NEW FIXED POINT THEOREM FOR CONTINUOUS MAPS OF THE CLOSED n-CELL

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1. INTRODUCTION

In this paper the authors prove two fixed point theorems for continuous maps of a closed n-cell η^n into the euclidean space $R^n \supset \eta^n$. Neither theorem requires that η^n be mapped into itself.

The main theorem is Theorem 1 in which it is proved that *a continuous mapping of a closed n-cell η^n into $R^n \supset \eta^n$ which maps the boundary of η^n into η^n, has a fixed point.* It is believed that this theorem is new and is stronger than Brouwer's classical fixed point theorem inasmuch as it implies the latter and has weaker hypotheses.

Although the same theorem can be proved in a much shorter way by using Tietze's extension theorem followed by the classical Brouwer's fixed point theorem, however, in the proofs given below no knowledge of these two theorems is presupposed.

In this paper the proofs of the theorems are based in part on use of homologies, and in part on the turning index (defined below), which is essentially a generalization to the n dimensional case of the idea involved in [1], pp. 251-5, for the case of a circular disc.

2. NOTATION

In what follows, R^n denotes an oriented euclidean n-space, fixed once and for all.

All closed solid n-spheres and $(n-1)$-spheres are assumed to be triangulated with solid n-spheres oriented to agree with

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R^n, and $(n-1)$-spheres oriented with orientations induced by their interiors.

Symbols c^{n-1}, g^{n-1}, ... denote oriented $(n-1)$-cycles in R^n; D^{n-1}, V^{n-1}, ... denote $(n-1)$-spheres in R^n. E^n denotes a closed solid n-sphere in R^n, and the boundary of E^n is denoted by S^{n-1}. η^n denotes a closed n-cell in R^n and the boundary of η^n is denoted by σ^{n-1}.

In this paper η^n is assumed to be the image of E^n under homeomorphism θ, and η^n and σ^{n-1} obtain their orientations from E^n and S^{n-1} respectively.

3. The Turning Index

Let c^{n-1} be an $(n-1)$-cycle in R^n and g a continuous map of c^{n-1} into R^n having no fixed point. Let D^{n-1} be an $(n-1)$-sphere with center 0, called a *direction sphere* [2]. Let c^{n-1} be mapped on D^{n-1} as follows. To a point $c \varepsilon c^{n-1}$ there corresponds a point $d \varepsilon D^{n-1}$ such that the line segment from 0 to d has the same sense and direction as that from c to $g(c)$. The resulting $(n-1)$-cycle g^{n-1} on D^{n-1} is called, in the sequel, *the $(n-1)$-cycle g^{n-1} resulting from g applied to c^{n-1}*, and the degree of the resulting map, that is, the multiple of D^{n-1} which is homologous to g^{n-1} (which is clearly independent of the radius of D^{n-1} and the location of 0) is called the *turning index* of c^{n-1} under g.

If p is a point not on c^{n-1}, the *index of p relative to c^{n-1}* is defined as the turning index of the map which maps every point of c^{n-1} into p. (For odd n, this is the negative of the corresponding definition given in [3], as shown by Theorem 1.5, page 105).

4. Preliminary Lemmas

Lemma 1. *Let g and h be two continuous maps into R^n of an $(n-1)$-cycle c^{n-1}, such that neither leaves any point of c^{n-1} fixed, and, for no point $c \varepsilon c^{n-1}$ are the directions from c to $g(c)$ and from c to $h(c)$ exactly opposite. Then the turning indices of c^{n-1} under g and h are equal.*

Proof. For each $c \varepsilon c^{n-1}$, the directions of the two vectors $\overline{c,g(c)}$ and $\overline{c,h(c)}$ are not opposite and hence, if not identical, determine a 2-plane P in which they make an angle of less than π radians. As a parameter p varies from 0 to 1, let the direction of $\overline{c,h}(c)$ change in P so that the angle between the two vectors $\overline{c,h}(c)$ and $\overline{c,g}(c)$ decreases uniformly to zero while their lengths remain fixed. If the angle is zero at the start, no change in direction takes place. For each value of p, $0 \leqq p \leqq 1$, the corresponding mapping as determined above in the definition of turning index, maps c^{n-1} on the direction sphere D^{n-1}, and the result, as p varies from 0 to 1, is to deform the $(n-1)$-cycle h^{n-1} on D^{n-1} resulting from h applied to c^{n-1} into the $(n-1)$-cycle g^{n-1} resulting from g applied to c^{n-1}. Hence h^{n-1} is homologous to g^{n-1}, and therefore to the same multiple of D^{n-1}, so that the turning indices under consideration are equal. Thus Lemma 1 is proved.

LEMMA 2. *Let g be a continuous map into R^n of an $(n-1)$-cycle c^{n-1}, such that c^{n-1} and $g(c^{n-1})$ are contained in different half-spaces into which R^n is separated by some $(n-1)$-plane. Then the turning index of c^{n-1} under g is zero.*

Proof. Since the $(n-1)$-cycle g^{n-1} resulting from g applied to c^{n-1} is clearly entirely on one hemisphere of D^{n-1}, we conclude that c^{n-1} cannot be homologous to any multiple of D^{n-1} other than zero. Thus Lemma 2 is proved.

LEMMA 3. *Let σ^{n-1} be the boundary of a closed n-cell $\eta^n \subset R^n$. Let e be a point in the inside of σ^{n-1}. Then the index of e relative to σ^{n-1} is 1 or -1.*

While this result is given in [3], page 109, Theorem 4.1, the following proof is given as shorter and obtained independently.

Proof. Let η^n and σ^{n-1} be respectively the homeomorphic images (under homeomorphism θ) of the closed solid n-sphere E^n with boundary S^{n-1}, i.e., $\eta^n = \theta(E^n)$ and $\sigma^{n-1} = \theta(S^{n-1})$. By use of the invariance of regionality, it is easy to show that $\eta^n = \theta(E^n)$ contains no point outside σ^{n-1} and contains every point inside σ^{n-1}.

Let V^{n-1} be an $(n-1)$-sphere with center at e, so small that V^{n-1} and its interior are inside σ^{n-1}, hence composed of points of η^n. Let $\beta^{n-1} = \theta^{-1}(V^{n-1})$ and $d = \theta^{-1}(e)$.

For each point $b \in \beta^{n-1}$ let the half-line beginning at d and passing through b intersect S^{n-1} at b'.

Now, for every t, with $0 \leq t \leq 1$, let $\beta^{n-1}(t)$ be the $(n-1)$-cycle determined as follows. For each point $b \in \beta^{n-1}$ there corresponds a point $b(t)$ of $\beta^{n-1}(t)$ on the closed segment from b to b' such that the distance from b to $b(t)$ is t times the distance from b to b'.

Let $V^{n-1}(t) = \theta[\beta^{n-1}(t)], \qquad 0 \leq t \leq 1.$

As t varies from 0 to 1, the cycle $V^{n-1}(t)$ undergoes a deformation from initial position $V^{n-1}(0) = V^{n-1}$ to final position $V^{n-1}(1)$. Since $V^{n-1}(1)$ is on σ^{n-1}, there is an integer x such that

$$(1) \qquad V^{n-1}(1) \sim x\,\sigma^{n-1} \qquad \text{on } \sigma^{n-1},$$

where $\sim$ stands for " is homologous to ".

For each t, let $k(t)$ be the mapping which maps every point of $V^{n-1}(t)$ into e, and let V^{n-1} serve as the direction sphere. As t varies from 0 to 1, the $(n-1)$-cycle $k^{n-1}(0)$ resulting from $k(0)$ applied to V^{n-1} is deformed on the direction sphere V^{n-1} into the $(n-1)$-cycle $k^{n-1}(1)$ resulting from $k(1)$ applied to $V^{n-1}(1)$. Thus these two $(n-1)$ cycles are homologous on V^{n-1}. Therefore the index of e relative to V^{n-1} equals the index of e relative to $V^{n-1}(1)$. However, since $k(0)$ maps every point of V^{n-1} into e, we derive that ([4], page 92)

$$(2) \qquad \text{the index of } e \text{ relative to } V^{n-1}(1) = (-1)^n.$$

Let y be the index of e relative to σ^{n-1}. From (1) we infer that xy is the index of e relative to $V^{n-1}(1)$. Hence, by (2), $xy = (-1)^n$. Consequently, $y = 1$ or $y = -1$. Thus Lemma 3 is proved.

LEMMA 4. *If a continuous map f of a closed n-cell $\eta^n \subset R^n$ into R^n has no fixed point, then the turning index of the boundary σ^{n-1} of η^n under f is zero.*

Proof. Let, as in the proof of Lemma 3, $\eta^n = \theta(E^n)$ and $\sigma^{n-1} = \theta(S^{n-1})$ be respectively the images under the homeomorphism θ of the closed solid n-sphere E^n and its boundary S^{n-1}.

Let u be the center of S^{n-1}. Since f has no fixed point, it is clear that we can choose $d > 0$ so small that a closed solid n-sphere H_d^n of radius d with center at $\theta(u)$ is entirely in η^n, and H_d^n and its image $f(H_d^n)$ are contained in different half-spaces into which R^n is separated by some $(n-1)$-plane.

Now, let S^{n-1} undergo a deformation by uniform radial shrinking toward u till it reaches a position S_2^{n-1} whose image σ_2^{n-1} under θ is contained in the interior of H_d^n. By means of θ, there results a deformation of σ^{n-1} into σ_2^{n-1} which by means of the mapping f induces a deformation, on the direction sphere, of the $(n-1)$-cycle f^{n-1} resulting from f applied to σ^{n-1} into the $(n-1)$-cycle f_2^{n-1} resulting from f applied to σ_2^{n-1}.

Thus the turning index of σ^{n-1} under f equals the turning index of σ_2^{n-1} under f, which by Lemma 2 equals zero. Thus Lemma 4 is proved.

5. THE THEOREMS

THEOREM 1. *Let $\eta^n \subset R^n$ be a closed n-cell and f a continuous mapping of η^n into R^n such that f maps the boundary σ^{n-1} of η^n into η^n. Then f has at least one fixed point.*

Proof. Assume no fixed points. Let, as in the case of Lemma 3, η^n and σ^{n-1} be respectively the images (under the homeomorphism θ) of the closed solid n-sphere E^n with boundary S^{n-1}, i.e., $\eta^n = \theta(E^n)$ and $\sigma^{n-1} = \theta(S^{n-1})$.

Let u be the center of S^{n-1}. Consider the mapping f' of σ^{n-1} which maps every point $\sigma \in \sigma^{n-1}$ into the point $\theta(u)$. Since f' is the mapping which appears in the definition of the index of $\theta(u)$ relative to σ^{n-1}, we see by Lemma 3 that the turning index of σ^{n-1} under f' is non-zero.

By hypothesis, $f(\sigma) \in \eta^n$ for every $\sigma \in \sigma^{n-1}$. Hence we may deform $f(\sigma^{n-1})$ as follows. As a parameter p varies from 0 to 1, the point σ' moves in η^n along the path $\theta[\overline{\theta^{-1} f(\sigma), u}]$ starting from σ and ending at $\theta(u)$.

For $p = 1$, the above deformation yields the mapping f'. Therefore, the (n-1)-cycle resulting from f applied to σ^{n-1} is homologous on the direction sphere (as a consequence of a deformation) to the $(n-1)$-cycle resulting from f' applied to

σ^{n-1}. Consequently, the turning index of σ^{n-1} under f equals the turning index of σ^{n-1} under f', and hence is not zero. But this contradicts Lemma 4. Thus, Theorem 1 is true.

THEOREM 2. *Let $\eta^n \subset R^n$ be a closed n-cell with boundary σ^{n-1} and f a continuous map of η^n into R^n which leaves no point of σ^{n-1} fixed. If there exists an inner point e of η^n and an angle α with $0 \leqq \alpha \leqq \pi$, such that for no point $\sigma \in \sigma^{n-1}$ is α an angle from the vector $\overline{\sigma, f(\sigma)}$ to the vector $\overline{e, \sigma}$ then f leaves at least one point fixed.*

Proof. Suppose f leaves no point fixed. We shall show that under the hypotheses of Theorem 2, either

i) for no point $\sigma \in \sigma^{n-1}$ is the direction from σ to $f(\sigma)$ opposite to that from e to σ,

or

ii) for no point $\sigma \in \sigma^{n-1}$ is the direction from σ to $f(\sigma)$ opposite to that from σ to e.

For, otherwise we would have points σ_1 and $\sigma_2 \in \sigma^{n-1}$ such that, as σ traverses a path from σ_1 to σ_2 on σ^{n-1}, the angle between $\overline{\sigma, f(\sigma)}$ and $\overline{\sigma, e}$ would change continuously from 0 to π, hence assume the value α, a contradiction.

If i) holds, we apply Lemma 1 taking the mapping g of Lemma 1 as the mapping f, and as the mapping h, we take a mapping which makes correspond to each point $\sigma \in \sigma^{n-1}$ the intersection of the half line starting at the point e and passing through the point σ, with an $(n-1)$-sphere V^{n-1} whose center is e and which is located completely outside of σ^{n-1}. We infer by Lemma 1 that the turning indices of σ^{n-1} under f and h are equal. Since the turning index of σ^{n-1} under h clearly equals the turning index of σ^{n-1} relative to V^{n-1}, we infer from Lemma 3 that the turning index of σ^{n-1} under f is non-zero.

If ii) holds, again by Lemmas 1 and 3 the turning index of σ^{n-1} under f is non-zero. (Here, for the mapping g of Lemma 1, we again take the mapping f, and for the mapping h, we take a mapping which makes correspond to each point $\sigma \in \sigma^{n-1}$ the intersection of the half line starting at the point e and passing through the point σ, with an $(n-1)$-sphere V^{n-1} whose center is e and which is located completely inside of σ^{n-1}).

In short, the turning index of σ^{n-1} under the assumption of the absence of fixed points is non-zero, a fact which contradicts Lemma 4. Hence f has at least one fixed point, and Theorem 2 is proved.

COROLLARY 1. *Let E^n be a closed solid n-sphere and f a continuous mapping of E^n into R^n such that f maps the boundary S^{n-1} of E^n into E^n. Then f has at least one fixed point.*

Proof. If no point of S^{n-1} is fixed, then the hypotheses of Theorem 2 are seen to be satisfied with e at the center of the sphere E^n and $\alpha = 0$.

Clearly, Corollary 1 also follows immediately from Theorem 1. Proofs of this corollary also appear in the literature ([3], page 115).

COROLLARY 2. *Let $\eta^n \subset R^n$ be a closed n-cell with boundary σ^{n-1}, and f and g two continuous maps of η^n into R^n such that for no point $\sigma \varepsilon \sigma^{n-1}$ is $f(\sigma) = g(\sigma)$. If there exists an inner point e of η^n and a constant angle β, $0 \leqq \beta \leqq \pi$, such that for no point $\sigma \varepsilon \sigma^{n-1}$ is β an angle between the vectors $\overline{e, \sigma}$ and $\overline{f(\sigma), g(\sigma)}$, then there is a point $\eta_0 \varepsilon \eta^n$ such that $f(\eta_0) = g(\eta_0)$.*

Proof. Consider the map h of η^n into R^n such that for every point $\eta \varepsilon \eta^n$ the vectors $\overline{\eta, h(\eta)}$ and $\overline{f(\eta), g(\eta)}$ are equal. By Theorem 2, the map h has a fixed point η_0. Consequently, $f(\eta_0) = g(\eta_0)$.

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