8. Algebras over Hopf algebras.

leads to the formula

$$(\partial f)x = \partial(fx) + (-1)^{r+1} f(\partial x), \qquad r = \deg f.$$

8. Algebras over Hopf algebras.

We have seen that a graded algebra is a graded R-module X and an R-mapping $\mu\colon X \otimes X \to X$. Suppose now that X is also an A-module where A is a Hopf algebra over R. Then $X \otimes X$ is an A-module as defined in section 7. We define X to be *an algebra over the Hopf algebra A* (briefly, an A-algebra) if the multiplication mapping $\mu\colon X \otimes X \to X$ is an A-mapping.

In terms of elements $a \in A$ and $x_1, x_2 \in X$, the condition for μ to be an A-mapping takes the form

$$(8.1) \qquad a(x_1 x_2) = \textstyle\sum_i (-1)^{pq_i} (a_i' x_1)(a_i'' x_2)$$

where

$$\Psi a = \Sigma_i a_i' \otimes a_i'', \qquad p = \deg x_1, \qquad q_i = \deg a_i''.$$

It is to be observed that this concept of an algebra over a Hopf algebra has arisen in a natural way. The discussion of section 7 demonstrates its inevitability. This being true there ought to be numerous examples.

The first, and for us the most important example, is the cohomology algebra of a space $H^*(X; Z_p)$ over the Hopf algebra $\mathscr{A}_p$ of reduced power operations. The cup-product formula

$$\mathscr{P}^k(x_1 x_2) = \sum_{i=0}^{k} (\mathscr{P}^i x_1)(\mathscr{P}^{k-i} x_2),$$

and the diagonal mapping $\Psi \mathscr{P}^k = \sum_{i=0}^{k} \mathscr{P}^i \otimes \mathscr{P}^{k-i}$ show that 8.1 is satisfied.

Another example is provided by the differential, graded, augmented algebras of Cartan [8]. In this case, X is an augmented chain complex (i.e. a module over $E(\partial, -1)$, see § 7), and a *chain* mapping $\mu\colon X \otimes X \to X$ defines an algebra structure in X.

A trivial example is provided by any algebra X over R. Note first that $\varphi: R \otimes R \to R$ defined by $\varphi(r_1 \otimes r_2) = r_1 r_2$ is an isomorphism (recall that $\otimes = \otimes_R$). Set $\Psi = \varphi^{-1}: R \to R \otimes R$, then φ, Ψ give a natural structure of a Hopf algebra to the ground ring R. It is easily checked that the natural R-structure in $X \otimes X$ coincides with that defined by Ψ. Thus any algebra over the ground ring is an algebra over the ground ring regarded as a Hopf algebra.

As another example, let X be an algebra over R, and let π be a group of automorphisms of the algebra X. Let A be the group ring of π over R with the usual multiplication. Define the diagonal $\Psi: A \to A \otimes A$ to be the mapping induced by the diagonal mapping $d: \pi \to \pi \times \pi$. Then A becomes a Hopf algebra. Since any $g \in \pi$ is an automorphism, $g(x_1 x_2) = (gx_1)(gx_2)$; and since $dg = (g, g)$, it follows that 8.1 holds. Thus any algebra is an algebra over the Hopf algebra of its automorphism group.

9. Universal A-algebras.

The foregoing examples of algebras over Hopf algebras arose naturally. We now show how to construct them in a wholesale fashion.

Let A be any Hopf algebra. It is easy to construct many modules over the algebra A (i.e. take quotients of A by left ideals, and then take direct sums of these). Let M be any graded A-module. Let M^n denote the tensor product of n copies of M. As in section 7, M^n is an A-module. Form the direct sum

$$T(M) = \sum_{n=0}^{\infty} M^n$$

where $M^0 = R$. Define $\mu: T(M) \otimes T(M) \to T(M)$ in terms of components $x \in M^r$, $y \in M^s$ by $\mu(x \otimes y) = x \otimes y \in M^{r+s}$ making use of the associative law $M^r \otimes M^s \approx M^{r+s}$. In this way $T(M)$ is an associative algebra. It is called the *free associative algebra* generated by M (also, the *tensor algebra* of M). Since the associative law $M^r \otimes M^s \approx M^{r+s}$ is an A-mapping, it follows that $T(M)$ is an algebra over the Hopf algebra A.