ON SOME VERSIONS OF TAYLOR'S THEOREM

ON SOME VERSIONS OF TAYLOR'S THEOREM

by R. P. Boas, Jr., Evanston

(Reçu le 10 juin 1959.)

A familiar form of Taylor's theorem with remainder states that, under suitable hypotheses, if $n > 1$,

$$(1) \quad f(a) = f(o) + a f'(o) + \cdots + \frac{a^{n-1}}{(n-1)!} f^{(n-1)}(o) + \frac{a^n}{n!} f^{(n)}(\xi), \; 0 < \xi < a$$

It is usual to suppose at least that f is continuous in $[0, a]$, that $f^{(n-1)}$ is continuous in $[0, a)$, and that $f^{(n)}(x)$ exists (finite or infinite) in $(0, a)$. The formula can, of course, be written down under less stringent hypotheses; a recent paper in this journal [1] shows that it is valid when the continuity of $f^{(n-1)}$ at 0 is omitted. This has been noticed before [2]. What I want to point out is that while the theorem with is true the weaker hypothesis, it is trivial. More precisely, we have the following result.

Theorem 1. *If* $f^{(n-1)}$ *is not continuous (on the right) at* 0, $f^{(n)}$ *assumes all real values in* $0 < x < a$ *and so (1) holds for some* ξ *whether the coefficients have Taylor's form or not.*

This was in fact proved long ago by Hobson [3, vol. 2, p. 203] with the unnecessary additional restriction that $f^{(n)}$ is never infinite in $(0, a)$.

The proof depends on two facts, the first of which is a well known corollary of the law of the mean.

Lemma 1. *If* f *is continuous and* $f'(x)$ *exists (finite or infinite) in* $p \leq x < q$ *(as a right-hand derivative at* p*), then if the limit* $f'(p^+)$ *exists (finite or infinite) it is equal to* $f'(p)$.

That is, f' cannot have a simple jump, finite or infinite.

Lemma 2. *If* f *is continuous and* f' *exists (finite or infinite) in* (p, q), *while* $f(p^+)$ *does not exist (finite or infinite) then* $f'(x)$ *assumes every finite value in* (p, q).

Lemma 2 is proved by Hobson [3, vol. 1, p. 363] with the unnecessary restriction that f' is finite in (p, q). Since the proof is short and the result is not well known, I give the proof.

If $f(x)$ does not approach a limit as $x \to p^+$, neither does the continuous function $H(x) = f(x) - \lambda x$, where λ is an arbitrary real number. Hence H is not monotonic in a right-hand neighborhood of 0, so it has extrema. At an extremum ξ, $H'(\xi) = 0$, i.e. $f'(\xi) = \lambda$.

Now consider Taylor's theorem when $f^{(n-1)}$ is not continuous at 0. Since $f^{(n)}$ is a derivative, by Lemma 1 it does not approach a limit; by Lemma 2, $f^{(n)}$ assumes every finite value; consequently Taylor's theorem (1) is trivial.

We can go further and exclude some other plausible weakened hypotheses for (1). There is, for example, nothing in the structure of (1) to require that $f^{(n-1)}$ is continuous if we admit infinite values for $f^{(n)}$. However, we can establish the following result.

THEOREM 2. *Formula (1) is trivial unless* $f^{(n-1)}$ *is continuous in* $[0, a)$ *and* $f^{(n)}$ *is (Lebesgue) integrable on every subinterval* $(0, b)$ *and bounded on one side.*

In fact, if $f^{(n)}(x)$ is finite in $(0, a)$, $f^{(n-1)}$ is continuous in $(0, a)$ and so in $[0, a)$ unless (1) is trivial. Suppose that $f^{(n)}(c)$ is infinite, $0 < c < a$. By Lemma 2, unless $f^{(n-1)}$ approaches limits from both sides as $x \to x_0$, $f^{(n)}$ assumes all real values and (1) is trivial. If $f^{(n-1)}$ approaches limits from both sides at c, it is continuous at c by Lemma 1.

Again, if $f^{(n)}$ is unbounded both above and below, it assumes all real values since a derivative has the Darboux property [3, vol. 1, p. 379]. If $f^{(n)}$ is bounded below, then $f^{(n-1)}(x) + \lambda x$, with a sufficiently large λ, is non-decreasing. It follows from Fatou's lemma that $f^{(n)}$ is integrable on every $(0, b)$.

There are a number of other forms of the remainder in Taylor's theorem, of the general type

$$R_n = A_n\, g(\xi)\, f^{(n)}(\xi), \quad 0 < \xi < a, \tag{2}$$

with a suitable auxiliary function g, and A_n independent of f and ξ.

THEOREM 3. *The propositions about the triviality of Taylor's theorem that we have established with* $g(x) \equiv 1$ *still hold with the remainder (2) provided that* g *is bounded away from* 0 *in every neighborhood of* a *and* $1/g$ *is a derivative.*

To verify this we need slight extensions of Lemma 2, and of the fact that derivatives possess the Darboux property.

LEMMA 2′. *If* f *is continuous and* f′ *exists (finite or infinite) in* (p, q), *while* $f(p^+)$ *does not exist (finite or infinite); if* G *is continuous in* [p, q), G′ *exists (finite) in* (p, q) *and* $G'(x) \neq 0$ *in* (p, q); *then* $f'(x)/G'(x)$ *assumes every finite value in* (p, q).

Since $f(p^+)$ does not exist, $H(x) = f(x) - \lambda G(x)$ does not approach a limit (since $G(p^+)$ does exist). Hence H is not monotonic and so possesses extrema. At an extremum ξ we have $H'(\xi) = 0$, so $f'(\xi) = \lambda G'(\xi)$. Since $G'(\xi)$ is neither 0 nor infinite, $f'(\xi)/G'(\xi) = \lambda$.

LEMMA 3. *If* f *and* G *are continuous in* [p, q]; *if* f′ *exists (finite or infinite) in* [p, q], *and* G′ *exists (finite) in* [p, q]; *if* f′ (p) *and* f′ (q) *are finite and* G′ *has a fixed sign (and hence is never* 0*) in* [p, q]; *and if*

$$f'(p)/G'(p) < c < f'(q)/G'(q),$$

then there is a ξ *in* (p, q) *such that* $f'(\xi)/G'(\xi) = c$.

This says in effect that f'/G', like f', has the Darboux property.

Consider $H(x) = f(x) - cG(x)$ and suppose for definiteness that $G'(p) > 0$. Then $H'(p) < 0$, $H'(q) > 0$, so the continuous function H cannot assume its minimum at p or q. If H assumes its minimum at ξ, we have $f'(\xi) = cG'(\xi)$ and so (since $G'(\xi)$ is neither zero nor infinite), $f'(\xi)/G'(\xi) = c$.

It now follows just as before that Theorems 1 and 2 hold, with $g = 1/G'$ in (2).

REFERENCES

1. GODEFROID, M., Remarque su la formule de Taylor. *Enseignement math.* (2) 4 (1958), 120-123.
2. HARTMAN, P. Remark on Taylor's formula. *Bull. Amer. Math. Soc.*, 51 (1945), 731-732.
3. HOBSON, E. W., *The theory of functions of a real variable and the theory of Fourier's series.* Cambridge University Press, vol. 1, 3d ed., 1927; vol. 2, 2d ed., 1926.