2. Umbral Relations.

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$$(11) \qquad U_k = (2^{2k} - 1) V_k = (-1)^k (2^{2k} - 1) \frac{B_k}{4k} .$$

As is well known, the double series occurring in (1), (4), (5), (6) are absolutely convergent for $k > 1$; for $k = 1$, the convergence is conditional. However, as has been shown by HURWITZ [2] in the case of (1), if the summation is first carried out with respect to m and then with respect to n, the resulting sum agrees with (2) with $k = 1$. For this case ($k = 1$) similar conditions hold for (4), (5) and (6). These matters are of relevance in studying certain modular transformations of these functions to be discussed later.

2. UMBRAL RELATIONS.

The functions defined in what precedes arise in a natural manner as a consequence of the well-known fact that the Jacobi theta functions are solutions of the partial differential equation

$$(12) \qquad \frac{\partial^2 z}{\partial s^2} = 2 \frac{\partial z}{\partial t} , \quad z = \theta_r (v, \tau) , \quad (r = 1, 2, 3, 4) ,$$

with $s = 2\pi v$ and $-t = 2\pi i \tau$, and, what appears to be less well-known, that the functions $u = ln\theta_r (v, \tau)$ satisfy the non-linear equation:

$$(13) \qquad \frac{\partial^2 u}{\partial s^2} = 2 \frac{\partial u}{\partial t} - \left(\frac{\partial u}{\partial s} \right)^2 .$$

Here, the notation for the theta function is that used in TANNERY-MOLK's treatise [4].

The arithmetical consequence of (13) can best be obtained through the use of the infinite product representation of $\theta_r (v, \tau)$. It is found that the calculations needed are greatly facilitated and the results obtained very simply expressed in a symbolic form through an application of the umbral calculus of BLISSARD and LUCAS [5]. It is not feasible to give details for all cases and we merely indicate briefly the nature of the calculations for the case $r = 4$. Thus, since,

$$(14) \qquad \theta_4(v, \tau) = Q_0 \prod_{n=1}^{\infty} (1 - q^{2n-1} e^{2\pi i v})(1 - q^{2n-1} e^{-2\pi i v}) ,$$

$$Q_0 = \prod_{n=1}^{\infty} (1 - q^{2n}) , \quad q = e^{\pi i \tau} ,$$

and taking into account the change in variables from (v, τ) to (s, t) it is found that if $u(s, t) = ln\, \theta_4(v, \tau)$, then

$$(15) \qquad \frac{\partial u}{\partial s} = 2 \sum_{n=1}^{\infty} \frac{q^n}{1 - q^{2n}} \sin ns , \quad q = e^{-t/_2} ,$$

and

$$(16) \qquad \frac{\partial^2 u}{\partial s^2} = 2 \sum_{n=1}^{\infty} \frac{nq^n}{1 - q^{2n}} \cos ns ;$$

moreover,

$$(17) \qquad \frac{\partial u}{\partial t} = \sum_{n=1}^{\infty} \frac{nq^n}{1 - q^{2n}} - 2 \frac{\partial}{\partial t} \left\{ \sum_{n=1}^{\infty} \frac{q^n}{1 - q^{2n}} \frac{\cos ns}{n} \right\}$$

provided $Re t \pm 2 Im\, s > 0$ in (15), (16) and (17).

Now, in (15) replace sin *ns* by its power series development and interchange the order of summation to obtain

$$(15)_1 \qquad \frac{\partial u}{\partial s} = 2 \sum_{k=1}^{\infty} (-1)^{k-1} \frac{s^{2k-1}}{(2k-1)!} \Psi_{2k-1}(t) .$$

Hence if Ψ is the umbra of the sequence $\{\Psi_{2k-1}(t)\}$ we may write symbolically:

$$(18) \qquad \frac{\partial u}{\partial s} \cong 2 \sin \Psi s , \quad \frac{\partial^2 u}{\partial s^2} \cong 2\, \Psi \cos \Psi s .$$

Similarly, a more extended calculation shows that

$$(19) \qquad \frac{\partial u}{\partial t} \cong \Psi^{(1)} + 2 \frac{\partial}{\partial t} \frac{1 - \cos \Psi s}{\Psi} .$$

In (18) and (19), in order to pass from symbolic equality to actual equality, the functions $\sin \Psi s$, $\cos \Psi s$ and $(1 - \cos \Psi s)/\Psi$ are to be expanded in powers of s and then the exponents in the powers of Ψ are lowered into subscripts; thus $\Psi^{(1)}$ would then be written $\Psi_1(t)$.

If (18) and (19) are substituted in (13), there results the following umbral identity:

$$\Psi(1 - \cos \Psi s) + 2 \frac{\partial}{\partial t}\left(\frac{1 - \cos \Psi s}{\Psi}\right) = 2 \sin \Psi s * \sin \Psi s \; , \tag{20}$$

where the asterisk (*) indicates umbral multiplication.

For the cases $r = 2, 3$, rather extensive calculations show that umbral identities of the same form exist. We may therefore state the following result which is implied by the non-linear equation (13).

Theorem 1: "Let Ψ, X, Φ be respectively the umbrae of the sequences $\{\Psi_{2k-1}(t)\}$, $\{X_{2k-1}(t)\}$, and $\{\Phi_{2k-1}(t)\}$. If γ is one of these umbrae, then the following umbral identity holds:

$$\gamma(1 - \cos \gamma s) + 2 \frac{\partial}{\partial t}\left(\frac{1 - \cos \gamma s}{\gamma}\right) = 2 \sin \gamma s * \sin \gamma s \; . \tag{21}$$

3. Recurrences.

It is clear that (20) implies a recurrence relation for the functions $\Psi_j(t)$, and indeed, Theorem 1 yields the following.

Theorem 2: "Let $\gamma_j(t)$ be $\Psi_j(t)$, $X_j(t)$ or $\Phi_j(t)$; then the following recurrence holds:

$$\frac{d}{dt}\gamma_{2n-1}(t) + \frac{1}{2}\gamma_{2n+1}(t) = \sum_{k=0}^{n-1}\binom{2n}{2k+1}\gamma_{2k+1}(t)\,\gamma_{2n-2k-1}(t) \; , \tag{22}$$

and hence $\gamma_{2n+1}(t)$ is a polynomial in $\gamma_1(t)$ and its derivatives up to order n."

This result, in turn, implies the following

Theorem 3: "Let $\rho_{2k-1}(n)$ be either of the arithmetical func tions $\beta_{2k-1}(n)$ or $\xi_{2k-1}(n)$ defined by (8) and (10) respectively; then $\rho_{2k-1}(n)$ satisfies a recurrence relation of the form:

$$\rho_{2k+1}(n) - n\,\rho_{2k-1}(n) = 2\sum_{s=0}^{k-1}\sum_{j=1}^{n-1}\binom{2k}{2s+1}\rho_{2s+1}(j)\,\rho_{2k-2s-1}(n-j), \tag{23}$$