1. Introduction.

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THE COHOMOLOGY ALGEBRA OF A SPACE [1])

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1. Introduction.

The history of the development of the concept of the cohomology algebra of a space is marked by quite a few wrong turns, blind alleys, and fallacious preconceptions. The purpose of this article is to trace this development with emphasis on the errors that were made. This may be useful since the story is not yet complete, and the final form of the concept is still to be determined.

The key to the existence of the multiplication of cohomology classes lay in the Alexander duality theorem, especially in the form it was given by Pontrjagin in 1934: If X is a closed subset of the n-sphere S^n, then, for any coefficient group G, the singular homology group $H_{n-q-1}(S^n - X; G)$ is isomorphic to char $H_q(X; \text{char } G)$ where char means character group, and H_q is Čech homology. This latter group coincides with what we now call the cohomology group, and we denote it by $H^q(X; G)$. Since $H_i(S^n) = 0$ for $0 < i < n$, it follows by exactness that

$$H^q(X; G) \approx H_{n-q-1}(S^n - X; G) \approx H_{n-q}(S^n, S^n - X; G)$$

for $0 < q < n - 1$, the latter isomorphism being given by the boundary operator. If G is a ring R, the relative groups $H_r(S^n, S^n - X; R)$ admit an intersection theory in the sense of Lefschetz, because S^n is a manifold. This induces a multiplication

$$H^p(X; R) \otimes H^q(X; R) \to H^{p+q}(X; R) . \qquad (1.1)$$

To obtain a fully satisfactory duality theorem, it was necessary to show that this multiplication is independent of the imbedding and of n. During the years 1935 to 1938 this was achieved in papers by Alexander, Čech, Gordon and Whitney. A direct

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1) Talk delivered at the Zurich Colloquium on Differential Geometry and Topology, June 1960.

internal description was given of the groups H^q, and of the multiplication 1.1. Their constructions were triumphs of ingenuity, and it was puzzling that something so basic should be so difficult. The difficulty was dissipated by Lefschetz in 1942 when he showed that the multiplication is the composition of the cross-product with values in $H^{p+q}(X\times X; R)$ followed by the homomorphism induced by the diagonal mapping $X\rightarrow X\times X$.

The upshot was a readily defined product, easily proved to be associative, to have a unit in dimension 0, and to satisfy the commutation law

$$(1.2) \qquad xy = (-1)^{pq} xy\ , \qquad x\in H^p,\ y\in H^q\ .$$

With this success there was great hope that the multiplication would lead to deeper insight into the topology of spaces. To give this hope a precise formulation, one spoke of the *cohomology algebra* $H^*(X; R)$. It was defined to be the direct sum $\Sigma_{q=0}^{\infty} H^q(X; R)$ with the multiplication determined by the multiplications of the component parts. It is clearly an associative algebra with unit. And now one can ask if the theory of algebras can be brought to bear on topological problems via $H^*(X; R)$.

The above definition of H^* constituted a wrong turn into a blind alley. It is an error which has not yet been fully erased. The mistake lay in forming the direct sum over the dimensional index q. No one has yet found a valid geometric reason for adding cohomology classes of different dimensions. The only algebraic reason for doing so was to make H^* into a familiar algebraic object. This was a forcing of the mathematics into a preconceived pattern. There was no gain in doing so. For example, the interesting part of the algebra is the sum $\Sigma_{q>0} H^q$; but if X is a finite complex, this part lies in the radical of H^*. Unfortunately, algebraic theory has little to say about the radical. Its major results concern the quotient by the radical. Even worse, H^* is badly non-commutative in spite of the fact that the rule 1.2 for commuting two elements is just as useful as the commutative law $xy = yx$.

It has now come to be recognized that the proper algebraic concept, to which the cohomology of a space conforms, is that of a graded algebra. A *graded algebra A* is, first of all, a sequence

$\{A^q\}$, $q = 0, 1, 2, \ldots$, of R-modules. Thus, an element of A is an element of some A^q, and q is called its degree. Elements may be added only if they have the same degree. In addition, homomorphisms $A^p \otimes A^q \to A^{p+q}$ are given for all $p, q \geqq 0$. These define a bilinear product xy for all $x, y \in A$. The product is required to be associative.

An ordinary algebra C is converted into a graded algebra A by setting $A^0 = C$ and $A^q = 0$ for $q > 0$. In this way, a graded algebra is a *generalization* of the notion of an algebra. Thus we are free to generalize the properties of algebras to graded algebras in any convenient manner which conforms in degree 0.

In particular, a graded algebra A is called commutative if $xy = (-1)^{pq}\, yx$ for all $x \in A^p$ and $y \in A^q$. Thus, what was once called the *anti*-commutative law is now called the commutative law. And the cohomology algebra of a space is an associative, commutative, graded algebra.

A unit of a graded algebra A is an element $1 \in A^0$ such that $1x = x = x1$ for all $x \in A$. An augmentation ε of A is a homomorphism $\varepsilon\colon A \to R$ of graded algebras with unit. Thus $\varepsilon\,(A^q)=0$ for $q > 0$, and $\varepsilon\,(1) = 1$. In case ε gives an isomorphism in degree 0, $A^0 \approx R$, then A is called *connected.*

If P is a space consisting of a single point, it is clear that $H^*(P; R) = R$ as a graded algebra. For any space X, the mapping $\eta\colon X \to P$ induces a monomorphism $\eta^*\colon R \to H^*(X; R)$ and $\eta^*\,(1)$ is the unique unit of $H^*(X; R)$. Finally, any mapping $P \to X$ induces an augmentation of $H^*(X; R)$. Clearly X is arcwise connected if and only if $H^*(X; R)$ is connected; and then the augmentation is unique.

2. Realizing a graded algebra as a cohomology algebra.

Let Z denote the ring of integers. It is well known that, if B is a graded Z-module such that $B^0 = Z$, B^1 is free, and B^n is finitely generated for each n, then there is a space X which realizes B in that $H^*(X; Z) \approx B$. One solves this problem, for a single dimension n, by a cluster C_n of n-spheres and $(n + 1)$-cells; and then the general case is solved by a union of the C_n's