1}$ is $f(\sigma)=g(\sigma)$. If there exists an inner point $e$ of $\eta^{n}$ and a constant angle $\beta, 0 \leq \beta \leq \pi$, such that for no point $\sigma \varepsilon \sigma^{n-1}$ is $\beta$ an angle between the vectors $\overline{e, \sigma}$ and $\overline{f(\sigma), g(\sigma)}$, then there is a point $\eta_{0} \varepsilon \eta^{n}$ such that $f\left(\eta_{0}\right)=g\left(\eta_{0}\right)$.

Proof. Consider the map $h$ of $\eta^{n}$ into $R^{n}$ such that for every point $\eta \varepsilon \eta^{n}$ the vectors $\overline{\eta, h(\eta)}$ and $\overline{f(\eta), g(\eta)}$ are equal. By Theorem 2, the map $h$ has a fixed point $\eta_{0}$. Consequently, $f\left(\eta_{0}\right)=g\left(\eta_{0}\right)$.

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