3. Al-Khwrizms solution.

five. AG times GB is twenty one. Then the remainder is HG multiplied by itself, or four; or HG is two. HB is five. Then GB remains as three and AG is seven."

Fig. 2

3. *Al-Khwārizmī's solution.*

Although al-Khwārizmī does not make use of the more abstract one line proof of Heron, nevertheless it is evident that he leans on the concrete concept of root [12] already known in ancient Babylonian times. In his discussion of the equation $x^2 + 21 = 10x$, al-Khwārizmī [13] makes it evident that he is utilizing a concept extremely practical in geometric terms.

> " When a square plus twenty one dirhems are equal to ten roots, we depict the square as a square surface AD of unknown sides. Then we join it to a parallelogram, HB, whose width, HN, is equal to one of the sides of AD. The length of the two surfaces together is equal to the side HC. We know its length to be ten numbers since every square has equal sides and angles; and if one of its sides is multiplied by one, this gives the root of the surface, and if by two, two of its roots. When it is declared that the square plus twenty one equals ten of its roots, we know that the length of the side HC equals ten numbers because the side CD is a root of the square figure. We divide the line CH into two halves on the point G. Then you know that line HG equals line GC, and that line GT equals line CD. Then we extend line GT a distance equal to the difference between line CG and line GT to make the quadrilateral. Then line TK equals line KM, making a quadrilateral MT of equal sides and angles. We know that the line TK and the other sides equals five. Its surface is twenty five obtained by the

multiplication of the half of the roots by itself, or five by five equals twenty five. We know that the surface HB is the twenty one that is added to the square. From the surface HB, we cut off line TK, one of the sides of the surface MT, leaving the surface TA. We take from the line KM line KL which is equal to line GK. We know that line TG equals line ML and that line LK cut from line MK equals line KG. Then the surface MR equals surface TA. We know that surface HT plus surface MR equals surface HB, or twenty one. But surface MT is twenty five. And so, we subtract from surface MT, surface HT and surface MR, both equal to twenty one. We have remaining a small surface, RK, or twenty five less twenty one, or four. Its root, line RG, is equal to line GA, or two. If we subtract it from line CG, which is half of the roots, there remains line AC, or three. This is the root of the first square. If it is added to line GC, which is half of the roots, it comes to seven, or line RC, the root of a larger square. If twenty one is added to it, the result is ten of its roots. This is the figure."

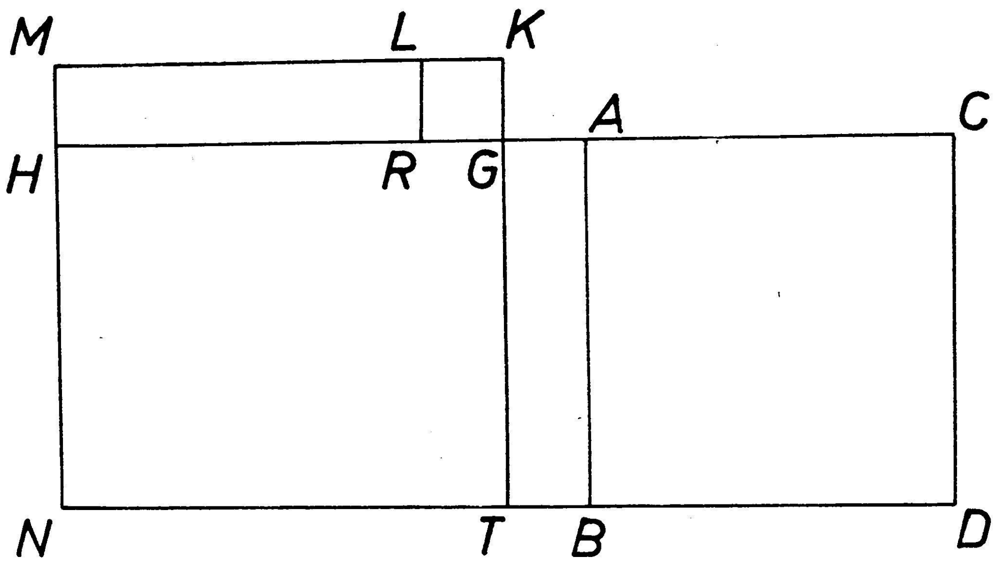

Fig. 3

The algebra of al-Khwārizmī admits the double solution and has a novel manner of utilizing geometry for algebra. Al-Khwārizmī shows the three Arabic types of quadratic equations. In reality, this classification is comparable with the standardized types which had long before been set up by the Babylonians [14]. It is interesting that Al-Khwārizmī shows no evidence of acquaintance with the work of the great Greek algebraist, Diophantus [15].

4. *Abū Kāmil's solution.*

Shūja' [16] also discusses the solution of the question $x^2 + 21 = 10x$, the problem treated by al-Khwārizmī. He solves the equation algebraically in the following steps. Modern symbols have been substituted for the sake of brevity.

$$x = \frac{10}{2} - \sqrt{\left(\frac{10^2}{2}\right) - 21} = 3$$

$$x = \frac{10}{2} + \sqrt{\left(\frac{10}{2}\right)^2 - 21} = 7 .$$

The equation is also solved directly for the two values of x^2:

$$x^2 = \frac{10^2}{2} - 21 - \sqrt{\left(\frac{10^2}{2}\right)^2 - 10^2 . 21} = 9$$

$$x^2 = \frac{10^2}{2} - 21 + \sqrt{\left(\frac{10^2}{2}\right)^2 - 10^2 . 21} = 49 .$$

Then he gives the following demonstration for the equation:

" I shall explain all this. I take the number, twenty one, which is together with the square and larger than the square. I construct the square as a square surface, ABGD, and add to it the twenty one which is the surface ABHL. This surface is greater than surface ABGD. Then, because of this, line BL is greater than line BD. The surface HD equals ten of the roots of ABGD. Then line LD is ten and the surface HB is twenty one, or equal to the product of LB and BD for BD equals BA. Line LD is then divided into two halves by the point X. It had already been divided into two unequal parts by point B. Therefore, the product of LB by BD added to the square on XB is equal to the square