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The exact convergence rate in the ergodic theorem of Lubotzky–Phillips–Sarnak and a universal lower bound on discrepancies

Antoine Pinochet Lobos and Christophe Pittet

Abstract. We establish a new result about the equidistribution of points on the two-dimensional round sphere. More precisely, we improve an upper bound of Lubotzky–Phillips–Sarnak on the discrepancies of some finite symmetric sets of isometries of the sphere defined with the help of Lipschitz quaternions. We show that a simple application of the spectral theorem leads to the best possible upper bound. Our proof relies on the deep result of Lubotzky–Phillips–Sarnak about spectral properties of some special free groups of isometries of the sphere. It leads to an upper bound on the discrepancy which matches *exactly* a general lower bound suspected by Lubotzky. We confirm Lubotzky's guess by proving a universal lower bound on the discrepancies of actions on atomless probability spaces. We also mention some facts relating discrepancies to standard deviations, spectral gaps, amenability, Kazdhan pairs and ϵ-good sets. In an appendix, we emphasize the relation between the computed discrepancies and the values of the Harish-Chandra function of a free group.

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Keywords. Convergence rate, discrepancy, equidistribution, Harish-Chandra function, Koopman representation, quasi-regular representation, von Neumann ergodic theorem.

1. Introduction

1.1. Birkhoff's sums and discrepancies. Let (X, ν) be a probability space and let Γ be a group acting on X by measure preserving transformations. Let $\gamma_1, \gamma_1^{-1}, \dots, \gamma_n, \gamma_n^{-1}$ be a finite symmetric family F of elements of Γ.

Definition 1.1. The *discrepancy* $\delta(F)$ of F relative to the given action of Γ on X is defined as

$$\delta(F) = \sup_{\|f\|_{L^2(X,\nu)}=1} \left\| x \mapsto \left(\int_X f(y) d\nu(y) - \frac{1}{2n} \sum_{k=1}^{n} f(\gamma_k^{-1} x) + f(\gamma_k x) \right) \right\|_{L^2(X,\nu)}.$$

(In the above statement, the expression $\|x \mapsto \varphi(x)\|_{L^2(X,\nu)}$ denotes the L^2-norm $\|\varphi\|_{L^2(X,\nu)}$ of a function φ. In what follows we will shorten the notation, writing $\|\varphi(x)\|_{L^2(X,\nu)}$ instead of $\|x \mapsto \varphi(x)\|_{L^2(X,\nu)}$.)

Decomposing f as $f = f_0 + f_1$, where

$$f_0(x) = f(x) - \int_X f(y)d\nu(y),$$

it is easy to check that

$$0 \leq \delta(F) \leq 1$$

(these inequalities also obviously follow from Definition 3.1 below). A heuristic necessary condition for the discrepancy to be small is that for almost all $x \in X$, the partial orbit

$$\gamma_1 x, \dots, \gamma_n x,$$

"spreads out" through the whole space X: Birkhoff's sums on partial orbits

$$\frac{1}{2n}\sum_{k=1}^{n} f(\gamma_k^{-1}x) + f(\gamma_k x)$$

are expected to be good approximations (when n is large) of integration over the whole space

$$\int_X f(y)d\nu(y),$$

for any $f \in L^2(X, \nu)$, only when the action is ergodic. In fact, the existence of a finite symmetric family F of elements of Γ such that

$$\delta(F) < 1$$

implies the ergodicity of the action of Γ on (X, ν), see Proposition 3.4 below for a proof. But the condition $\delta(F) < 1$, is stronger than ergodicity: according to Corollary 3.10 below, if the measure ν has no atom, then any symmetric finite family F which generates an amenable group satisfies

$$\delta(F) = 1,$$

so if we consider for example the group $\Gamma = \mathbb{Z}$ generated by a rotation T of infinite order acting on the circle $(S^1, d\theta)$, and if we choose the family $F = \{T, T^{-1}\}$, then $\delta(F) = 1$, although the action of Γ is ergodic. There is a special class of groups whose actions are ergodic if and only if their discrepancies are less than 1: if the group Γ generated by F has Kazhdan property (T) – and if the space $L^2(X, \nu)$ is separable – then the ergodicity of the action implies that $\delta(F) < 1$, see Proposition 3.4 below.

See also Remark 3.5 below for the equivalence between the condition $\delta(F) < 1$ and *strong ergodicity*. Notice that if we allow the measure ν to have atoms, then actions of amenable groups with zero discrepancy do exist, as the following simple example shows. Consider a finite group

$$\Gamma = \{\gamma_1, \dots, \gamma_n\}$$

of cardinality $|\Gamma| = n$, acting transitively on a finite set X. In this case, the invariant probability measure ν on X has to be the uniform measure

$$\nu = \frac{1}{|X|} \sum_{x \in X} \delta_x.$$

The discrepancy of the family $\gamma_1, \gamma_1^{-1}, \dots, \gamma_n, \gamma_n^{-1}$ equals zero: for any $x \in X$, and any $f \in l^2(X, \nu)$,

$$\frac{1}{|X|} \sum_{y \in X} f(y) = \frac{1}{2n} \sum_{k=1}^{n} f(\gamma_k^{-1} x) + f(\gamma_k x).$$

(In order to prove the above equality, it is enough to check it for f a characteristic function of a point of X.)

1.2. Smallest possible discrepancies on atomless probability spaces. Any finite symmetric family $\gamma_1, \gamma_1^{-1}, \dots, \gamma_n, \gamma_n^{-1}$ of elements of $\mathrm{SO}(3, \mathbb{R})$ acting on the round sphere $X = S^2$, endowed with its normalized uniform measure, satisfies

$$\delta(\gamma_1, \gamma_1^{-1}, \dots, \gamma_n, \gamma_n^{-1}) \geq \frac{\sqrt{2n-1}}{n}, \tag{1}$$

and when $n = \frac{p+1}{2}$, with p prime such that $p \equiv 1 \bmod 4$, the above lower bound is reached by some families constructed by Lubotzky Phillips and Sarnak (see [30, Theorem 1.3]). But a generic finite symmetric family of elements of $\mathrm{SO}(3, \mathbb{R})$ does *not* reach the above lower bound [30, Theorem 1.4]. In contrast, any family $\gamma_1, \dots, \gamma_n$ of elements of $\mathrm{SL}(2, \mathbb{Z})$, acting naturally on $T^2 = \mathbb{R}^2/\mathbb{Z}^2$, which generates a free group of rank n, satisfies

$$\delta(\gamma_1, \gamma_1^{-1}, \dots, \gamma_n, \gamma_n^{-1}) = \frac{\sqrt{2n-1}}{n},$$

see Theorem 2.5 below. Lubotzky writes in [29, p. 121] "It seems likely that there should be a general result like [the above lower bound (1)] for much more general X." This is indeed the case: for any finite symmetric family $\gamma_1, \gamma_1^{-1}, \dots, \gamma_n, \gamma_n^{-1}$, of any group, acting by measure preserving transformations, on any atomless probability space,

$$\delta(\gamma_1, \gamma_1^{-1}, \ldots, \gamma_n, \gamma_n^{-1}) \geq \frac{\sqrt{2n-1}}{n}. \tag{2}$$

Moreover, if the above lower bound is reached, then the group generated by

$$\gamma_1, \ldots, \gamma_n$$

is free of rank n. These facts are consequences of Theorem 3.6 below, combined with results from Kesten's thesis [27, Theorem 3] on the norm of Markov operators associated to the regular representation. (Lubotzky's question has been answered, without a proof, in exactly the same terms by Shalom in [41, Theorem 4.13], where the reader is referred to an unpublished manuscript entitled "Hecke operators of group actions and weak containment of unitary representations".)

1.3. Monte-Carlo estimators and discrepancies. We explain here in what sense standard deviations of Monte-Carlo estimators generalize discrepancies of group actions and discuss the question whether Monte-Carlo estimators may have standard deviations smaller than discrepancies of groups actions. The reader interested only in the dynamical aspects of discrepancies may safely skip this section. Let (X, ν) be a probability space. Let $Z_1, \ldots, Z_n$ be a family of n random variables with values in X with the same law ν. Let $f \in L^2(X, \nu)$. The Monte-Carlo estimator of the mean of f is the complex random variable

$$\frac{1}{n}\sum_{k=1}^{n} f(Z_k).$$

Its expectation is

$$\mathbb{E}\left[\frac{1}{n}\sum_{k=1}^{n} f(Z_k)\right] = \int_X f(y)d\nu(y).$$

In the case the random variables are *pairwise independent*, the variance of the Monte-Carlo estimator of the mean can be computed for any $f \in L^2(X, \nu)$ as

$$\mathbb{V}\left[\frac{1}{n}\sum_{k=1}^{n} f(Z_k)\right] = \frac{\|f\|_2^2 - \left|\int_X f(y)d\nu(y)\right|^2}{n} \leq \frac{\|f\|_2^2}{n}. \tag{3}$$

Hence the standard deviations σ (i.e., the positive square roots of the variances) satisfy:

$$\sup_{\|f\|_{L^2(X,\nu)}=1} \sigma\left[\frac{1}{n}\sum_{k=1}^{n} f(Z_k)\right] = \frac{1}{\sqrt{n}}.$$

If (X, ν) has no atom then

$$\sup_{\|f\|_{L^2(X,\nu)}=1} \sigma\left[\frac{1}{n}\sum_{k=1}^{n} f(Z_k)\right] \geq \frac{1}{\sqrt{n}}$$

holds true *without any independence hypothesis*. See [36].

Let us now explain in what sense the discrepancy of a group action is a special case of the discrepancy of a Monte-Carlo estimator of the mean. Let Γ be a group acting by measure preserving transformations on a probability space (X, ν). Let $\gamma_1, \gamma_1^{-1}, \dots, \gamma_n, \gamma_n^{-1}$ be a finite symmetric family of elements of Γ. As the action of Γ on X preserves ν, the random variables defined for all $x \in X$ as

$$Z_1(x) = \gamma_1 x,\ Z_2(x) = \gamma_1^{-1} x,\ \dots,\ Z_{2n-1}(x) = \gamma_n x,\ Z_{2n}(x) = \gamma_n^{-1} x,$$

all have the same law ν and we recover the discrepancy of the family in terms of standard deviations of Monte-Carlo estimators of the mean:

$$\sup_{\|f\|_{L^2(X,\nu)}=1} \sigma\left[\frac{1}{2n}\sum_{k=1}^{2n} f(Z_k)\right] = \delta(\gamma_1, \gamma_1^{-1}, \dots, \gamma_n, \gamma_n^{-1}).$$

To sum up, let (X, ν) be an atomless probability space; on the one hand, among all choices of $2n$ random variables with law ν, the choices when the random variables are pairwise independent, lead to the smallest possible standard deviations of Monte-Carlo estimators of the mean, and Formula (3) above shows that the value $\frac{1}{\sqrt{2n}}$ is achieved; on the other hand, any action of a family of $2n$ elements of a group has a discrepancy bounded below by $\frac{\sqrt{2n-1}}{n}$ and this lower bound is reached by some actions of free groups of rank n. The conclusion is that Monte-Carlo estimators using pairwise independent random variables do almost twice better than what can be obtained with groups actions. But from a practical point of view, producing independent random variables may be more difficult than producing an action with low discrepancy.

2. Statement of the main results

2.1. Convergence rates on the sphere. In [30] and [31] the theory of modular forms and the theory of unitary representations are applied to compute the discrepancy of orbit points of Lipschitz quaternions on the 2-sphere. We recall the construction from [31] of free subgroups of isometries of the round sphere. Let $\mathbb{H} = \{q = x_0 + x_1 i + x_2 j + x_3 k : x_0, x_1, x_2, x_3 \in \mathbb{R}\}$ be the field of quaternions. Let

$$\tau(q) = \overline{x_0 + x_1 i + x_2 j + x_3 k} = x_0 - x_1 i - x_2 j - x_3 k$$

denote the conjugate of q. Let $N(q) = q\overline{q}$ be the norm of q and let $|q| = \sqrt{N(q)}$ be its module. The multiplicative group $\mathbb{H}^*$ acts on $\mathbb{H}$ by conjugation and, if $q \in \mathbb{H}^*$ and $v \in \mathbb{H}$, then $|qvq^{-1}| = |v|$. As this action preserves the subspace $\operatorname{Im}\mathbb{H} = \{x_1 i + x_2 j + x_3 k : x_1, x_2, x_3 \in \mathbb{R}\}$, it defines a homomorphism

$$\operatorname{Ad} : \mathbb{H}^* \to \mathrm{SO}(3, \mathbb{R}),\ q \mapsto (v \mapsto qvq^{-1})$$

with values in the orientation preserving isometry group of the round sphere $\mathbb{S}^2$. (A word about notation and terminology. Identifying $\operatorname{Im}\mathbb{H}$ with the Lie algebra of the Lie group of quaternions of norm 1, we see that the map Ad just defined, evaluated at a quaternion q of norm 1, is the conjugation by q and is linear in the above model, hence coincides with its derivative at the identity. This means that, up to pre-factorization by the center $Z(\mathbb{H}^*) \simeq \mathbb{R}^*$, we are dealing with the usual adjoint representation from Lie group theory.) The ring

$$\mathbb{H}(\mathbb{Z}) = \{q = x_0 + x_1 i + x_2 j + x_3 k : x_0, x_1, x_2, x_3 \in \mathbb{Z}\}$$

of Lipschitz quaternions has 8 units:

$$\mathbb{H}(\mathbb{Z})^\times = \{\pm 1, \pm i, \pm j \pm k\}.$$

Let $n \in \mathbb{N}$. According to Jacobi (see for example [8, p. 27] or [13, Theorem 2.4.1] for odd integers), the cardinality of the set of Lipschitz quaternions of norm n is

$$|N^{-1}(n) \cap \mathbb{H}(\mathbb{Z})| = 8 \sum_{4 \nmid d | n} d.$$

Hence, if $n = p$ is prime, the set $N^{-1}(p) \cap \mathbb{H}(\mathbb{Z})$ splits as the disjoint union of the $p + 1$ orbits of the action of $\mathbb{H}(\mathbb{Z})^\times$. In the case $p \equiv 1 \bmod 4$, it is easy to check that each orbit contains a unique quaternion $q = x_0 + x_1 i + x_2 j + x_3 k$ with $x_0 > 0$ and $x_0 \equiv 1 \bmod 2$ and that the set

$$\{q = x_0 + x_1 i + x_2 j + x_3 k : x_0, x_1, x_2, x_3 \in \mathbb{Z},\ N(q) = p,\ x_0 > 0,\ x_0 \equiv 1 \bmod 2\}$$

splits into $\frac{p+1}{2}$ orbits of the involution τ, each containing two elements. Let $\Sigma_p \subset \mathrm{SO}(3, \mathbb{R})$ denote the image of this set under the homomorphism Ad. Let ν be the Lebesgue probability measure on the round sphere $\mathbb{S}^2$.

Theorem 2.1 (Lubotzky–Phillips–Sarnak [30, Theorem 1.3, Theorem 1.5]). *Let p be a prime such that $p \equiv 1 \bmod 4$. The subgroup Γ of $\mathrm{SO}(3, \mathbb{R})$ generated by the symmetric set Σ_p of cardinality $p + 1$ is free of rank $\frac{p+1}{2}$ and*

$$\sup_{\|f\|_{L^2(\mathbb{S}^2,\nu)}=1} \left\| \frac{1}{|\Sigma_p|} \sum_{\gamma \in \Sigma_p} f(\gamma x) - \int_{\mathbb{S}^2} f(y) d\nu(y) \right\|_{L^2(\mathbb{S}^2,\nu)} = \frac{2\sqrt{p}}{p+1}.$$

Let E_n be either the sphere or the ball of of radius n around $e \in \Gamma$ with respect to the word metric defined by Σ_p. There is a constant $C > 0$ such that for all $n \in \mathbb{N}$,

$$\sup_{\|f\|_{L^2(\mathbb{S}^2,\nu)}=1} \left\| \frac{1}{|E_n|} \sum_{\gamma \in E_n} f(\gamma x) - \int_{\mathbb{S}^2} f(y) d\nu(y) \right\|_{L^2(\mathbb{S}^2,\nu)} \leq C n p^{-n/2}.$$

The next theorem is our main result. It generalizes [30, Theorem 1.3] (choosing $n = 1$ in the first equality of Theorem 2.2 below, one recovers [30, Theorem 1.3]). It also strengthens [30, Theorem 1.5]. Indeed, [30, Theorem 1.5] establishes an upper bound depending on an unknown constant, and claims no lower bound, whereas the second equality in Theorem 2.2 below is an exact formula for the rate of convergence, resulting from the combination of a new upper bound, smaller than the upper bound from Theorem [30, Theorem 1.5], with the universal lower bound resulting from Theorem 3.6 below.

Theorem 2.2. *Let p be a prime such that $p \equiv 1 \bmod 4$. Let Γ be the free subgroup of rank $\frac{p+1}{2}$ of $\mathrm{SO}(3,\mathbb{R})$ generated by the symmetric set Σ_p. Let S_n, respectively B_n, be the sphere, respectively the ball, of radius n around $e \in \Gamma$ with respect to the word metric defined by Σ_p. Then*

$$\sup_{\|f\|_{L^2(\mathbb{S}^2,\nu)}=1} \left\| \frac{1}{|S_n|} \sum_{\gamma \in S_n} f(\gamma x) - \int_{\mathbb{S}^2} f(y) d\nu(y) \right\|_{L^2(\mathbb{S}^2,\nu)} = \left(1 + \frac{p-1}{p+1} n\right) p^{-n/2},$$

$$\sup_{\|f\|_{L^2(\mathbb{S}^2,\nu)}=1} \left\| \frac{1}{|B_n|} \sum_{\gamma \in B_n} f(\gamma x) - \int_{\mathbb{S}^2} f(y) d\nu(y) \right\|_{L^2(\mathbb{S}^2,\nu)} = \frac{p-1}{p+1-\frac{2}{p^n}} \left(1 + \left(1 + \frac{1}{\sqrt{p}}\right) n\right) p^{-n/2}.$$

2.2. Upper bounds: Weil and Deligne. Let Γ be the free group of rank r and let $\mathbb{C}[\Gamma]$ be the its group algebra. Let $s_1, \ldots, s_r$ be a free generating set of Γ and let $S = \{s_1^{\pm 1}, \ldots, s_r^{\pm 1}\}$. Each element $\gamma \in \Gamma$ has a word length $\|\gamma\|_S$ defined by S. For each $n \in \mathbb{N} \cup \{0\}$, we consider the *Hecke element* of $\mathbb{C}[\Gamma]$ defined as

$$T_n = \sum_{\|\gamma\|_S = n} \gamma.$$

The upper bounds are obtained with the help of three main ingredients. The first ingredient is an equality between the spectra of two operators. The two operators are defined with the help of the same Hecke element T_n and two unitary representations of the same free subgroup of $\mathrm{SO}(3,\mathbb{R})$ generated by Lipschitz quaternions. The first representation is the Koopman representation π_0 on the Hilbert space $L^2_0(\mathbb{S}^2,\nu)$ of square integrable functions on the sphere with zero integral. The second representation is the regular representation λ on $l^2(\Gamma)$. See inclusion (6) below for the precise statement. To the best of our knowledge, the only known proof of the inclusion of the spectra

$$\sigma\big(\pi_0(T_n)\big) \subset \sigma\big(\lambda(T_n)\big)$$

is the one from [30, S153–S158] and [31, Theorem 4.1], which uses the theory of automorphic forms and Deligne's solution to the Weil conjecture. The second ingredient is an application of the spectral theorem to Hecke elements. The third one is an identity between the norm of operators defined by the regular representation of a free group and values of the Harish-Chandra function of a free group; see Proposition A.2 below.

2.3. Lower bounds: A general fact. The lower bounds follow from Theorem 3.6 below. We give a short self-contained proof of this theorem. Alternatively, we also explain how it can be deduced from a general result of Shalom announced in [41, Theorem 4.14] and also stated in [16, Proposition 7]: if a countable group Γ acts by measure-preserving transformations on an atomless probability space (X,ν), then there is a subgroup H of Γ, such that the quasi-regular representation of Γ on $l^2(\Gamma/H)$ is weakly contained in the restriction of the Koopman representation of Γ to the orthogonal complement $L^2_0(X,\nu)$ of the constant functions. In the following proposition, we spell out the consequence of Theorem 3.6 we need.

Proposition 2.3. *Assume that Γ is a free group of rank $r \geq 1$ acting by measure-preserving transformations on an atomless probability space (X,ν). Let $\{a_1,\dots,a_r\}$ be a free generating set of Γ. Let $S=\{a_1^{\pm 1},\dots,a_r^{\pm 1}\}$. Let $q=2r-1$. Let S_n, respectively B_n, be the sphere, respectively the ball of radius n around $e \in \Gamma$ with respect to the word metric defined by S. Then*

$$\sup_{\|f\|_{L^2(X,\nu)}=1} \left\| \frac{1}{|S_n|}\sum_{\gamma\in S_n} f(\gamma x) - \int_X f(y)\,d\nu(y) \right\|_{L^2(X,\nu)} \geq \left(1+\frac{q-1}{q+1}n\right) q^{-n/2},$$

$$\sup_{\|f\|_{L^2(X,\nu)}=1} \left\| \frac{1}{|B_n|}\sum_{\gamma\in B_n} f(\gamma x) - \int_X f(y)\,d\nu(y) \right\|_{L^2(X,\nu)}$$
$$\geq \left(1+2q^{-n}\sum_{k=0}^{n-1} q^k\right)^{-1} \left(1+\left(1+\frac{1}{\sqrt{q}}\right)n\right) q^{-n/2}.$$

Remark 2.4. Notice that both lower bounds evaluated at $q = 1$ give the value 1 and in this case both inequalities are equalities. When $q > 1$, applying the formula for the sum of the first n terms in the geometric series gives

$$\left(1 + 2q^{-n} \sum_{k=0}^{n-1} q^k\right)^{-1} = \frac{q-1}{q+1-\frac{2}{q^n}}.$$

2.4. Convergence rates on the torus. It follows from [30, Theorem 1.4] that a generic finitely generated free subgroup of $\mathrm{SO}(3, \mathbb{R})$ does *not* realize the lower bounds of Proposition 2.3. This is in contrast with the group of automorphisms of the torus where any finitely generated free subgroup realizes the fastest possible convergence rate, as stated in the following theorem which easily follows from [41, Theorem 4.17] or [14, Remark 20 on p. 16], or ideas presented in [18], or [20].

Theorem 2.5. *Let $\mathbb{T}^2 = \mathbb{R}^2/\mathbb{Z}^2$ be the 2-torus with its normalized Haar measure ν and let $\mathrm{GL}(2, \mathbb{Z}) \simeq Aut(\mathbb{T}^2) = G$ be its automorphism group. Assume that $\Gamma < G$ is a free subgroup of rank $r \geq 1$ freely generated by $\{a_1, \ldots, a_r\} \subset G$. Let $S = \{a_1^{\pm 1}, \ldots, a_r^{\pm 1}\}$. Let $q = 2r - 1$. Let S_n, respectively B_n, be the sphere, respectively the ball, of radius n around $e \in \Gamma$ with respect to the word metric defined by S. Then*

$$\sup_{\|f\|_{L^2(\mathbb{T},\nu)}=1} \left\| \frac{1}{|S_n|} \sum_{\gamma \in S_n} f(\gamma x) - \int_{\mathbb{T}^2} f(y) d\nu(y) \right\|_{L^2(\mathbb{T},\nu)} = \left(1 + \frac{q-1}{q+1} n\right) q^{-n/2},$$

$$\sup_{\|f\|_{L^2(\mathbb{T},\nu)}=1} \left\| \frac{1}{|B_n|} \sum_{\gamma \in B_n} f(\gamma x) - \int_{\mathbb{T}^2} f(y) d\nu(y) \right\|_{L^2(\mathbb{T},\nu)}$$
$$= \left(1 + 2q^{-n} \sum_{k=0}^{n-1} q^k\right)^{-1} \left(1 + \left(1 + \frac{1}{\sqrt{q}}\right) n\right) q^{-n/2}.$$

We present the general case of actions by automorphisms on compact abelian groups in Theorem 3.11 below.

2.5. Structure of the paper. Section 3 is devoted to the relevant definitions and notation from representation theory needed for the proofs of the results and facts stated in the introduction. It contains the proof of the universal lower bound for discrepancies (Theorem 3.6) as well as the proof that Bernoulli shifts realize this universal lower bound (Corollary 3.12). In Section 4, we prove the remaining results stated in the introduction, assuming the formulae for the norms of the operators defined with the help of the regular representation. In the appendix, the reader will find a detailed explanation of a geometric method for establishing the formulae used in Section 4.

2.6. Related references.

About the works of Lubotzky–Phillips–Sarnak and the relations between equidistribution on the sphere and Ramanujan graphs. The spectral theory developed in the two papers [30] and [31] brings information on the equidistribution of points on the sphere (there is a Séminaire Bourbaki [12] on these two papers by Colin de Verdière) and it is very close to the spectral theory developed in [32] where the aim is the construction of Ramanujan graphs (there is a Séminaire Bourbaki [13] on [32] by A. Valette). The relation between the two subjects are explained in Lubotzky's book [29]: the main results from [30] and [31] about equidistributions of points and the main results from [32] about Ramanujan graphs are two applications of a theorem of Deligne [29, Theorem 6.1.2] about automorphic representations and its implications about the growth of Fourier coefficients of modular forms. The result of Deligne and its implications about Fourier coefficients of modular forms may be viewed as part of the Langlands program. Other advances in the Langlands program (Lafforgue's proof of the Ramanujan–Petersson conjecture for GL_d in positive characteristic and the establishment of the global Jacquet–Langlands correspondence for GL_d in positive characteristic by Badulescu and Roche) have been applied to the construction of Ramanujan complexes [19]. In a different direction, Parzanchevski and Sarnak have shown that the optimal generating rotations $\Sigma_p \subset \mathrm{SO}(3, \mathbb{R})$ can be used to construct efficient gates needed as building blocks for quantum algorithms [35].

About ergodicity and discrepancies of subgroups of compact Lie groups. The five pages paper of Arnol'd and Krylov [1] is one of the earlier reference on free groups of rotations and their ergodic properties. Clozel [9] has obtained sharp bounds (up to multiplicative constants) on the discrepancies of some subsets of $\mathrm{SO}(2n)$. In a series of paper, Bourgain and Gamburd (see [5] and the references therein) construct finite symmetric sets in $\mathrm{SU}(d)$ whose averaging operators have norms smaller than 1 (see Definition 3.1 below which identifies discrepancies with norms of averaging operators).

About convergence rate for ergodic theorems, spectral transfers and the Harish-Chandra function. In [10], Clozel, Oh, and Ullmo, express convergence rates for ergodic theorems on locally symmetric spaces, in terms of Harish-Chandra functions. The survey [22] of Gorodnik and Nevo includes many convergence rates estimates. The *spectral transfer principle*, as formulated in [34], explains how information on the decay of coefficients of unitary representations may bring convergence rates estimates in ergodic theorems.

About random points on spheres. In [17], Ellenberg, Michel and Venkatesh, discuss and improve Linnik's work on the distribution of point sets on the 2-sphere, obtained from the representation of a large integer as a sum of three integer squares. In [7] and [6], Bourgain, Rudnick, Sarnak, evaluate this distribution through different "statistics" and compare it with what's happening in higher dimensions, and with the case of random points. Shalom's survey [41] presents estimates of the discrepancy of random points.

3. Discrepancies and representation theory

In this section we recall the relevant definitions and notation from representation theory needed for the proofs of the results and facts stated in the introduction.

3.1. Three involutive algebras. Let Γ be a group. Recall that the formal linear combinations of elements of Γ with complex coefficients

$$\sum_{\gamma\in\Gamma} a_\gamma \gamma$$

form the group algebra $\mathbb{C}[\Gamma]$ of Γ over the complex numbers. Each element of $\mathbb{C}[\Gamma]$ has a norm defined as

$$\Big\|\sum_{\gamma\in\Gamma} a_\gamma \gamma\Big\|_1 = \sum_{\gamma\in\Gamma} |a_\gamma|.$$

The space

$$l^1(\Gamma) = \Big\{f : \Gamma \to \mathbb{C} : \|f\|_1 = \sum_{\gamma\in\Gamma} |f(\gamma)| < \infty\Big\}$$

of summable functions on Γ is a convolution algebra for the law

$$(f * g)(x) = \sum_{\gamma\in\Gamma} g(\gamma^{-1}x) f(\gamma), \quad \forall x \in \Gamma.$$

There is a unique embedding of involutive unital algebras of $\mathbb{C}[\Gamma]$ into $l^1(\Gamma)$, which sends each $\gamma \in \mathbb{C}[\Gamma]$ to the characteristic function $\delta_\gamma \in l^1(\Gamma)$ of γ. Let $\pi : \Gamma \to \mathrm{U}(\mathcal{H})$ be a unitary representation of Γ on a Hilbert space $\mathcal{H}$. Let $B(\mathcal{H})$ be the involutive algebra of bounded operators on $\mathcal{H}$. If $T \in B(\mathcal{H})$ we denote by $\|T\|$ its operator norm. There is a unique morphism of involutive unital algebras from $l^1(\Gamma)$ to $B(\mathcal{H})$ whose restriction to Γ equals π. We also denote this morphism by π.

Let E be a finite subset of Γ and let $|E|$ be its cardinality. We denote the element of $\mathbb{C}[\Gamma]$ defined as the sum over elements of E by

$$\mathbf{1}_E = \sum_{\gamma\in E} \gamma$$

and we define m_E to be $\frac{1}{|E|}\mathbf{1}_E$.

Let $l^2(\Gamma)$ be the Hilbert space of square integrable functions on Γ. Let $\rho_\Gamma : \Gamma \to \mathrm{U}(l^2(\Gamma))$ be the right regular representation:

$$\rho_\Gamma(\gamma)f(x) = f(x\gamma), \quad \forall x, \gamma \in \Gamma,$$

and let $\lambda_\Gamma : \Gamma \to \mathrm{U}(l^2(\Gamma))$ be the left regular representation:

$$\lambda_\Gamma(\gamma)f(x) = f(\gamma^{-1}x), \quad \forall x, \gamma \in \Gamma.$$

3.2. Koopman representations. Let (X, ν) be a measured space. Let Γ be a group acting on X by measure-preserving transformations. Let $L^2(X, \nu)$ be the Hilbert space of complex square integrable functions on X. If $f \in L^2(X, \nu)$ we denote its norm by $\|f\|_2$. Let $\pi : \Gamma \to \mathrm{U}(L^2(X, \nu))$ be the Koopman representation:

$$\pi(\gamma)f(x) = f(\gamma^{-1}x).$$

3.3. Discrepancies and Laplace operators. Unless stated otherwise, we will assume that ν is a probability measure. Let $\mathcal{H} = L^2(X, \nu)$. Let $\mathbf{1}_X \in \mathcal{H}$ be the constant function on X equal to 1. Let $P \in B(\mathcal{H})$ be the orthogonal projection onto the complex line generated by $\mathbf{1}_X$. We have $P^2 = P$, $P = P^*$, $P\pi(\gamma) = \pi(\gamma)P = P$ for all $\gamma \in \Gamma$. Let π_0 denote the restriction of π to the kernel

$$\mathrm{Ker} P = L^2_0(X, \nu) = \left\{ f \in L^2(X, \nu) : \int_X f(x)d\nu(x) = 0 \right\}.$$

We will also call π_0 a Koopman representation.

Definition 3.1. Let $m = \sum_\gamma m(\gamma)\gamma$ be an element of $\mathbb{C}[\Gamma]$. We assume that $\sum_\gamma m(\gamma) = 1$. The discrepancy of m is defined as the norm $\|\pi_0(m)\|$ of the operator $\pi_0(m) : L^2_0(X, \nu) \to L^2_0(X, \nu)$.

Notice that the condition $\sum_\gamma m(\gamma) = 1$, implies that

$$\pi(m) - P = \pi(m)(\mathrm{id}_{L^2(X,\nu)} - P).$$

As the operator $\mathrm{id}_{L^2(X,\nu)} - P$ is the orthogonal projection from $L^2(X, \nu)$ onto $L^2_0(X, \nu)$, we deduce that

$$\|\pi_0(m)\| = \sup_{\|f\|_2 = 1} \left\| \sum_{\gamma \in \Gamma} f(\gamma^{-1}x)m(\gamma) - \int_X f(y)d\nu(y) \right\|_2, \tag{4}$$

where the supremum is taken over the unit sphere of the whole Hilbert space $L^2(X, \nu)$. Choosing

$$m = \frac{1}{2n} \sum_{k=1}^{n} (\gamma_k^{-1} + \gamma_k),$$

we recover the discrepancy of a finite symmetric family $\gamma_1, \gamma_1^{-1}, \ldots, \gamma_n, \gamma_n^{-1}$ of elements of Γ:

$$\delta(\gamma_1, \gamma_1^{-1}, \ldots, \gamma_n, \gamma_n^{-1}) = \|\pi_0(m)\|. \tag{5}$$

For m as above (or more generally for m a symmetric probability measure on Γ), it is fruitful to consider the Markov operator $K = \pi_0(m)$ and to define the associated Laplace operator $\Delta = \mathrm{id}_{L_0^2(X,\nu)} - K$. Notice that $\|K\| \leq 1$, so Δ is a positive operator. The following proposition is well known.

Proposition 3.2. (1) *For all $f \in L_0^2(X, \nu)$,*

$$\frac{1}{n}\sum_{k=1}^{n} \|\pi_0(\gamma_k) f - f\|^2 = 2\langle \Delta f, f\rangle.$$

(2) *The operator norm of K can be computed with the help of Rayleigh quotients:*

$$\|K\| = 1 - \inf_{f \neq 0} \frac{\langle \Delta f, f\rangle}{\|f\|}.$$

Definition 3.3 ([39]). Let $\gamma_1, \ldots, \gamma_n$ be a finite family of elements of Γ. Let $\epsilon > 0$. The family is ϵ-*good* if, for any $f \in L_0^2(X, \nu)$ such that $\|f\| = 1$, there exists $1 \leq k \leq n$ such that

$$\|\pi_0(\gamma_k) f - f\| \geq \epsilon.$$

Proposition 3.4. *Let Γ be a group acting by measure preserving transformations on a probability space (X, ν).*

(1) *Let $m \in \mathbb{C}[\Gamma]$ such that $\sum_\gamma m(\gamma) = 1$. If $\|\pi_0(m)\| < 1$ then the action of Γ on (X, ν) is ergodic.*

(2) *Let F be a finite symmetric family of elements of Γ. If $\delta(F) < 1$ then the action of Γ on (X, ν) is ergodic.*

(3) *Let $\gamma_1, \ldots, \gamma_n$ be a finite family of elements of Γ which is ϵ-*good. *Then*

$$\delta(\gamma_1, \gamma_1^{-1}, \ldots, \gamma_n, \gamma_n^{-1}) \leq 1 - \frac{\epsilon^2}{2n}.$$

(4) *Let $\gamma_1, \gamma_1^{-1}, \ldots, \gamma_n, \gamma_n^{-1}$ be a finite symmetric family of elements of Γ. Let $0 < \lambda < 1$. If*

$$\delta(\gamma_1, \gamma_1^{-1}, \ldots, \gamma_n, \gamma_n^{-1}) \leq 1 - \lambda,$$

then the family $\gamma_1, \ldots, \gamma_n$ is $\sqrt{2\lambda}$-good.

(5) *Let F be a finite symmetric family of elements of Γ. Let H be the subgroup of Γ generated by the family F. If H has Kazdhan property* (T) *and if the action of H on (X, ν) is ergodic, then $\delta(F) < 1$.*

Proof. Let $f \in L^2(X, \nu)$. If f is invariant under the action of Γ, so are the components $f = f_0 + f_1$, where

$$f_0(x) = f(x) - \int_X f(y)d\nu(y).$$

Hence $\pi_0(m)(f_0) = \sum_\gamma m(\gamma) f_0 = f_0$, so that

$$\|f_0\| \leq \|\pi_0(m)\| \|f_0\|.$$

Under the hypothesis $\|\pi_0(m)\| < 1$, the above inequality forces $\|f_0\| = 0$. Hence f is constant. This proves that the action is ergodic. Applying equality (5) above, we see that the second statement in the proposition is a special case of the first statement. In order to prove the third statement in the proposition, let $f \in L^2_0(X, \nu)$ such that $\|f\| = 1$. As the family is ϵ-good, there exists $1 \leq i \leq n$ such that

$$\|\pi_0(\gamma_i)f - f\| \geq \epsilon.$$

Applying Proposition 3.2, we obtain that

$$\langle \Delta f, f \rangle = \frac{1}{2n} \sum_{k=1}^{n} \|\pi_0(\gamma_k)f - f\|^2 \geq \frac{1}{2n} \|\pi_0(\gamma_i)f - f\|^2 \geq \frac{\epsilon^2}{2n}$$

and

$$\delta(\gamma_1, \gamma_1^{-1}, \ldots, \gamma_n, \gamma_n^{-1}) = \|K\| \leq 1 - \frac{\epsilon^2}{2n}.$$

In order to prove the fourth statement in the proposition, let $f \in L^2_0(X, \nu)$ such that $\|f\| = 1$. By hypothesis,

$$\langle \Delta f, f \rangle \geq \lambda.$$

That is,

$$\frac{1}{2n} \sum_{k=1}^{n} \|\pi_0(\gamma_k)f - f\|^2 \geq \lambda.$$

Hence there exists $1 \leq k \leq n$ such that

$$\|\pi_0(\gamma_k)f - f\| \geq \sqrt{2\lambda}.$$

We now prove the last statement of the proposition. The hypothesis that the group H generated by $\gamma_1, \ldots, \gamma_n$ has property (T) implies the existence of $\epsilon > 0$, such that

$(\{\gamma_1, \dots, \gamma_n\}, \epsilon)$ is a Kazdhan pair for H (see [2, Definition 1.1.3]). If the action of H on (X, ν) is ergodic, the representation π_0 restricted to H has no non-trivial invariant vector. The definition of a Kazdhan pair then implies that for any $f \in L^2_0(X, \nu)$ such that $\|f\| = 1$, there exists $1 \le k \le n$, such that $\|\pi_0(\gamma_k)f - f\| > \epsilon$. ■

Remark 3.5. In the case Γ is countable and m is a symmetric probability measure on Γ whose support generates Γ, the condition $\|\pi_0(m)\| < 1$ is equivalent to the *strong ergodicity* of the action of Γ on (X, ν). See [41, Theorem 4.2].

3.4. A universal lower bound on discrepancies. The following theorem generalizes and strengthens [40, Théorème 3.3], [9, Theorem 2], [37, Theorem 4], [30, Theorem 1.3].

Theorem 3.6 (A universal lower bound on discrepancies in terms of the regular representation). *Let (X, ν) be an atomless probability space. Let Γ be a group acting on X by measure-preserving transformations. Let $m \in \mathbb{C}[\Gamma]$ be a positive element (i.e., $m = \sum_{\gamma \in \Gamma} a_\gamma \gamma$ with $a_\gamma \ge 0$). Then*

$$\|\pi_0(m)\| \ge \|\rho_\Gamma(m)\|.$$

Remark 3.7. Let $\mathcal{H}$ be a Hilbert space and let $\pi : \Gamma \to \mathrm{U}(\mathcal{H})$ be a unitary representation. The unique morphism $\pi : l^1(\Gamma) \to B(\mathcal{H})$ of unital involutive algebras extending π is continuous. (More precisely it satisfies $\|\pi(f)\| \le \|f\|_1$ for all $f \in l^1(\Gamma)$.) As a consequence, Theorem 3.6 extends to positive elements in $l^1(\Gamma)$.

Remark 3.8. Without the positivity hypothesis, the inequality $\|\pi_0(m)\| \ge \|\rho_\Gamma(m)\|$ obviously fails. As a simple example, consider the cyclic group of order two $\Gamma = \{e, \sigma\}$ acting trivially on a probability space. If we choose $m = e - \sigma$ in $\mathbb{C}[\Gamma]$, then $\pi_0(m) = 0$ but $\|\rho(m)\| = 2$.

For any countable group Γ, there exist Koopman representations on atomless probability spaces, realizing the universal lower bound from Theorem 3.6. Corollary 3.12 below shows that Bernoulli shifts provide such representations in the case Γ is infinite.

We give two proofs of Theorem 3.6. In the first proof, we make the additional assumption that (X, ν) is a standard probability space. This proof is based on a stronger result due to Shalom [41, Theorem 4.14], published without a proof, which was rediscovered and proved by Dudko and Grigorchuk in [16, Proposition 14] with the assumption that (X, ν) is standard. The second proof is short, self-contained, and does not require (X, ν) to be standard. It follows an idea of B. Sévennec.

First proof of Theorem 3.6. By considering the subgroup of Γ generated by the support of the positive element m, we can assume that Γ is finitely generated. According to [16, Proposition 14], there exists a subgroup H of Γ such that the quasi-regular representation of Γ on $l^2(\Gamma/H)$ is weakly contained in the restriction of the Koopman representation of Γ to the orthogonal complement $L^2_0(X,\nu)$ of the constant functions. As the quasi-regular representation of Γ on $l^2(\Gamma/H)$ contains positive vectors, the theorem follows from [42, Lemma 2.3], and the definition of weak containment [2, Definition F.1.1]. ■

Second proof of Theorem 3.6. If $m = 0$ the inequality is trivial. If $m \neq 0$, multiplying m by $\|m\|_1^{-1}$, we may assume that $\|m\|_1 = 1$. As $\|T\|^2 = \|TT^*\|$ for any bounded operator T on a Hilbert space, we may moreover assume $m = m^*$.

Let $e \in \Gamma$ be the identity element. We claim that for any $n \in \mathbb{N}$, $\|\pi_0(m^n)\| \geq m^{(n)}(e)$, where $m^{(n)} \in l^1(\Gamma)$ is the n-th convolution power of m (although m has finite support it is convenient here and in what follows to view m in the convolution algebra $l^1(\Gamma)$ of summable functions).

To prove this claim, let F be the support of $m^{(n)}$. We choose a measurable subset B_+ of X such that

$$0 < \nu(B_+) < \frac{1}{2|F|}.$$

Such a set obviously exists because ν is finite without atom. As the action preserves the measure,

$$\nu(FB_+) \leq |F|\nu(B_+) < 1/2.$$

A finite atomless measure has the intermediate value property (see [3, 1.12.10 Corollary]) hence there exists a measurable subset B_- of $X \setminus FB_+$ satisfying $\nu(B_+) = \nu(B_-)$. Let

$$\varphi = \frac{\mathbf{1}_{B_+} - \mathbf{1}_{B_-}}{\|\mathbf{1}_{B_+} - \mathbf{1}_{B_-}\|_2} \in L^2_0(X,\nu).$$

The proof of the claim follows then from the Cauchy–Schwarz inequality, the symmetry of F, and the positivity of m:

$$\begin{aligned}
\|\pi_0(m^n)\| &\geq \langle \pi_0(m^n)\varphi, \varphi\rangle \\
&= \frac{1}{\nu(B_+) + \nu(B_-)} \sum_{\gamma\in\Gamma} \big[\nu(\gamma B_+ \cap B_+) + \nu(\gamma B_- \cap B_-)\big] m^{(n)}(\gamma) \\
&\geq \frac{1}{\nu(B_+) + \nu(B_-)} \sum_{\gamma=e} \big[\nu(\gamma B_+ \cap B_+) + \nu(\gamma B_- \cap B_-)\big] m^{(n)}(\gamma) \\
&= m^{(n)}(e).
\end{aligned}$$

This finishes the proof of the claim.

Applying the claim, we obtain

$$\begin{aligned}\|\pi_0(m)\| &= \lim_{n\to\infty} \|\pi_0(m)^n\|^{1/n} \\ &\geq \limsup_{n\to\infty} m^{(n)}(e)^{1/n} \\ &= \|\rho_\Gamma(m)\|.\end{aligned}$$

The last equality above goes back to [27, Lemma 2.2]. ■

Remark 3.9. As mentioned to us by P.-E. Caprace, the existence of measure preserving actions of locally compact *amenable* groups with a spectral gap (see for example [33, Corollary 1.10, 1, Chap. III]) prevents an obvious generalization of the above result to locally compact groups. But we believe the following statement is true.

Let μ be a Haar measure on a locally compact group G. Let ρ_G be the right regular representation of G. Let X be a Hausdorff topological space. Let ν be an atomless Borel regular probability measure on X. Suppose that G acts continuously on X, and preserves ν. Assume there exists a point x_0 in the support of ν, with compact stabilizer, such that the orbit of x_0, under any compact subset K of G, has zero measure: $\nu(Kx_0) = 0$. Let $\pi_0 : L^1(G,\mu) \to B(L_0^2(X,\nu))$ be the Koopman representation, restricted to the subspace of functions with zero integral. Then, for any continuous function $f : G \to \mathbb{R}$ with compact support and taking only nonnegative values, we have

$$\|\pi_0(f)\| \geq \|\rho_G(f)\|.$$

If moreover G is is second countable, the inequality extends to nonnegative functions belonging to $L^1(G,\mu)$. (We hope to say more on these questions in a forthcoming paper.)

The following corollary together with Proposition 2.3 illustrate Theorem 3.6 with two opposite situations: the discrepancy is maximal in the case the transformations generate an amenable group whereas the discrepancy may be small in the case the transformations generate a free group.

Corollary 3.10. *Let $E \subset \mathrm{Aut}(X,\nu)$ be finite and symmetric. Let Γ be the group generated by E. If Γ is amenable then*

$$\sup_{\|f\|_2=1} \left\| \frac{1}{|E|} \sum_{\gamma\in E} f(\gamma x) - \int_X f(y)\,d\nu(y) \right\|_2 = 1.$$

Proof. Without any hypothesis on Γ we always have $\|\pi_0(m_E)\| \leq \|m_E\|_1 = 1$. According to Theorem 3.6, $\|\pi_0(m_E)\| \geq \|\rho_\Gamma(m_E)\|$. According to [26], the group Γ is amenable if and only if $\|\rho_\Gamma(m_E)\| = 1$. ■

3.5. Actions on compact abelian groups.

Theorem 3.11. *Let A be a compact abelian group. Let ν be its normalized Haar measure. Let Γ be a group acting by continuous automorphisms on A (hence preserving ν). We assume that for any non-trivial continuous homomorphism $\chi : A \to \mathbb{T}$ with values in the circle group $\mathbb{T}$, the corresponding stabilizer subgroup of Γ,*

$$\Gamma_\chi = \{\gamma \in \Gamma : \forall a \in A, \chi(\gamma^{-1}a) = \chi(a)\},$$

is amenable. Let π_0 be the Koopman representation of Γ on $L^2_0(A, \nu)$. For any $m \in \mathbb{C}[\Gamma]$,

$$\|\pi_0(m)\| \leq \|\rho_\Gamma(m)\|.$$

(In other words, π_0 is weakly contained in ρ_Γ.) Moreover, in the case $m \in \mathbb{C}[\Gamma]$ is a positive element, and if A is nontrivial, the opposite inequality

$$\|\pi_0(m)\| \geq \|\rho_\Gamma(m)\|$$

also holds true, hence

$$\|\pi_0(m)\| = \|\rho_\Gamma(m)\|.$$

Proof. Let us first prove the second inequality. We claim that if A is a finite abelian group, then $\|\pi_0(m)\|$ equals 1. Indeed, if $\|\pi_0(m)\|$ was strictly smaller than 1, then the action of Γ would be ergodic according to Proposition 3.4 (1), hence transitive, and it can't be because the zero element of A is fixed by every automorphism of A. If A is not finite, then since it is compact, the Haar measure ν does not have atoms, and the second inequality follows immediately from Theorem 3.6.

We now explain how the first inequality follows from the properties of the Fourier transform and from the continuity of the induction. Let

$$\hat{A} = \mathrm{Hom}(A, \mathbb{T})$$

be the Pontryagin dual group of A, that is the group of all continuous homomorphisms from A to $\mathbb{T}$ endowed with the compact-open topology. Recall the following general facts from the theory of Fourier analysis on locally compact abelian groups (see [43] or [25]). The dual group $\hat{A}$ is discrete (because A is compact) hence the Haar measure on $\hat{A}$ is the counting measure, the Fourier transform

$$\mathcal{F} : L^2(A, \nu) \to l^2(\hat{A}),$$

defined by the formula

$$\mathcal{F}(f)(\chi) = \int_A f(a)\overline{\chi(a)}d\nu(a),$$

is an isometric isomorphism of Hilbert spaces. The group Γ acts on $\hat{A}$ *via* the dual action, that is, for all $\gamma \in \Gamma$, $\chi \in \hat{A}$ and $a \in A$, we define

$$(\gamma\chi)(a) := \chi(\gamma^{-1}a).$$

Since the Haar measure on $\hat{A}$ is the counting measure, it is preserved by the dual action. Let $\hat{\pi}$ be the Koopman representation associated to the dual action. The Fourier transform intertwines the two representations. More precisely, let $\varphi \in L^2(A, \nu)$, $\chi \in \hat{A}$ and $\gamma \in \Gamma$, we have

$$\begin{aligned}\hat{\pi}(\gamma)\big(\mathcal{F}(\varphi)\big)(\chi) &= \int_A \varphi(a)\overline{\gamma^{-1}\chi(a)}d\nu(a)\\ &= \int_A \varphi(a)\overline{\chi(\gamma a)}d\nu(a)\\ &= \int_A \varphi\big(\gamma^{-1}a\big)\overline{\chi(a)}d\nu(a)\\ &= \mathcal{F}\big(\pi(\gamma)(\varphi)\big)(\chi).\end{aligned}$$

Notice that $\hat{A} \setminus \{\mathbf{1}_A\}$ is stable, since Γ acts trivially on $\mathbf{1}_A$. The Koopman representation associated to this action is denoted by $\hat{\pi}_0$ and is in fact unitarily equivalent, again *via* the Fourier transform, to π_0 (the constant functions are exactly those for which all the Fourier coefficients but the $\mathbf{1}_A$-th vanish). Let T be a transversal for the action of Γ on $\hat{A} \setminus \{\mathbf{1}_A\}$, i.e. a subset of $\hat{A} \setminus \{\mathbf{1}_A\}$ which intersects each orbit of the action at exactly one point. Then the family of orbits

$$(\Gamma \cdot \chi)_{\chi \in T}$$

is a partition of $\hat{A} \setminus \{\mathbf{1}_A\}$; this partition yields the decomposition

$$l^2\big(\hat{A} \setminus \{\mathbf{1}_A\}\big) \simeq \bigoplus_{\chi \in T} l^2(\Gamma \cdot \chi)$$

as a Hilbert direct sum of sub-representations of $\hat{\pi}_0$. For all $\chi \in T$, let Γ_χ be the stabilizer of χ (the notation should not be confused with the orbit $\Gamma \cdot \chi$). Then for all $\chi \in T$,

$$l^2(\Gamma \cdot \chi) \simeq l^2(\Gamma/\Gamma_\chi);$$

to sum up, we have proved the following unitary equivalence

$$\pi_0 \simeq \bigoplus_{\chi \in T} \lambda_{\Gamma/\Gamma_\chi},$$

where $\lambda_{\Gamma/\Gamma_\chi}$ denotes the quasi-regular representation on $l^2(\Gamma/\Gamma_\chi)$. By assumption, Γ_χ is amenable, so we have a weak containment of representations

$$\mathbf{1}_{\Gamma_\chi} \prec \lambda_{\Gamma_\chi}.$$

By continuity of the induction, it follows that the quasi-regular representation $\lambda_{\Gamma/\Gamma_\chi}$, itself equivalent to $\mathrm{Ind}_{\Gamma_\chi}^{\Gamma} \mathbf{1}_{\Gamma_\chi}$, is weakly contained in $\mathrm{Ind}_{\Gamma_\chi}^{\Gamma} \lambda_{\Gamma_\chi}$, which is itself equivalent to λ_Γ. Since this holds for all $\chi \in T$, the direct sum

$$\bigoplus_{\chi \in T} \lambda_{\Gamma/\Gamma_\chi},$$

and hence, π_0, which is equivalent to it, is weakly contained in λ_Γ. This is equivalent to say (see [2, Theorem F.4.4]) that for all $f \in l^1(\Gamma)$,

$$\|\pi_0(f)\| \leq \|\lambda_\Gamma(f)\|.$$

Finally, the left regular representation and the right regular representation of Γ are equivalent, and this concludes the proof. ∎

Corollary 3.12 (Bernoulli shifts). *Let Γ be an infinite countable group. Let $\mathbb{Z}/2\mathbb{Z}$ be the cyclic group of order two, with the discrete topology. Consider the direct product*

$$A = \prod_{\gamma \in \Gamma} (\mathbb{Z}/2\mathbb{Z})_\gamma$$

endowed with the product topology. Let ν be the normalized Haar measure on A. Let

$$g \cdot (a_\gamma) = (a_{g\gamma}), \quad \forall g \in \Gamma,\ \forall (a_\gamma) \in A,$$

be the action of Γ on A by Bernoulli shift automorphisms. Let $m \in \mathbb{C}[\Gamma]$ be a positive element. Let π_0 be the Koopman representation of Γ on $L_0^2(A, \nu)$. Then

$$\|\pi_0(m)\| = \|\rho_\Gamma(m)\|.$$

Proof. As Γ is infinite, the direct product A is infinite and the normalized Haar measure on this compact abelian group has no atom. Let $\chi : A \to \mathbb{T}$ be a continuous homomorphism. In order to prove the corollary it is enough to show that the stabilizer Γ_χ is finite (hence amenable). As any non trivial element of A has order two, we may view χ as a continuous homomorphism taking values in the discrete group $\mathbb{Z}/2\mathbb{Z}$. Hence $\chi^{-1}(\{0\})$ is open in A. By definition of the product topology, there exists a finite set $F \subset \Gamma$ such that

$$\{(a_\gamma) \in A : a_\gamma = 0,\ \forall \gamma \in F\} \subset \chi^{-1}(\{0\}).$$

Let us denote this open set as

$$U_F = \{(a_\gamma) \in A : a_\gamma = 0,\ \forall \gamma \in F\},$$

and let

$$V_F = \{(a_\gamma) \in A : a_\gamma = 0,\ \forall \gamma \in \Gamma \setminus F\}.$$

For each element $\gamma \in \Gamma$, let

$$p_\gamma : A \to \mathbb{Z}/2\mathbb{Z}$$

be the projection onto the direct factor $(\mathbb{Z}/2\mathbb{Z})_\gamma \cong \mathbb{Z}/2\mathbb{Z}$ of A. Also let us write $\mathbf{1}_g$ the element (a_γ) of A such that $a_\gamma = 0$ if $\gamma \neq g$ and $a_g = 1$. Each $a \in A$ can be written

$$a = u + v$$

with $u \in U_F$ and $v \in V_F$. It follows that

$$\chi = \sum_{\gamma \in F} \chi(\mathbf{1}_\gamma) p_\gamma.$$

As the action on Γ on itself by left translations is free, we see that Γ_χ is finite (bounded by $|F^{-1}F|$), hence amenable. ■

4. Proofs of the main results

We complete the proofs of the results stated in Section 2.

4.1. Lower bounds on the discrepancies of actions of free groups. We prove Proposition 2.3.

Proof. Let $n \in \mathbb{N}$ be given. Let $S_n \subset \Gamma$ be the sphere around e of radius n. As explained in Proposition A.2 from the Appendix, or according to [11] or [44, Formula 12.17],

$$\|\rho_\Gamma(m_{S_n})\| = \left(1 + \frac{q-1}{q+1} n\right) q^{-n/2}.$$

Let $E = S_n$ and let $H < \Gamma$ be the subgroup generated by E. Let $m_E \in \mathbb{C}[H] \subset \mathbb{C}[\Gamma]$. Decomposing Γ into its right H-cosets, it is easy to check that ρ_Γ restricted to H decomposes as a direct sum of unitary representations, all unitary equivalent to ρ_H, and that consequently

$$\|\rho_\Gamma(m_E)\| = \|\rho_H(m_E)\|.$$

Applying Theorem 3.6 we obtain

$$\|\pi_0(m_E)\| \geq \|\rho_H(m_E)\|.$$

Applying Formula (4) finishes the proof of the corollary in the case $E = S_n$. The case of the ball of radius n is similar. ■

4.2. Exact convergence rate for some isometries of the sphere. We prove Theorem 2.2.

Proof. It follows from [30, S153–S158] and [31, Theorem 4.1], that the spectrum of π_0 satisfies

$$\sigma\big(\pi_0(\mathbf{1}_{\Sigma_p})\big) \subset [-2\sqrt{p}, 2\sqrt{p}]. \tag{6}$$

(In fact, it is shown in [30, S153–S158] that $\sigma\left(\pi_0\left(\mathbf{1}_{\Sigma_p}\right)\right) = [-2\sqrt{p}, 2\sqrt{p}]$. We will "only" use the inclusion $\sigma(\pi_0(\mathbf{1}_{\Sigma_p})) \subset [-2\sqrt{p}, 2\sqrt{p}]$ but this is by far the hardest to prove; the main ingredient in its proof is the inequality [31, Theorem 4.1] which relies in particular on [15].) Applying inclusion (6) and Theorem 3.6 we deduce the inequalities

$$2\sqrt{p} \geq \left\|\pi_0\left(\mathbf{1}_{\Sigma_p}\right)\right\| \geq \left\|\rho_\Gamma\left(\mathbf{1}_{\Sigma_p}\right)\right\|.$$

It then follows from Kesten's spectral characterization of free groups (see [30, S157] and [27]) that Γ is free of rank $r = \frac{p+1}{2}$ (and freely generated by any subset A of Σ_p containing $\frac{p+1}{2}$ elements and satisfying $A \cap A^{-1} = \emptyset$). On Γ we consider the word metric defined by Σ_p, and for each $n \in \mathbb{N} \cup \{0\}$, the Hecke element

$$T_n = \sum_{\|\gamma\|_{\Sigma_p}=n} \gamma \in \mathbb{C}[\Gamma].$$

We have

$$T_0 = e, \quad T_1 = \sum_{\gamma \in \Sigma_p} \gamma = \mathbf{1}_{\Sigma_p}, \quad T_1 T_1 = T_2 + 2rT_0.$$

If $n \geq 2$ we have

$$T_n T_1 = T_{n+1} + pT_{n-1}.$$

There is a unique morphism of unital rings, from the ring $\mathbb{Z}[X]$ of polynomials in one variable with integer coefficients, to $\mathbb{C}[\Gamma]$, sending X to T_1. The above recursion relations show that T_n is in the image of this morphism for any $n \geq 0$. In other words for each $n \geq 0$, there exists $P_n \in \mathbb{Z}[X]$ such that $T_n = P_n(T_1)$.

We first prove the theorem in the case of a sphere $S_n \subset \Gamma$. The lower bound on the discrepancy follows from Proposition 2.3. For the upper bound, applying the spectral theorem for bounded self-adjoint operators, inclusion (6), and Kesten's computation [27] of the spectrum of the regular representation

$$\sigma\big(\rho_\Gamma(T_1)\big) = [-2\sqrt{p}, 2\sqrt{p}],$$

we deduce that

$$\|\pi_0(T_n)\| = \big\|\pi_0\big(P_n(T_1)\big)\big\|$$

$$
\begin{aligned}
&= \left\| P_n\big(\pi_0(T_1)\big) \right\| \\
&= \sup_{\lambda\in\sigma(\pi_0(T_1))} |P_n(\lambda)| \\
&\leq \sup_{\lambda\in[-2\sqrt{p},2\sqrt{p}]} |P_n(\lambda)| \\
&= \sup_{\lambda\in\sigma(\rho_\Gamma(T_1))} |P_n(\lambda)| \\
&= \|\rho_\Gamma(T_n)\|.
\end{aligned}
$$

Applying Proposition A.2 or [44, Formula 12.17], we conclude that

$$
\sup_{\|f\|_2=1} \left\| \frac{1}{|S_n|} \sum_{\gamma\in S_n} f(\gamma x) - \int_{\mathbb{S}^2} f(y)d\nu(y) \right\|_2 \leq \left(1 + \frac{p-1}{p+1}n\right) p^{-n/2}.
$$

The proof, in the case of a ball B_n, follows exactly the same lines. ■

Remark 4.1. The recurrence properties of the polynomials defined in the proof are reminiscent of the definition of Chebyshev's polynomials. The interested reader may consult [23] for further details (we thank Pierre de la Harpe for pointing out this reference).

4.3. Exact convergence rate for automorphisms of the torus. We prove Theorem 2.5.

Proof. According to Proposition 2.3, the lower bounds on the discrepancies are true and the cases with $q = 1$ have already been discussed in Corollary 3.10 and Remark 2.4. As explained in [14, Remark 20 on p. 16] or in [41, Theorem 4.17], the restriction of the Koopman representation defined by the action of G on $\mathbb{T}^2$

$$
\pi_0 : G \to \mathrm{U}\big(L_0^2(\mathbb{T}^2, \nu)\big)
$$

is weakly contained in the regular representation ρ_G. Hence, according to [14, Theorem 7], for any $m \in \mathbb{C}[G]$,

$$
\|\pi_0(m)\| \leq \|\rho_G(m)\|.
$$

This inequality also follows from Theorem 3.11 above. Choosing $m = m_{E_n} \in \mathbb{C}[\Gamma] \subset \mathbb{C}[G]$, where E_n is either a sphere or a ball, we get (as explained in the proof of Proposition 2.3):

$$
\|\rho_G(m_{E_n})\| = \|\rho_\Gamma(m_{E_n})\|.
$$

Applying Proposition A.2 or [44, Formula 12.17] finishes the proof of the theorem. ■

A. Appendix

The aim of this appendix is to recall well-known facts about the regular and quasi-regular representations of the automorphism group of a regular tree. Most relevant for this paper are explicit formulae for the norms of Markov operators, defined by the regular representation of the automorphism group of the tree. Although the formulae from Proposition A.2 below follow from [11], or [44, Formula 12.17] which is based on [44, Theorem 12.10] (see also [38] for a more general setting), it seems worthwhile to emphasize that these formulae can also be deduced and expressed with the help of the Harish-Chandra function of the quasi-regular representation of the automorphism group of a homogeneous tree. (Both approaches are equivalent; the modular functions of cocompact amenable subgroups in [38] correspond to Radon-Nikodym cocycles of quasi-regular representations.) One reason for relying on the quasi-regular representation and the Harish-Chandra function is that the method also applies to lattices in semi-simple groups.

A.1. The boundary of a tree and Busemann cocycles. Let (X, d) be the regular tree of degree $q+1$ equipped with its geodesic path metric d for which each edge is isometric to the unit interval $[0,1] \subset \mathbb{R}$. Let x_0 be a vertex of X. Let ∂X be its boundary at infinity (we refer the reader to [4] for more details). Let $b \in \partial X$ and let $\beta : [0,\infty) \to X$ be a geodesic ray representing b. Let $x, y \in X$. Let

$$B_b(x,y) = \lim_{t\to\infty}\left[d\big(x,\beta(t)\big) - d\big(y,\beta(t)\big)\right]$$

be the Busemann cocycle defined by $b \in \partial X$. Let $a, b \in \partial X$ and let $\alpha, \beta : [0,\infty) \to X$ be geodesic rays representing a and b. Their Gromov product relative to the base point x_0 is defined as

$$(a|b)_{x_0} = \frac{1}{2}\lim_{t\to\infty}\left[d\big(x_0,\alpha(t)\big) + d\big(x_0,\beta(t)\big) - d\big(\alpha(t),\beta(t)\big)\right].$$

The formula

$$d_{x_0}(a,b) = e^{-(a|b)_{x_0}}$$

defines an ultra-metric on ∂X.

A.2. Conformal transformations and Radon-Nikodym derivatives. The group $\mathrm{Aut}(X)$ of isometries of X acts on ∂X by conformal transformations. The Hausdorff dimension of $(\partial X, d_{x_0})$ equals $\log q$ and the normalized Hausdorff measure ν on $(\partial X, d_{x_0})$ is the unique Borel probability measure on ∂X invariant under the action of the stabilizer $K = \mathrm{Aut}(X)_{x_0}$ of x_0. The Radon-Nikodym derivative of $g \in \mathrm{Aut}(X)$ at $b \in \partial X$ is

$$\frac{dg_*\nu}{d\nu}(b) = q^{B_b(x_0, gx_0)}.$$

A.3. The Koopman representation and the Harish-Chandra function. Let us define a unitary representation

$$\lambda_\nu : \mathrm{Aut}(X) \to \mathrm{U}\big(L^2(\partial X, \nu)\big)$$

as

$$\big(\lambda_\nu(g) f\big)(b) = f(g^{-1}b)\sqrt{\frac{dg_*\nu}{d\nu}(b)}.$$

We still call it a Koopman representation, although the action of $\mathrm{Aut}(X)$ on ∂X does not preserve the measure ν, but only its class.

(The representation λ_ν is unitary equivalent to the quasi-regular representation $\lambda_{G/P}$, where $G = \mathrm{Aut}(X)$ and $P = G_b$, with $b \in \partial X$ any base point at infinity.)

Let $\mathbf{1}_{\partial X} \in L^2(\partial X, \nu)$ be the constant function equal to 1. The Harish-Chandra function

$$\Xi : \mathrm{Aut}(X) \to (0, \infty)$$

is the coefficient of λ_ν defined by $\mathbf{1}_{\partial X}$ that is,

$$\Xi(g) = \langle \lambda_\nu(g)\mathbf{1}_{\partial X}, \mathbf{1}_{\partial X}\rangle.$$

As the action of K preserves the measure and as λ_ν is unitary, the Harish-Chandra function is K-bi-invariant and symmetric. (In [24, Part II, Section 16] Harish-Chandra introduces the function Ξ on a connected reductive Lie group. The function Ξ can be viewed as the spherical function associated to a quasi-regular representation. The definitions make sense for G a locally compact (separable) unimodular group with a compact subgroup K such that (G, K) is a Gelfand pair, and an irreducible unitary representation π with one-dimensional K-invariant subspace [21, Section 1.5]. In our case $G = \mathrm{Aut}(X)$, $K = \mathrm{Aut}(X)_{x_0}$ and $\pi = \lambda_\nu$.)

A.4. A formula for the Harish-Chandra function. The length function

$$L : \mathrm{Aut}(X) \to \mathbb{N} \cup \{0\}$$

$$L(g) = d(x_0, gx_0)$$

is also K-bi-invariant and symmetric (notice that the elements of length 0 are the elements of K). As K acts transitively on each sphere of X with center x_0, if $g, g' \in \mathrm{Aut}(X)$ satisfy $L(g) = L(g')$ then there exist $k, k' \in K$ such that $kg = g'k'$. This implies that Ξ is constant on the level sets of L. For each $n \in \mathbb{N} \cup \{0\}$, we will write $\Xi(n)$ for the common value of the Harish-Chandra function on all $g \in \mathrm{Aut}(X)$ such that $L(g) = n$. We claim that for any $n \in \mathbb{N} \cup \{0\}$,

$$\Xi(n) = \left(1 + \frac{q-1}{q+1}n\right) q^{-n/2}. \tag{7}$$

If $L(g) = 0$, that is if $g \in K$, then $\Xi(g) = 1$. If $L(g) > 0$, let $[x_0, gx_0]$ be the (image of the) unique geodesic segment of X between x_0 and gx_0. For each vertex x of X at distance exactly 1 from $[x_0, gx_0]$, consider the ball U_x of ∂X of radius $e^{-d(x_0,x)}$ consisting of the points at infinity of the geodesic rays of X starting from x_0 and passing through x. We obtain the partition

$$\partial X = \bigcup_{d(x,[x_0,gx_0])=1} U_x.$$

As for each $r \in \mathbb{N}$ the measure of a ball of radius e^{-r} equals $[(q+1)q^{r-1}]^{-1}$, it is easy to check that

$$\begin{aligned}
\Xi(g) &= \int_{\partial X} q^{\frac{1}{2}B_b(x_0,gx_0)} d\nu(b) \\
&= \sum_{d(x,[x_0,gx_0])=1} \int_{U_x} q^{\frac{1}{2}B_b(x_0,gx_0)} d\nu(b) \\
&= \sum_{d(x,[x_0,gx_0])=1} \int_{U_x} q^{\frac{1}{2}(d(x_0,x)-d(gx_0,x))} d\nu(b) \\
&= \sum_{d(x,[x_0,gx_0])=1} q^{\frac{1}{2}(d(x_0,x)-d(gx_0,x))} \nu(U_x) \\
&= \left(1 + \frac{q-1}{q+1} n\right) q^{-n/2}.
\end{aligned}$$

A.5. Computing operator norms with the Harish-Chandra function. We first compute the norms of some operators defined by the Koopman representation. We then explain how spectral transfer applies to compute the norms of the corresponding operators defined by the regular representation.

Proposition A.1. *Let $r \in \mathbb{N}$. Let Γ be the free group of rank r. Let $a_1, \ldots, a_r$ be a free generating set of Γ. Let X be the Cayley graph of Γ with respect to $\{a_1^{\pm 1}, \ldots, a_r^{\pm 1}\}$. Let $e = x_0 \in X$ be a base point. Let $G = Aut(X)$ and let $L(g) = d(x_0, gx_0)$ be the length function on G defined by x_0. For each integer $n \geq 0$, let $S_n = L^{-1}(n) \cap \Gamma$ and let $B_n = L^{-1}([0,n]) \cap \Gamma$ (where $\Gamma \subset G$ is the natural embedding). Then*

$$\|\lambda_\nu(m_{S_n})\| = \Xi(n),$$

$$\|\lambda_\nu(m_{B_n})\| = \frac{1}{|B_n|} \sum_{k=0}^{n} \Xi(k)|S_k|.$$

Proof. We first consider the case of the sphere S_n. Let $\mathbf{1}_{\partial X}$ be the constant function equal to 1 on ∂X. Applying the Cauchy–Schwarz inequality and the fact that the function Ξ is constant on S_n, we obtain

$$\begin{aligned}\|\lambda_\nu(m_{S_n})\| &\geq \|\lambda_\nu(m_{S_n})\mathbf{1}_{\partial X}\|_2 = \|\lambda_\nu(m_{S_n})\mathbf{1}_{\partial X}\|_2\|\mathbf{1}_{\partial X}\|_2 \\ &\geq \langle \lambda_\nu(m_{S_n})\mathbf{1}_{\partial X}, \mathbf{1}_{\partial X}\rangle = \frac{1}{|S_n|}\sum_{\gamma\in S_n}\langle \lambda_\nu(\gamma)\mathbf{1}_{\partial X}, \mathbf{1}_{\partial X}\rangle \\ &= \Xi(n).\end{aligned}$$

To prove the other inequality, we first notice that for $p \in \{1, 2, \infty\}$, the operator $\lambda_\nu(m_{S_n}) : L^p(X, \nu) \to L^p(X, \nu)$ is bounded. In the case $p = 2$ it is self-adjoint. Thanks to Riesz–Thorin's theorem,

$$\|\lambda_\nu(m_{S_n})\|_{2\to 2} \leq \|\lambda_\nu(m_{S_n})\|_{\infty\to\infty}.$$

As $\lambda_\nu(m_{S_n})$ preserves positive functions, it is obvious that

$$\|\lambda_\nu(m_{S_n})\|_{\infty\to\infty} = \|\lambda_\nu(m_{S_n})\mathbf{1}_{\partial X}\|_\infty.$$

We claim that the function $\lambda_\nu(m_{S_n})\mathbf{1}_{\partial X}$ is constant equal to $\Xi(n)$. To prove the claim we first show that the function is K-invariant (hence constant as K acts transitively on ∂X). Let $b \in \partial X$ and $k \in K$. We have

$$\begin{aligned}\lambda_\nu(m_{S_n})\mathbf{1}_{\partial X}(kb) &= \frac{1}{|S_n|}\sum_{\gamma\in S_n} q^{\frac{1}{2}B_{kb}(x_0,\gamma x_0)} \\ &= \frac{1}{|S_n|}\sum_{\gamma\in S_n} q^{\frac{1}{2}B_b(k^{-1}x_0,k^{-1}\gamma x_0)} \\ &= \frac{1}{|S_n|}\sum_{\gamma\in S_n} q^{\frac{1}{2}B_b(x_0,k^{-1}\gamma x_0)} \\ &= \frac{1}{|S_n|}\sum_{\gamma\in S_n} q^{\frac{1}{2}B_b(x_0,\gamma x_0)} \\ &= \lambda_\nu(m_{S_n})\mathbf{1}_{\partial X}(b).\end{aligned}$$

The claim is proved because ν is a probability measure and by definition of Ξ we have

$$\int_{\partial X}\lambda_\nu(m_{S_n})\mathbf{1}_{\partial X}(b)d\nu(b) = \Xi(n).$$

In the he case of the ball B_n, applying Cauchy–Schwarz's inequality and Riesz–Thorin's theorem in similar ways proves that

$$\|\lambda_\nu\left(\mathbf{1}_{B_n}\right)\| = \sum_{\gamma\in B_n}\Xi(\gamma).$$

The result follows because Ξ is constant on spheres. ∎

Proposition A.2. *Let $r \in \mathbb{N}$. Let Γ be the free group of rank r. Let $a_1, \dots, a_r$ be a free generating set of Γ. Let $S = \{a_1^{\pm 1}, \dots, a_r^{\pm 1}\}$. For each integer $n \geq 0$, let S_n, respectively B_n, be the sphere, respectively the ball, around $e \in \Gamma$ of radius n with respect to the word metric on Γ defined by S. Let $q = 2r - 1$. Then*

$$\|\rho_\Gamma(m_{S_n})\| = \left(1 + \frac{q-1}{q+1} n\right) q^{-n/2},$$

$$\|\rho_\Gamma(m_{B_n})\| = c(q,n)\left(1 + \left(1 + \frac{1}{\sqrt{q}}\right) n\right) q^{-n/2},$$

where $c(q,n) = \left(1 + 2q^{-n} \sum_{k=0}^{n-1} q^k\right)^{-1}$.

Proof. We claim that for any positive element $m \in \mathbb{C}[\Gamma]$, we have

$$\|\rho_\Gamma(m)\| = \|\lambda_\nu(m)\|.$$

The inequality $\|\rho_\Gamma(m)\| \leq \|\lambda_\nu(m)\|$ follows from [42, Lemma 2.3] because $\mathbf{1}_{\partial X}$ is a positive vector for λ_ν. The inequality $\|\rho_\Gamma(m)\| \geq \|\lambda_\nu(m)\|$ is true for any element $m \in \mathbb{C}[\Gamma]$ (not only positive ones), because the action of Γ on ∂X is amenable, see [28] and [14, Theorem 7]. To prove the proposition, we apply Proposition A.1 and Formula (7). In the case of the sphere S_n, this immediately proves the statement. The case of the ball B_n requires some computation. If $q = 1$, the formula is obvious. If $q > 1$, we have

$$\begin{aligned}
\|\rho_\Gamma(m_{B_n})\| &= \frac{1}{|B_n|} \sum_{k=0}^{n} \Xi(k) |S_k| \\
&= \frac{1}{|B_n|} \left(1 + \frac{q + q^{1/2}}{q} n\right) q^{n/2} \\
&= \left(\frac{q+1}{q-1} q^n - \frac{2}{q-1}\right)^{-1} \left(1 + \frac{q + q^{1/2}}{q} n\right) q^{n/2} \\
&= c(q,n)\left(1 + \left(1 + \frac{1}{\sqrt{q}}\right) n\right) q^{-n/2},
\end{aligned}$$

where $c(q,n) = \left(1 + 2q^{-n} \sum_{k=0}^{n-1} q^k\right)^{-1}$. ■

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