THE NORM OF A REAL LINEAR TRANSFORMATION IN MINKOWSKI SPACE

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1. The definition of norm

By the Minkowski space $l^p(n)$ we mean the space of vectors $x = (\xi_1, \ldots, \xi_n)$ with the norm of x defined by

$$\| x \|_p = \Big(\sum_i | \xi_i |^p \Big)^{1/p} .$$

Here it is supposed that $p \geqq 1$, so that $\| x \|_p$ is a norm on $l^p(n)$.

If $l^p(n)$ and $l^q(m)$ are Minkowski spaces of dimensions n and m, respectively, a linear transformation A of $l^p(n)$ into $l^q(m)$ is determined by a matrix (a_{jk}) of constants $(j = 1, \ldots, m, k = 1, \ldots, n)$; if A transforms x into $y = (\eta_1, \ldots, \eta_m)$, the η's are given in terms of the ξ's by the equations

$$\eta_j = \sum_{k=1}^{n} a_{jk} \xi_k \qquad j = 1, \ldots, m .$$

If we write $y = \mathrm{A}x$, the norm of A is defined as

$$\| \mathrm{A} \| = \max_{x \neq 0} \frac{\| \mathrm{A}x \|_q}{\| x \|_p} = \max_{x \neq 0} \frac{\Big(\sum_j \Big| \sum_k a_{jk} \xi_k \Big|^q \Big)^{1/q}}{\Big(\sum_k | \xi_k |^p \Big)^{1/p}} .$$

We may consider all of these things with respect to the complex field, letting the vector components $\xi_1, \ldots, \xi_n$ and the matrix elements a_{jk} be complex numbers. In this case we call $l^p(n)$ a *complex* Minkowski space and A a *complex* linear transformation. But we may equally well confine our attention to real scalars, in which case the space and the transformation are called *real*.

Now, if A is a transformation determined by a matrix of real elements a_{jk}, the transformation can be considered either as a real transformation or as a complex transformation, and accordingly there are two possible definitions of its norm. If in (1) we allow x to vary over all nonzero elements of the complex space $l^p(n)$, we get the norm of A as a complex transformation, whereas if we restrict the vector x to have real components, we get the norm of A as a real transformation. Let us denote these two norms by

$$\| A \|_c \quad \text{and} \quad \| A \|_r .$$

2. The theorem

We shall prove the following result:

THEOREM *Let* A *be a transformation of* $l^p(n)$ *into* $l^q(m)$ *determined by a matrix* (a_{jk}) *of real constants. Suppose* $q \geqq p \geqq 1$. *Then*

$$\| A \|_c = \| A \|_r . \tag{2}$$

Proof. We first observe that when p is fixed and q varies subject to $q \geqq p$, $\| A \|$ is a continuous function of q at $q = p$, regardless of whether we have a real or a complex transformation. For, let the dependence on q be exhibited by writing $\|A\| = M(q)$. It is well known that $\| y \|_q$ does not decrease as q decreases. Hence $p < q$ implies

$$\frac{\| Ax \|_q}{\| x \|_p} \leqq \frac{\| Ax \|_p}{\| x \|_p},$$

whence also $M(q) \leqq M(p)$. Now suppose that x is chosen $(x \neq 0)$ so that

$$\frac{\| Ax \|_p}{\| x \|_p} = M(p).$$

Keeping this x fixed, let $q \to p$. Then

$$\frac{\| Ax \|_q}{\| x \|_p} \to M(p) ,$$

for, $\| Ax \|_q$ certainly depends continuously on q. But

$$\frac{\| Ax \|_q}{\| x \|_p} \leqq M(q) \leqq M(p) ;$$

hence $M(q) \to M(p)$ when q approaches p from above.

It follows from this that it suffices to prove the theorem on the assumption that $q > p \geqq 1$, for the theorem will then also be true in the limiting case $q = p$.

It is evident that $\| A \|_r \leqq \| A \|_c$. Hence, to prove the theorem, it is enough to prove that $\| A \|_r < \| A \|_c$ is impossible. We therefore start with the assumption that $q > p$ and that $\| A \|_r < \| A \|_c$. Let $x = (\xi_1, \ldots, \xi_n)$ be a vector with complex components such that $x \neq 0$ and

$$\frac{\| Ax \|_q}{\| x \|_p} = \| A \|_c .$$

Let

$$\sum_k a_{jk} \xi_k = A_j + i B_j ,$$

where A_j and B_j are real. Observe that $| A_j + i B_j | \neq 0$ for at least one j, for otherwise we would have $\| A \|_c = 0$, contrary to $\| A \|_r < \| A \|_c$. Next, there is no real θ such that $e^{-i\theta} \xi_k$ is real for each k. For, if there were, we could let $u = (\rho_1, \ldots, \rho_k)$, where $\rho_k = e^{-i\theta} \xi_k$; then u would be a real vector such that

$$\frac{\| Au \|_q}{\| u \|_p} = \frac{\| Ax \|_q}{\| x \|_p} = \| A \|_c ,$$

and this would imply $\| A \|_r = \| A \|_c$. Now let $\xi_k = \alpha_k + i \beta_k$, where α_k and β_k are real. Then the real vectors $(\alpha_1, \ldots, \alpha_n)$ and $(\beta_1, \ldots, \beta_n)$ are linearly independent, for if the contrary were the case there *would* be a θ such that $e^{-i\theta} \xi_k$ is real for each k.

Now consider the function F of two real variables s, t defined as follows:

$$F(s,t) = \frac{\left(\sum_{j=1}^{m} | A_j + (s+it) B_j |^q\right)^{1/q}}{\left(\sum_{k=1}^{n} | \alpha_k + (s+it) \beta_k |^p\right)^{1/p}} .$$

If $w = (\omega_1, \ldots, \omega_n)$ is the complex vector with components

$$\omega_k = \alpha_k + s\beta_k + it\beta_k ,$$

we see that

$$F(s,t) = \frac{\| Aw \|_q}{\| w \|_p} .$$

Hence $F(s,t) \leqq \| A \|_c$ for all s and t, while $F(0,1) = \| A \|_c$. Thus F attains its absolute maximum value at the point (0, 1).

This being the case, we arrive at a contradiction, and hence complete the proof of the theorem, by proving the following lemma:

LEMMA. *Let* $(\alpha_1, \ldots, \alpha_n)$ *and* $(\beta_1, \ldots, \beta_n)$ *be two linearly independent vectors in real n-dimensional space. Let* $(A_1, \ldots, A_m)$ *and* $(B_1, \ldots, B_m)$ *be two vectors, not both zero, in real m-dimensional space.*
Suppose $0 < p < q$ *and let*

$$F(s,t) = \frac{\left(\sum_{j=1}^{m} | A_j + (s+it) B_j |^q\right)^{1/q}}{\left(\sum_{k=1}^{n} | \alpha_k + (s+it) \beta_k |^p\right)^{1/p}} ,$$

where s and t are real variables. Then F *cannot have a relative maximum at the point* $s = 0$, $t = 1$.

Proof. Let $F_1 = \frac{\partial F}{\partial s}$, $F_2 = \frac{\partial F}{\partial t}$, $F_{11} = \frac{\partial^2 F}{\partial s^2}$, $F_{22} = \frac{\partial^2 F}{\partial t^2}$.

The proof will be accomplished if we assume that F *does* have a relative maximum at (0,1) and then show that $F_{11} + F_{22} > 0$ at (0,1). For, with a relative maximum at (0,1) we necessarily have $F_{11} \leqq 0$ and $F_{22} \leqq 0$ at (0,1).

Let us write

$$P(s, t) = \sum_k |\alpha_k + (s + it)\beta_k|^p,$$

$$Q(s, t) = \sum_j |A_j + (s + it)B_j|^q.$$

Then

$$F(s, t) = \frac{Q^{1/q}}{P^{1/p}},$$

and we readily find that

$$F_1 = F\left\{\frac{1}{q}\frac{Q_1}{Q} - \frac{1}{p}\frac{P_1}{P}\right\}, \tag{3}$$

with a similar formula for F_2. The calculation of F_{11} yields the result

$$F_{11} = F\left\{\frac{1}{q}\frac{QQ_{11} - Q_1^2}{Q^2} - \frac{1}{p}\frac{PP_{11} - P_1^2}{P^2}\right\} + F_1\left\{\frac{1}{q}\frac{Q_1}{Q} - \frac{1}{p}\frac{P_1}{P}\right\}.$$

The formula for F_{22} is similar.

If we now assume that F has a relative maximum at (0,1), we know that $F_1 = F_2 = 0$ at (0,1), and hence in this case the value of $F_{11} + F_{22}$ at (0,1) takes the form

$$F_{11} + F_{22} = F\left\{\frac{Q(Q_{11} + Q_{22}) - (Q_1^2 + Q_2^2)}{q\,Q^2} - \frac{P(P_{11} + P_{22}) - (P_1^2 + P_2^2)}{p\,P^2}\right\}. \tag{4}$$

Everything now depends on a careful evaluation of this expression.

For convenience in notation it is now well to assume that $\alpha_k^2 + \beta_k^2 \neq 0$ for each k, and that $A_j^2 + B_j^2 \neq 0$ for each j. There is no loss in generality in these assumptions. For certainly $\alpha_k^2 + \beta_k^2 \neq 0$ for *some* k. By re-indexing and diminishing n, if necessary, we insure that $\alpha_k^2 + \beta_k^2 \neq 0$ for every value of k that is considered. Similar remarks apply to $A_j^2 + B_j^2$.

Observing that

$$\frac{\partial}{\partial t}\alpha_k + (s + it)\beta_k| = \frac{(\alpha_k + s\beta_k)\beta_k}{|\alpha_k + (s + it)\beta_k|},$$

and

$$\frac{\partial}{\partial t} | \alpha_k + (s + it) \beta_k | = \frac{t \beta_k^2}{| \alpha_k + (s + it) \beta_k |},$$

we easily find

$$P_1 (s, t) = \sum_k p | \alpha_k + (s + it) \beta_k |^{p-2} (\alpha_k + s \beta_k) \beta_k ,$$

$$P_2 (s, t) = \sum_k p | \alpha_k + (s + it) \beta_k |^{p-2} t \beta_k^2 ,$$

$$P_{11} (s, t) = p \sum_k | \alpha_k + (s + it) \beta_k |^{p-2} \beta_k^2 + p (p - 2) \sum_k | \alpha_k + (s + it) \beta_k |^{p-4} (\alpha_k + s \beta_k)^2 \beta_k^2 ,$$

$$P_{22} (s, t) = p \sum_k | \alpha_k + (s + it) \beta_k |^{p-2} \beta_k^2 + p (p - 2) \sum_k | \alpha_k + (s + it) \beta_k |^{p-4} t^2 \beta_k^4 .$$

On putting $s = 0$, $t = 1$, we find that

$$P_{11} + P_{22} = p P_2 \tag{5}$$

at this point. Likewise

$$Q_{11} + Q_{22} = q Q_2 . \tag{6}$$

We now use (5) and (6) to simplify (4) by eliminating the second derivatives. Also, we eliminate the Q terms from (4) by means of (3), together with the fact that $F_1 = 0$ at (0,1), and a similar use of the fact that $F_2 = 0$. In this way we arrive at the formula

$$F_{11} + F_{22} = \left\{ p PP_2 - \left(P_1^2 + P_2^2\right) \right\} \frac{q - p}{p^2} \frac{F}{P^2} \cdot \cdot \tag{7}$$

We repeat: this holds at (0,1) as a consequence of having $F_1 = F_2$ there.

Since $q > p$ and F (0,1) > 0, all that remains is to prove the inequality

$$P_1^2 + P_2^2 < p PP_2 \tag{8}$$

at the point (0,1).

For convenience let us write

$$c_k = | \alpha_k + i \beta_k |^{\frac{p-2}{2}} .$$

Then, at (0,1),

$$P_1 = p \sum_k c_k^2 \alpha_k \beta_k , \quad P_2 = p \sum_k c_k^2 \beta_k^2 ,$$

$$P = \sum_k c_k^2 (\alpha_k^2 + \beta_k^2) \cdot$$

Now, since $c_k \neq 0$ and the vectors $(\alpha_1, ..., \alpha_n)$, $(\beta_1, ..., \beta_n)$ are linearly independent, the vectors $(c_1 \alpha_1, ..., c_n \alpha_n)$ and $(c_1 \beta_1, ..., c_n\beta_n)$ are also linearly independent. Consequently, by Cauchy's inequality,

$$|P_1| < p \left(\sum_k c_k^2 \alpha_k^2\right)^{1/2} \left(\sum_k c_k^2 \beta_k^2\right)^{1/2} ,$$

the inequality being strict because of the linear independence. Consequently

$$P_1^2 < p P_2 \sum_k c_k^2 \alpha_k^2 \cdot$$

Then

$$P_1^2 + P_2^2 < p P_2 \sum_k c_k^2 \alpha_k^2 + p P_2 \sum c_k^2 \beta_k^2 \cdot$$

The right side here is exactly $p P_2 P$, and so the proof of (8) is complete. This finishes the proof of the lemma. We have already pointed out how the lemma leads to a proof of the theorem.

3. Concluding remarks

In conclusion, we point out that the results we have described were known to M. Riesz when he wrote his paper on convexity and bilinear forms.[1] He made a brief sketch of the arguments in support of the results. But the intended form of Riesz's argument has seemed obscure to some people, and the results themselves are apparently not much known outside the circle of those who are thoroughly familiar with Riesz's paper. Hence it has seemed to be worth while to emphasize the results and to put the details of the proof on record.

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[1]) M. Riesz, *Sur les maxima des formes bilinéaires et sur les fonctionelles linéaires*, Acta Math. 49, 465-497 (1927).