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Unknotting with a single twist

Samantha Allen and Charles Livingston

Abstract. Given a knot $K \subset S^3$, is it possible to unknot it by performing a single twist, and if so, what are the possible linking numbers of such a twist? We develop obstructions to unknotting using a twist of a specified linking number. The obstructions we describe are built using classical knot invariants, Casson–Gordon invariants, and Heegaard Floer theory.

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1. Introduction

Figure 1 presents three illustrations of the right-handed trefoil knot, $T(2,3)$. In each, performing a full twist on the parallel strands that pass through the small circle results in an unknot. In the first two cases the required twist is negative, and in the last it is positive. The linking numbers of the twists, which by convention are always positive, are 2, 3, and 0, respectively. Thus, we say the set of unknotting twist indices, denoted $\mathcal{U}$, satisfies $\{2^-, 3^-, 0^+\} \subset \mathcal{U}(T(2,3))$. The reader is invited to show that for the figure eight knot, 4_1, $\{2^-, 0^-, 0^+, 2^+\} \subset \mathcal{U}(4_1)$. The results of this paper will imply that these two containments are, in fact, equalities.

Our goal is to consider the question of which knots can be unknotted with a single twist and, more generally, to describe tools for analyzing $\mathcal{U}(K)$. One of our basic results, a consequence of Corollary 8.4, is the following.

Theorem. *For each knot $K \subset S^3$, $\mathcal{U}(K)$ is a subset of a set of the form $\{a^-, (a+1)^-, 0^-, 0^+, b^+, (b+1)^+\}$ for some $a, b > 0$. With the exception of the case $a = 1, b = 1$, the containment is always proper.*

We will see that the values of a and b are determined by the Heegaard Floer complex $\mathrm{CFK}^\infty(K)$. Much of our work is identifying and organizing invariants that can be used to further restrict $\mathcal{U}(K)$.

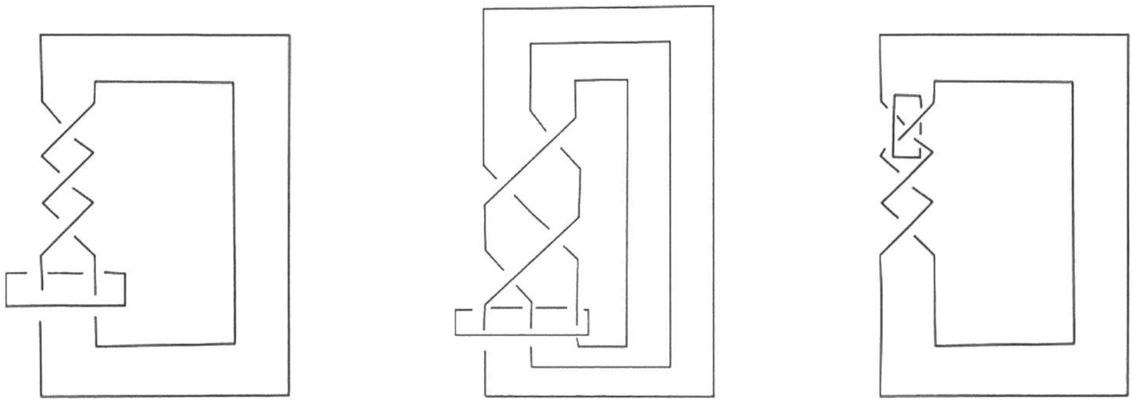

FIGURE 1
Unknotting the trefoil

With a few exceptions, finding knots for which the unknotting set has several elements is difficult. For instance, for large $k \geq 2$ we don't know of any examples for which $\{k^-, (k+1)^+\} \subset \mathcal{U}(K)$. The following result, stated as Corollary 5.8, indicates the challenge of building examples and the impossibility of finding examples among low-crossing number knots.

Theorem. *If* $\{k^-, (k+1)^+\} \subset \mathcal{U}(K)$ *for* $k \geq 1$, *then*

$$g(K) \geq \frac{2k^3 + 3k^2 - 11k + 6}{6}.$$

This general problem of understanding unknotting with a single twist has been extensively studied, often in the more general setting in which the operation consists of introducing perhaps more than one full twist on the parallel strands. A sampling of references includes [AN, ANMM1, ANMM3, ANY, GoS, HM97, Mat, MY, Ter]. Of particular note is work of Ince [Inc1, Inc2] which applies Heegaard Floer theory in the case of linking number 0 and that of Sato [Sat], which considers a related slicing problem in $\mathbb{CP}^2$.

There is an interesting generalization of these questions related to slicing knots in $\pm\mathbb{CP}^2$. In the case that $l \notin \mathcal{U}(K)$, one can ask for the minimum genus of a surface bounded by K in $\pm\mathbb{CP}^2$ that represents the homology class $l \in H_2(\pm\mathbb{CP}^2 \setminus B^4, \partial) \cong \mathbb{Z}$. Progress on this problem has recently been made by Pichelmeyer [Pic].

From the perspective of classical knot theory, a theorem of Ohyama [Ohy] heightens the interest in unknotting with one twist: every knot can be unknotted with two full twists. In [Liv2] it is observed that the linking numbers of the two twists can be any consecutive pair of integers, and if one requires that the linking numbers be 0, then up to $2g$ twists might be required, where g is the three-genus of the knot.

Some of our results concerning $\mathcal{U}(K)$ overlap with previous ones, albeit with alternative and at times simpler proofs. Other results, especially those based on Heegaard Floer theory, are new, as is our examination of new ways to use the various approaches in conjunction.

The basis of much of our work is the observation that if K can be unknotted with a single twist, then three-manifolds built by surgery on K bound four-manifolds with special properties. This in turn lets us apply a range of tools to the problem, including those that arise in classical knot theory, Casson–Gordon theory, and Heegaard Floer theory. Of special relevance is the work of Aceto and Golla [AG] applying Heegaard Floer theory to the question of which surgeries on a given knot K bound rational homology four-balls.

We now summarize a few of the main results presented in the paper.

Outline. Section 2 presents the basic geometric observations that form the basis of our later work. This includes the observation that if K can be unknotted with a single twist of sign s and linking number l, then $S^3_{-sl^2-s}(K) \cong S^3_{sl^2+s}(J)$ for some knot J. We also note the well-known fact that for such knots K, $S^3_{-sl^2}(K)$ bounds a four-manifold W with $H_1(W) \cong \mathbb{Z}_l$ and, if $l \neq 0$, $H_2(W) = 0$; in addition, we note that in fact $H_1(W) \cong \pi_1(W)$ and that the map $\pi_1(S^3_{sl^2}(K)) \to H_1(W)$ is surjective.

Sections 3, 4, 5, and 6 present results related to homological invariants associated to branched cyclic covers and the infinite cyclic cover. In Section 3 the focus is on the ranks of the homology groups. As we describe, our results that arise from $\mathbb{Z}[t,t^{-1}]$-coefficients instead of $\mathbb{Q}[t,t^{-1}]$-coefficients depend on the use of Gröbner bases. In cases in which the rank of the homology is not sufficient to provide necessary obstructions, we observe in Section 4 that the $\mathbb{Q}/\mathbb{Z}$-valued linking form can provide stronger obstructions.

Section 5 explores the use of the knot signature function, using an approach developed by Casson and Gordon; our results include a short proof of a theorem of Aït Nouh and Yasuhara [ANY] which they stated for torus knots. In combination, these results place strong limits on the possibility of the set $\mathcal{U}(K)$ containing both positive and negative entries; for instance, if $\{4^-, 5^+\} \subset \mathcal{U}(K)$, then the genus of K is at least 23; another new application of the signature result is that, with a few exceptions, if $\gcd(l_1, l_2) \neq 1$, then $\{l_1, l_2\} \not\subset \mathcal{U}(K)$ for any knot K (with any choice of signs).

Finally, Section 6 complements the signature results of Section 5 with a discussion of the Arf invariant. Applications to torus knots are described. In addition, among classical knot invariants, the Arf invariant is to our knowledge the strongest one that can address the possibility of $1^\pm \in \mathcal{U}(K)$. Later we will see that Heegaard Floer obstructions offer alternative obstructions, but even for low-crossing alternating knots of fewer than 13 crossings, the Arf invariant provides an obstruction in over 600 cases in which the Heegaard Floer obstructions vanish.

Section 7 provides background on Heegaard Floer theory, summarizing the essential properties of the knot invariants $\nu^+(K)$ and $V_i(K)$. We also describe the needed properties of the three-manifold correction terms, $d(Y, \mathfrak{s})$, restricting to the case of $Y = S^3_m(K)$.

Section 8 presents an obstruction to unknotting based on the invariants $V_i(K)$. The obstruction itself was first presented by Sato [Sat] and our result also follows from work of Aceto–Golla [AG]. We include our own proof; in our setting we are able to give very short, and hopefully accessible, arguments. In addition, we present new applications of these obstructions. Section 9 presents much stronger constraints on the invariants $V_i(K)$. These are specific to the unknotting problem and do not apply in the more general settings of [AG, Sat].

In Section 10 we present obstructions based on the Upsilon invariant of Ozsváth-Stipsicz-Szabó [OSS]. As we will make clear, the Upsilon invariant is theoretically no stronger than the V_i-invariants, but it has the advantage of being much more computable; this is illustrated with an example of connected sums of torus knots. Section 11 presents obstructions based on the Heegaard Floer invariants associated to cyclic covers of a knot K, more specifically, $d(M_2(K), \mathfrak{s})$.

Section 12 discusses the case of alternating knots, in which computations are most accessible, and Section 13 illustrates a construction of Aït Nouh that provides examples of knots that can be unknotted with positive and negative twists, both of linking number 1.

Section 14 addresses the initial question: given a knot, can it be unknotted with a single twist? That is, for what knots is $\mathcal{U}(K) = \varnothing$?

Section 15 presents some comments and open problems concerning the sets $\mathcal{U}(K)$.

In the appendices we present a few technical results and present a summary of an analysis of prime knots of eight or fewer crossings.

2. Geometric results related to unknotting twists

Throughout this paper we will use surgery descriptions of three-manifolds, knots, and their branched covering spaces. A basic reference is the text by Rolfsen [Rol, Chapter 9H]. More details can be found in [GS] and original sources such as [Kir].

2.1. Three-dimensional aspects of surgery diagrams and unknotting. Let $K \subset S^3$ be a knot. We denote by $S^3_n(K)$ the three-manifold formed by n-

surgery on K. If (K, J) is a link, we write $S^3_{n,m}(K, J)$ for the three-manifold formed by performing n- and m-surgery on K and J, respectively.

In the case of the unknot U and $s = \pm 1$ we have $S^3_s(U) \cong S^3$. It follows that $S^3_{n,s}(K, U) \cong S^3_{n'}(K')$ for some n' and K'. As described in [Rol, Chapter 9G], K' is the knot formed by performing a full twist to the strands of K passing through U, twisting left or right depending on whether $s = 1$ or $s = -1$, respectively. The new surgery coefficient is $n' = n - sl^2$, where $l = \text{link}(K, U)$. (In general, the linking number is defined for *oriented* links. Here, we chose orientations so that the linking number is nonnegative.) In summary, we have the next result.

Theorem 2.1. *Suppose that K can be unknotted with a single left-handed or right-handed twist, corresponding to signs $s = -1$ and $s = 1$, respectively, and linking number l. Then for all n,*

$$S^3_n(K) \cong S^3_{n+sl^2,s}(U_1, U_2),$$

where U_1 and U_2 are both unknotted and have linking number l.

Corollary 2.2. *In the setting of the theorem:*

(1) $S^3_{-sl^2}(K) \cong S^3_{0,s}(U_1, U_2)$, *where* $link(U_1, U_2) = l$.

(2) $S^3_{-sl^2-s}(K) \cong S^3_{-s,s}(U_1, U_2) \cong S^3_{sl^2+s}(J)$, *for some knot* J.

2.2. Four-dimensional aspects of surgery diagrams and unknotting. There are homeomorphisms

$$S^3_0(U) \cong S^1 \times S^2 \cong \partial(S^1 \times B^3),$$

but note that these are not canonical and this fact makes discussing framings somewhat complicated. Akbulut [Akb] developed a diagrammatic method of illustrating $S^1 \times B^3$, its handlebody structure as a manifold built from B^4 by adding a single one-handle, and framed curves in its boundary; these framed curves determine attaching maps for two-handles. In these diagrams, an unknotted circle C with a dot represents $S^1 \times B^3$, its boundary is identified with 0-surgery on the same unknotted circle, and framed curves in that boundary are represented by curves in the complement of C marked with integers. A full exposition is contained in [GS]. Before stating the theorem that builds on the geometric connections between the three-dimensional and four-dimensional perspective, we present an example.

Example 2.3. The right-handed trefoil knot K can be unknotted by performing a single negative twist of two-strands. Thus $S^3_4(K) = S^3_{0,-1}(U_1, U_2)$ for some link of unknotted circles $\{U_1, U_2\}$. Standard surgery diagrams (see [Rol, Chapter 9G])

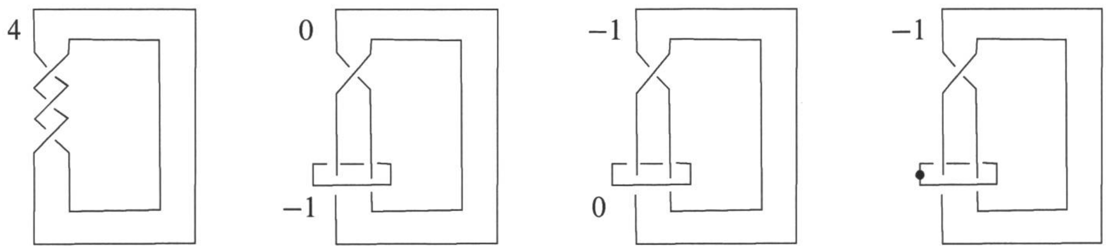

FIGURE 2
Four diagrams of $S^3_4(T(2,3))$. The two middle diagrams are isotopic.

appear in the first three frames of Figure 2. The fourth frame of Figure 2 provides the dotted-circle diagram (see [GS, Section 5.4]), in which $S^1 \times B^3$ is represented by the curve with a dot on it and the attaching curve for a two-handle addition is represented by a curve with framing -1. This final diagram then represents a four-manifold bounded by $S^3_4(K)$.

In general, for any link of unknotted circles (U_1, U_2), we have $S^3_{0,s}(U_1, U_2) \cong \partial W$, where W is a four-manifold that can be built from $S^1 \times B^3$ by adding a single two-handle. The next theorem then follows readily; we write $\mathbb{Z}_l$ for $\mathbb{Z}/l\mathbb{Z}$, so in the special case $l = 0$ we have $\mathbb{Z}_l \cong \mathbb{Z}$.

Theorem 2.4. *Suppose that K can be unknotted with a single twist of sign s and linking number l.*

(1) *$S^3_{-sl^2}(K) = \partial W$ where W can be built from $S^1 \times B^3$ by adding a single two-handle, added with framing s in a diagram for $S^1 \times B^3$.*
(2) *The attaching curve for the two-handle represents $l \in H_1(S^1 \times B^3)$, so $\pi_1(W) \cong H_1(W) \cong \mathbb{Z}_l$.*
(3) *The map induced by inclusion, $H_1(S^3_{-sl^2}(K)) \to H_1(W)$, corresponds to the surjection $\mathbb{Z}_{l^2} \to \mathbb{Z}_l$.*

3. Single twist unknotting: Homological constraints

The simplest obstructions to unknotting with a single twist arise from homological properties of the cyclic branched covers and the infinite cyclic cover of the knot. To describe these, we let $M_q(K)$ denote the q-fold cyclic branched cover of S^3 with branching set K and let $M_\infty(K)$ denote the infinite cyclic cover. We begin with a definition.

Definition 3.1. A triple (U_1, U_2, s) where (U_1, U_2) is a link with unknotted components and $s = \pm 1$ is called a *surgery diagram* for a knot K if U_1 represents the knot K in $S^3_s(U_2) \cong S^3$.

Theorem 3.2. *If K can be unknotted with a twist of linking number l and l is divisible by q, then $H_1(M_q(K),\mathbb{Z})$ is generated by q elements. Additionally, if $l=0$, then the homology group $H_1(M_\infty(K),\mathbb{Z})$ is generated by a single element as a $\mathbb{Z}[t,t^{-1}]$-module.*

Proof. Details of the construction of branched covers of knots from surgery diagrams are presented in [Rol, Chapter 6C]. Starting with the surgery diagram for K, (U_1,U_2,s), one can construct a surgery diagram of $M_q(K)$; the surgery link consists of the components of the preimage of U_2 in the q-fold branched cover of S^3 over U_1, $M_q(U_1)\cong S^3$. There are q components. In particular, the homology of $M_q(K)$ has a presentation with q generators.

The case of $l=0$ is similar; in brief, the infinite cyclic cover is given by surgery on the set of translates of a single curve in the infinite cyclic cover of the unknot. Rolfsen's illustration of infinite cyclic covers [Rol, Chapter 7C] makes the result apparent. □

Note. Recall that $\operatorname{rank}(H_1(M_q(K),\mathbb{Z}))\le 2g(K)$. This follows from a theorem of Seifert [Sei]; see also [Rol, Chapter 8, D9] and [Gil]. Thus, the obstruction arising from Theorem 3.2 can provide information only in the case $q\le 2g$.

Example 3.3. For low-crossing prime knots, this result is of limited value. Among prime knots of 12 or fewer crossings, for only seven is $\operatorname{rank}(H_1(M_2(K),\mathbb{Z}))>2$. These are $12a_{554}, 12a_{750}$, $12n_{553}$, $12n_{554}$, $12n_{555}$, $12n_{556}$, and $12n_{642}$. Thus, only these seven are obstructed from being unknotted with a single twist with even linking number using 2-fold branched covers.

For connected sums, the theorem offers stronger results. For instance, for the trefoil knot, $T(2,3)$, $H_1(M_2(T(2,3)))\cong\mathbb{Z}_3$ and thus $3T(2,3)$ cannot be unknotted with a single twist of even linking number. In this context, it is worth noting that there are examples of composite knots that can be unknotted with a single twist, [MS, MY, Ter], and an open conjecture is that all such examples have exactly two prime components.

3.1. Alexander ideals and Gröbner bases. The homology of the infinite cyclic cover of a knot has a presentation as a $\mathbb{Z}[t,t^{-1}]$-module of the form $A=V-tV^{\mathsf{T}}$, where V is a square Seifert matrix of size $2g$. For $0\le k<2g$, the k-elementary ideal (or Alexander ideal) $E_k(K)$ is defined to be the ideal in $\mathbb{Z}[t,t^{-1}]$ that is generated by the $(2g-k)\times(2g-k)$ minors of A. These ideals are independent of the choice of Seifert matrix V and are invariants of the underlying module. In general $E_0(K)$ is principal, generated by the Alexander polynomial. If $H_1(M_\infty(K))$ is generated by a single element, then $E_1(K)=\langle 1\rangle$.

A reduced Gröbner basis of a multi-variable polynomial ideal is a generating set of a specific form. A basic reference is [DF]. For us, the relevant properties are that such bases are readily computable by computer packages (we use Wolfram Mathematica [Wol]) and permit one to determine whether two given ideals are equal. To apply Gröbner bases to our work, we note that there is a surjection

$$\mathbb{Z}[t,s] \to \mathbb{Z}[t,s]/\langle 1-ts\rangle \cong \mathbb{Z}[t,t^{-1}],$$

and thus ideals in $\mathbb{Z}[t,t^{-1}]$ can be analyzed via their preimages in the polynomial ring $\mathbb{Z}[t,s]$.

Example 3.4. In considering the rank of $H_1(M_\infty(K))$ we can use rational coefficients, in which case the obstruction is more easily computed, or we can work with integer coefficients, in which case the computation is more complicated but the results are much stronger. For instance, there are 84 prime knots of 9 or fewer crossings. Of those, only two, 8_{18} and 9_{40}, have infinite cyclic cover with noncyclic homology using $\mathbb{Q}[t,t^{-1}]$-coefficients. If one switches to $\mathbb{Z}[t,t^{-1}]$-coefficients, an additional seven knots are obstructed from being unknotted with a twist of linking number 0 ($9_{35}, 9_{37}, 9_{41}, 9_{46}, 9_{47}, 9_{48}, 9_{49}$). Among these examples is 9_{46}; see [Rol, Chapter 8C], where showing that this knot does not have cyclic Alexander module is presented as an exercise. From what is developed there, it is easily seen that the second Alexander ideal is $\langle 3, 1-t\rangle \subset \mathbb{Z}[t,t^{-1}]$. For the rest of the examples, we used Mathematica to find the Gröbner basis for $E_1(K)$, in each case showing that the module is nontrivial.

4. Single twist unknotting: Two-fold covers, linking forms and signatures

In the case that the modules $H_1(M_q(K))$ do not obstruct an unknotting twist, the linking form on $H_1(M_2(K))$ can offer much stronger constraints. Recall that for any three-manifold M with $H_1(M,\mathbb{Q}) = 0$, there is a linking form $\mathrm{lk}\colon H_1(M) \times H_1(M) \to \mathbb{Q}/\mathbb{Z}$. In the case that $M = \partial W$, where $H_1(W) = 0$ and the intersection form of W is presented by a matrix Q, the matrix Q is also a presentation matrix for $H_1(M)$ and Q^{-1} presents the linking form of M with respect to a corresponding generating set of $H_1(M)$. (Our sign convention, which differs from that of some references, is chosen so that if $M = S^3_n(K)$, then the meridian to K has self-linking $1/n \in \mathbb{Q}/\mathbb{Z}$.) More generally, if M is given as surgery on a link, then the linking form with respect to the meridians is presented by the inverse of the associated surgery matrix, formed from the linking matrix by using the surgery coefficients as the diagonal entries.

Theorem 4.1. *If K can be unknotted with a single twist of linking number $2k$ and sign s, then the two-fold branched cover $M_2(K)$ is given by surgery on a two-component link with surgery matrix*

$$Q = \begin{pmatrix} a & b \\ b & a \end{pmatrix},$$

where $\left|a^2 - b^2\right| = \det(K)$ *and* $a + k \equiv 1 \mod 2$.

Proof. The statement that the surgery matrix is 2×2 with the diagonal entries equal follows from the discussion of Section 3. The determinant of a knot is the order of the homology of the 2-fold branched cover, giving the condition that $\left|a^2 - b^2\right| = \det(K)$.

A theorem of Nagami [Nag] states that for a closed four-manifold W with $H_1(W, \mathbb{Z}_2) = 0$, the two-fold branched cover over a surface that represents $2k$ for some homology class $k \in H_2(W)$ is Spin if and only if the mod 2 reduction of k is dual to the second Stiefel–Whitney class of W. As described after the completion of this proof, the bounding manifold we have constructed is the two-fold branched cover of a punctured $\pm\mathbb{CP}^2$; because the boundary is S^3, Nagami's result applies: to move to the setting of closed manifolds simply cap off the punctured manifold with a four-ball and the surface with an orientable surface.

Thus, the two-fold cover is Spin if and only if k is odd. The intersection form of the two-fold cover is given by Q, and so the cover is Spin if and only if a is even. Combining these observations we have that the parities of k and a must differ, or more concisely, $a + k \equiv 1 \mod 2$. □

Kauffman and Taylor [KT] proved that if a knot K bounds a surface F in a four-manifold W and the two-fold branched cover of (S^3, K) extends over (W, F), then the signature of K is determined by invariants of W, the normal bundle to F, and the two-fold branched cover of W over F. Restricting to our setting, we consider a knot K that can be unknotted with a single twist of linking number l and sign $s = \pm 1$. Such a knot bounds a disk $\Delta \subset s\mathbb{C}P^2 \setminus B^4$ with Euler class satisfying $\chi^2 = sl^2$. The pair $(\mathbb{C}P^2 \setminus B^4, \Delta)$ is built from the unknotted ball pair (B^4, B^2) by adding a two-handle with framing s along an unknotted J in the complement of ∂B^2. If l is even, then the two-fold cover branched over Δ is thus built from B^4 by adding two two-handles to B^4 with the same framings. Notice that such a handlebody description of a four-manifold yields a surgery diagram for its boundary, and in the present case this surgery diagram is identical to the one presented in Section 3.

A restatement of [KT, Theorem 3.1] in this special case immediately yields the following result.

Theorem 4.2. *If a knot $K \subset S^3$ can be unknotted with a single twist of even linking number l and sign $s = \pm 1$, then*

$$\sigma(K) = \sigma(N) - 2s + \frac{1}{2}sl^2,$$

where N is the two-fold cover of $s\mathbb{C}P^2 \setminus B^4$ branched over some disk Δ such that $\partial\Delta = K$.

Theorems 4.1 and 4.2 have the following corollary.

Corollary 4.3. *If a knot $K \subset S^3$ can be unknotted with a single twist of even linking number $l = 2k$ and sign $s = \pm 1$, then the two-fold cover of S^3 branched over K bounds a four-manifold N with second Betti number $b_2(N) = 2$, signature*

$$\sigma(N) = \sigma(K) + 2s(1 - k^2),$$

and intersection pairing with matrix of the form

$$Q = \begin{pmatrix} a & b \\ b & a \end{pmatrix}.$$

The value of a is even or odd depending on whether k is odd or even, respectively. The value of $|a^2 - b^2| = \det(K)$ and Q is negative definite, indefinite, or positive definite depending on whether $\sigma(K) + 2s - \frac{1}{2}sl^2$ is $-2, 0$ or 2, respectively.

Proof (*Corollary 4.3*). The equation $\sigma(N) = \sigma(K) + 2s(1-k^2)$ is a restatement of the equation in Theorem 4.2, replacing l with $2k$. The matrix Q is the intersection form of the two-fold branched cover N; the conditions on its determinant and the parity of a follow from Theorem 4.1.

Finally, a form of rank two is negative definite if and only if it has signature -2; in our case the signature is given by $\sigma(K) + 2s - \frac{1}{2}sl^2$. The argument is similar for the indefinite and positive definite cases. □

Corollary 4.3 implies that the signature of a knot places very strong constraints on the possible even linking numbers of unknotting twists. Here is one simply stated result. Recall that if K is unknotted with a left-handed twist then the sign is $s = -1$.

Corollary 4.4. *If $K \subset S^3$ can be unknotted with a single twist of linking number $2k$, $k \geq 0$ and sign s, then*

$$k^2 - 2 \leq s\sigma(K)/2 \leq k^2.$$

Proof. Using the set up of Corollary 4.3, we have $s\sigma(K)/2 = (k^2-1)+s\sigma(N)/2$. The result follows from the observation that $-2 \leq \sigma(N) \leq 2$. □

Corollary 4.5. *If* $K \subset S^3$ *and* $\det(K) \equiv 3 \mod 4$, *then* $\{0^-, 0^+\} \not\subset \mathcal{U}(K)$.

Proof. According to Corollary 4.4, if K can be unknotted with twists of linking number $l = 0$ of both signs $s = -1$ and $s = 1$, then $\sigma(K) = 0$. However, in Murasugi's paper in which the knot signature was first defined, he proved that if $\det(K) \equiv 3 \mod 4$ then $\sigma(K) \equiv 2 \mod 4$; see [Mur, Theorem 5.6]. In particular, $\sigma(K) \neq 0$. □

Example 4.6. Consider the knot $K = -7_7$. This is a two-bridge knot $B(21, 13)$ and $M_2(K) = L(21, 13)$. It satisfies $\sigma(K) = 0$ and $\det(K) = 21$. This knot has unknotting number 1, and a quick examination of its diagram shows that it can be unknotted with a left-handed twist, so $0^- \in \mathcal{U}(K)$. We will show that $0^+ \notin \mathcal{U}(K)$. Suppose that K could be unknotted with a single positive twist of linking number 0. Then Corollary 4.3 implies that $M_2(K)$ bounds a positive definite four-manifold N with $b_2(N) = 2$ and intersection pairing

$$Q = \begin{pmatrix} a & b \\ b & a \end{pmatrix}$$

with determinant 21. Up to change of basis, there are only two such matrices:

$$Q_1 = \begin{pmatrix} 11 & 10 \\ 10 & 11 \end{pmatrix} \text{ and } Q_2 = \begin{pmatrix} 5 & 2 \\ 2 & 5 \end{pmatrix}.$$

This would imply that $H_1(M_2(K)) \cong \mathbb{Z}_{21}$ is generated by an element with self-linking either $11/21$ or $5/21$. The set of all self-linking numbers of generators would be given by the set of residues $11(i^2) \mod 21$ or $5(i^2) \mod 21$, where $\gcd(i, 21) = 1$. These two sets are $\{2, 8, 11\}$ and $\{5, 17, 20\}$. On the other hand, as an oriented manifold, $\Sigma(K) = L(21, 13)$ and the set of self-linking numbers of generators is given by $\{10, 13, 19\}$.

Example 4.7. Consider the question: for a given knot K, is $\{0^-, 0^+\} \subset \mathcal{U}(K)$? The results of this section imply that if K can be unknotted with a single twist of linking number 0, then $\sigma(K) \in \{-2, 0, 2\}$. Furthermore, if $\det K \equiv 3 \mod 4$, then by Corollary 4.5 at most one of 0^- and 0^+ are in $\mathcal{U}(K)$. Thus, if $\sigma(K) \neq 0$ the answer is "no." There are 35 prime knots of 8 or fewer crossings, and this condition reduces the number of possible "yes" cases to 13.

Of these remains 13 knots, four are unknotting number 1 and amphichieral, starting with the figure eight knot, 4_1. Thus, for these the answer is "yes." The answer is also "yes" for 8_{13}; we leave it to the reader to find crossing changes of each sign that unknot 8_{13}. The remaining set of eight knots is $\{6_1, 7_7, 8_1, 8_3, 8_8, 8_{12}, 8_{18}, 8_{20}\}$. In addition to 7_7 that was described in this section, other tools developed in this paper will rule out 8_8 and 8_{18}. This leaves five unknown cases: $\{6_1, 8_1, 8_3, 8_{12}, 8_{20}\}$.

In Appendix C we have a more extended discussion of low-crossing number knots and Table 1 presents a summary for all prime knots of eight an fewer crossings, considering all linking numbers and not just $l = 0$.

5. Single twist unknotting: Casson–Gordon invariants and signatures

We begin by reviewing Casson–Gordon invariants, restricting to the generality needed for our applications. Suppose that M^3 is a closed oriented three-manifold, $l > 0$, and $\phi\colon H_1(M) \to \mathbb{Z}_l$ is a surjective homomorphism. Suppose further that ϕ extends to a map $\overline{\phi}\colon H_1(W \setminus F) \to \mathbb{Z}_l$, where W is an oriented four-manifold with $\partial W = M$ and F is an embedded, possibly empty, surface. Then, for $0 < r < l$, we have the definition

$$\sigma_r(M, \phi) = \operatorname{sign}(W) - \epsilon_r(\widetilde{W}) - \frac{2[F]^2 r(l-r)}{l^2}. \tag{1}$$

Here $\operatorname{sign}(W)$ is the signature of W and $\widetilde{W}$ is the l-fold cyclic branched cover of W associated to $\overline{\phi}$. The integer $\epsilon_r(\widetilde{W})$ is the signature of the intersection form on $H_2(\widetilde{W}, \mathbb{C})$ restricted to the $\omega_l = e^{2\pi i r/l}$-eigenspace of the action of the generator of the group of deck transformations acting on $H_2(\widetilde{W}, \mathbb{C})$. Lastly, $[F]^2$ is the self-intersection number of F. (The fact that $\sigma_r(M, \phi)$ is a well-defined invariant of the triple (M, ϕ, r) is one of the accomplishments of [CGl]. There it is only required that there is a four-manifold and homomorphism pair $(W, \overline{\phi})$ such that $\partial(W, \overline{\phi}) = n(M, \phi)$ for some $n > 0$. In all our work, such a pair exists for $n = 1$, so we are restricting to that setting.)

The result [CGl, Lemma 3.1] can be applied to the case of $S^3_m(K)$ with ϕ the quotient map to $\mathbb{Z}_l$ for a divisor l of m, in which case it states:

$$\sigma_r\big(S^3_m(K), \phi\big) = \operatorname{sign}(m) - \operatorname{sign}\big((1 - \omega_l^{-r})V + (1 - \omega_l^{r})V^t\big) - \frac{2mr(l-r)}{l^2}. \tag{2}$$

Here $\operatorname{sign}(A)$ denotes the signature of a complex hermitian matrix A; if A is one-dimensional, that is, if $A = m$ for some real number, $\operatorname{sign}(m)$ is simply the sign of m. The matrix V is a Seifert matrix for K. In standard notation, the signature of the hermitianized Seifert form is called the *Tristram–Levine* r/l-signature of K, denoted $\sigma_{r/l}(K)$.

The next theorem and its proof are closely related to results of Aceto–Golla–Lecuona in [AGL]. The proofs there are more complicated that ours, necessarily so because they are working with four-manifolds with handlebody decompositions that might have three-handles. In our special case the results are slightly stronger in that we are not restricted to the case that r and l are relatively prime.

Theorem 5.1. *Suppose that $m \neq 0$ and $S^3_m(K)$ bounds a four-manifold W built from $S^1 \times B^3$ by adding a two-handle along a curve representing $l \in H_1(S^1 \times S^2)$. Then $m = \pm l^2$ for some $l > 0$ and for all r, $0 < r < l$,*

$$\left|\sigma_{r/l}(K) - \left(s - 2sr(l-r)\right)\right| \leq 1,$$

where $s = \frac{m}{l^2} = \pm 1$.

Proof. First observe that the handle decomposition of W yields a surgery description of $S^3_m(K)$ as $S^3_{0,a}(J_1, J_2)$ for some link (J_1, J_2) and some integer a. The surgery matrix is

$$\begin{pmatrix} 0 & l \\ l & a \end{pmatrix}.$$

In our situation, a will be seen to be ± 1, but for now we simply observe that since $H_1(S^3_m(K))$ is cyclic, $\gcd(a, l) = 1$, $l^2 = \pm m$, and the map induced by inclusion $H_1(S^3_m(K)) \to H_1(W)$ corresponds to the quotient map $\mathbb{Z}_{|m|} \to \mathbb{Z}_l$.

The manifold W can be used to compute $\sigma_r(S_m(K), \phi)$, where ϕ is the quotient map $\phi\colon \mathbb{Z}_{|m|} \to \mathbb{Z}_l$. There is no branching surface. Observe that W is a rational homology ball and so has signature 0.

We will explain in the final paragraph of this proof that the ω_l^r-eigenspace in $H_2(\widetilde{W}, \mathbb{C})$ is 1–dimensional if r is not divisible by l. Thus, $|\epsilon_r(\widetilde{W})| \leq 1$. From the definition of σ_r we see that for each value of r,

$$\left|\sigma_r(S^3_m(K), \phi)\right| \leq 1.$$

Equation (2) then can be written as

$$\left|\operatorname{sign}(m) - \sigma_{r/l}(K) - 2sr(l-r)\right| \leq 1,$$

as desired.

It remains to show that the ω_l^r-eigenspace in $H_2(\widetilde{W}, \mathbb{C})$ is 1-dimensional if $0 \leq r \leq l-1$. The space W has the homotopy type of a CWcomplex built with a single 0-cell, 1-cell, and 2-cell: call them $d, e,$ and f. Thus the l-fold cover $\widetilde{W}$ has the homology of a chain complex with l cells in each of these dimensions:

call them $\{d_i\}, \{e_i\}$, and $\{f_i\}$, for $0 \leq i \leq l-1$. In each dimension there is a decomposition of the chain complex into ω_l^r-eigenspaces for $0 \leq r \leq l-1$; in the case of dimension 2, the ω_l^r-eigenspace contains $\sum_{i=0}^{l-1} \omega_l^{-ri} f_i$. A dimension count shows that each of these is in fact a generator of the eigenspace. The fact that if $r \neq 0$ this chain is a cycle follows from the facts that $\partial f_i = \sum d_i$ and $\sum_{i=0}^{l-1} \omega_l^{ri} = 0$ if $r \neq 0$. $\square$

The following corollary is similar to results proved in [MY] and a related result in [ANY], which was presented in the case of torus knots.

Corollary 5.2. *If K can be unknotted with a single twist of linking number $l > 0$, then for all r, $0 < r < l$, and for s either 1 or -1*

$$\left|\sigma_{r/l}(K) + s - 2sr(l-r)\right| \leq 1,$$

where $s = -1$ or $s = 1$, depending on whether the twist is left-handed or right.

Proof. Except for the sign of s, this is an immediate consequence of Theorem 5.1. Suppose that K can be unknotted with a negative twist. In this case, the three-manifold of interest is $S^3_{l^2}(K)$ and in Equation (2), the term $\frac{m}{l^2} = 1$. Similarly for the right-handed twist. $\square$

The next result is similar, only we consider the case of $S^3_0(K)$.

Theorem 5.3. *Suppose that $S^3_0(K)$ bounds a four-manifold W built from $S^1 \times B^3$ by adding a two-handle along a curve representing $0 \in H_1(S^1 \times S^2)$. Then W is a definite manifold. For all $l > 0$, and all r, $0 < r \leq l$,*

$$\left|\sigma_{r/l}(K) + s\right| \leq 1,$$

where $s = 1$ if W is positive definite and $s = -1$ if W is negative definite.

Proof. In this case, the handle decomposition of W yields a surgery description of $S^3_0(K)$ as $S^3_{0,s}(J_1, J_2)$ for some link (J_1, J_2) and some integer s. The surgery matrix is

$$\begin{pmatrix} 0 & 0 \\ 0 & s \end{pmatrix}.$$

Because $H_1(S^3_0(K))$ is cyclic, we have that $s = \pm 1$.

Let $\phi\colon H_1(S^3_0(K)) \to \mathbb{Z}_l$ be a surjection. Then the manifold W can be used to compute $\sigma_r(S_0(K), \phi)$. There is no branching surface. In this case, W has signature $\operatorname{sign}(s) = \pm 1$, depending on whether it is positive or negative definite. Also, $\widetilde{W}$ is built from $S^1 \times B^3$ by adding l two-handles; it follows that each eigenspace is 1-dimensional and $|\epsilon_r(\widetilde{W})| \leq 1$. (The computation of the dimension of the eigenspace follows similarly as in the proof of Theorem 5.1.) Thus, the definition of the σ_r yields

$$\left|\sigma_r(S_0^3(K), \phi) - s\right| \leq 1.$$

Equation (2) then implies that

$$\left|\operatorname{sign}(m) - \sigma_{r/l}(K) - \frac{2 \cdot 0 \cdot r(l-r)}{l^2} - s\right| \leq 1.$$

Here $m = 0$, so this can be rewritten as

$$\left|\sigma_{r/l}(K) + s\right| \leq 1,$$

as desired. □

Corollary 5.4. *If K can be unknotted with a single twist of linking number $l = 0$, then for all $q > 0$ and all r,*

$$\left|\sigma_{r/q}(K) + s\right| \leq 1,$$

where $s = -1$ if it is a left-handed twist and $s = 1$ if it is a right-handed twist.

Corollary 5.5. *The positive torus knot $K = T(p,q)$ cannot be unknotted with a positive twist of linking number greater than 1.*

Proof. The signature function satisfies $\sigma_{r/l}(K) \leq -2$ for all $r/l > 1/pq$. (A proof is left to the appendix, Theorem A.4.) If K could be unknotted with a positive twist of some linking number $l \geq 2$, then the terms on the left in Corollary 5.2 would have absolute value at least three. □

Corollary 5.6. *Suppose a knot K can be unknotted with twists of linking numbers $l_1, l_2 \geq 2$ and signs s_1 and s_2, respectively. Then one of the following holds:*

(1) $\gcd(l_1, l_2) = 1$,

(2) $l_1 = l_2 = 2$ *with* $s_1 \neq s_2$, *or*

(3) $l_1 = l_2$ *and* $s_1 = s_2$.

Proof. Let $l_1, l_2 \geq 2$ and $\gcd(l_1, l_2) = n \neq 1$. Then $\frac{1}{n} = \frac{l_1/n}{l_1} = \frac{l_2/n}{l_2}$. We let $r_i = l_i/n \in \mathbb{Z}$ for $i = 1, 2$. Suppose that K can be unknotted with twists of sign s_i and linking number l_i for $i \in \{1, 2\}$. Applying Corollary 5.2, we see that, on the one hand,

$$\sigma_{1/n}(K) = -s_1 + 2s_1 r_1 (l_1 - r_1) + \eta_1 = \left(-1 + 2{l_1}^2\left(\frac{n-1}{n^2}\right)\right) s_1 + \eta_1,$$

for some $\eta_1 \in \{0, 1, -1\}$. On the other hand,

$$\sigma_{1/n}(K) = -s_2 + 2s_2r_2\,(l_2 - r_2) + \eta_2 = \left(-1 + 2\,l_2{}^2\left(\frac{n-1}{n^2}\right)\right)s_2 + \eta_2,$$

for some $\eta_2 \in \{0, 1, -1\}$. Note that if $s_1 \neq s_2$, these two equations imply that $\sigma_{1/n}(K)$ is both nonnegative and nonpositive. In this case, $\sigma_{1/n}(K) = 0$ and thus $l_1 = l_2 = 2$. Now assume that $s_1 = s_2$. From the formulas for $\sigma_{1/n}(K)$ given above, we have

$$s_1\left(-1 + 2l_1{}^2\frac{n-1}{n^2}\right) - s_1\left(-1 + 2l_2{}^2\frac{n-1}{n^2}\right) = \eta_2 - \eta_1.$$

Simplifying,

$$(l_2{}^2 - l_1{}^2)\frac{n-1}{n^2} = \frac{\eta_2 - \eta_1}{2s_1} \in \left\{0, \pm\frac{1}{2}, \pm 1\right\}.$$

Multiplying by $\frac{n^2}{n-1}$, we see that

$$l_2{}^2 - l_1{}^2 \in \left\{0, \pm\frac{n^2}{2(n-1)}, \pm\frac{n^2}{n-1}\right\}.$$

Note that $\frac{n^2}{n-1}$ is an integer only when $n = 2$ and $\frac{n^2}{n-1} = 4$. It is easily checked that neither 2 nor 4 is a difference of two squares. Therefore, we have that that $l_2{}^2 - l_1{}^2 = 0$ and $l_1 = l_2$. □

Example 5.7. The unknot U has $\mathcal{U}(U) = \{2^-, 1^-, 0^-, 0^+, 1^+, 2^+\}$. There are no known example of knots K for which $\{k^-, (k+1)^+\} \subset \mathcal{U}(K)$ and $k > 1$. If such an example exists, then Corollary 5.2 implies that the signature function alternates between positive and negative entries at k-roots of unity and $(k+1)$-roots of unity. This implies that the Alexander polynomial has multiple zeroes between these unit roots, and thus we get a bound on the degree of the Alexander polynomial. This in turn provides a lower bound on the genus of the knot. Since the result in Corollary 5.2 is stated in terms of a quadratic function, the calculations are not difficult, and results such as the following appear: If $\{3^-, 4^+\} \in \mathcal{U}(K)$ then $g(K) \geq 9$, and if $\{4^-, 5^+\} \in \mathcal{U}(K)$ then $g(K) \geq 23$.

In general, the bound on $g(K)$ is determined by summing quadratic polynomials and is thus given by a cubic equation. Having observed this, that cubic can be found explicitly by interpolating the first four values. We get the following result.

Corollary 5.8. *If $k \geq 1$ and $\{k^-, (k+1)^+\} \subset \mathcal{U}(K)$, then*

$$g(K) \geq \frac{2k^3 + 3k^2 - 11k + 6}{6}.$$

We note that if one considers pairs such as $\{k^-, (k+2)^+\}$ the computation becomes unmanageable; in this case the k-roots of unity and $(k+2)$-roots of unity do not alternate around the unit circle.

6. Single twist unknotting: Arf invariant

If M is a closed three-manifold and $H_1(M,\mathbb{Z}_2)=0$, then the Rochlin invariant $\mu(M)\in\mathbb{Z}_{16}$ is defined as follows. There exists a parallelizable four-manifold W with $\partial W = M$ and $H_1(W,\mathbb{Z}_2)=0$; $\mu(M)$ is defined to be the signature of the intersection form of $-W$, reduced modulo 16.

For knots $K\subset S^3$ there is an Arf invariant, $\mathrm{Arf}(K)\in\mathbb{Z}_2$, which can be defined as follows. If K has determinant $\det(K)$, then $\mathrm{Arf}(K)=0$ when $\det(K)\equiv\pm 1 \mod 8$ and $\mathrm{Arf}(K)=1$ when $\det(K)\equiv\pm 3\mod 8$. This is often stated in terms of the Alexander polynomial, using the fact that $\det(K)=\left|\Delta_K(-1)\right|$.

Background for these invariants is included in [GAn, Rob] and especially [Gor, Theorem 2], which, in the current setting, implies

$$\mu\big(S^3_n(K)\big)\equiv\mu\big(S^3_n(U)\big)+8\mathrm{Arf}(K)\in\mathbb{Z}_{16}, \tag{3}$$

for n odd. (Note that in [Gor] the invariants take value in $\mathbb{Q}/\mathbb{Z}$, in which $\mathbb{Z}/16$ embeds.)

Theorem 6.1. *If K can be unknotted with a single twist of linking number l with l odd, then:*

$$l\equiv\begin{cases}\pm 1 \mod 8 & \textit{if } \mathrm{Arf}(K)=0\in\mathbb{Z}_2,\\ \pm 3 \mod 8 & \textit{if } \mathrm{Arf}(K)=1\in\mathbb{Z}_2.\end{cases}$$

Proof. If l is odd and K can be unknotted with a twist of linking number l and sign s, then $S^3_{-sl^2}(K)$ is a $\mathbb{Z}_2$-homology sphere that bounds a $\mathbb{Z}_2$-homology B^4. Thus, $\mu(S^3_{-sl^2}(K))=0$. From Equation 3

$$\mu\big(S^3_{-sl^2}(U)\big)\equiv -8\mathrm{Arf}(K)\mod 16. \tag{4}$$

Notice that at this point, all the terms are either 0 or 8 modulo 16, so we can disregard signs. Furthermore, since $\mu(M)=-\mu(-M)$, the sign of the surgery coefficient on the unknot is not relevant. Thus, at this point, we need to determine the μ-invariant of the space $S^3_n(U)$ for $n=l^2$. Notice that as unoriented manifolds, this is a lens space: $S^3_n(U)\cong L(n,1)$.

There is a homeomorphism $L(n,1)\cong L(n,n-1)$ and $L(n,n-1)$ is constructed as $n/(n-1)$ rational surgery on the unknot. As described, for instance, in [Rol,

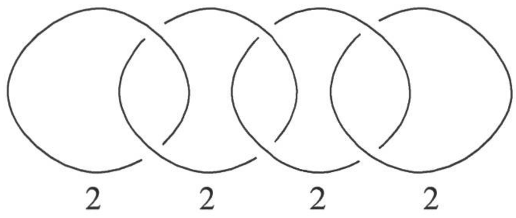

FIGURE 3
Surgery diagram for $L(5,1) \cong L(5,4)$

Section 9H], this space can be described as integer surgery on a link, where the surgery coefficients are determined by a continued fraction expansion of $n/(n-1)$. For example, we have

$$\frac{5}{4} = 2 - \cfrac{1}{2 - \cfrac{1}{2 - \cfrac{1}{2}}},$$

and thus, up to orientation, $L(5,1) \cong L(5,4)$ has a surgery diagram as shown in Figure 3.

In general, for $L(n, n-1)$ there is a chain of $n-1$ components, all of framing 2. Such a surgery diagram provides an even (and thus parallelizable) four-manifold bounded by $L(n, n-1)$. An easy argument shows the surgery matrix is positive definite and thus has signature $n-1$. Applying this to the case of $n = l^2$, Equation 4 becomes

$$8\mathrm{Arf}(K) \equiv l^2 - 1 \mod 16.$$

The value of $l^2 - 1$ modulo 16 depends only on $l \mod 8$: for odd l, $l^2 - 1 \equiv 0 \mod 16$ if and only if $l \equiv \pm 1 \mod 8$ and $l^2 - 1 \equiv 8 \mod 16$ if and only if $l \equiv \pm 3 \mod 8$. □

Corollary 6.2. *Let K be the torus knot $T(p,q)$. If p and q are odd and K can be unknotted with single twist of odd linking number l, then $l \equiv \pm 1 \mod 8$. If the torus knot $T(2p,q)$ can be unknotted with a single twist of odd linking number l, then $l \equiv \pm q \mod 8$.*

Proof. The Alexander polynomial of the torus knot is given by

$$\Delta_{T(p,q)}(t) = \frac{(t^{pq} - 1)(t-1)}{(t^p - 1)(t^q - 1)}.$$

In the case that p and q are both odd, the evaluation at $t = -1$ is immediately seen to be 1. If p is even, then evaluating $(t^{pq} - 1)/(t^p - 1)$ at $t = -1$ can be accomplished, for instance, using L'Hospital's rule and is seen to equal q. □

Example 6.3. The torus knot $T(2k, 2k \pm 1)$ can be unknotted with a single twist of linking number $2k \pm 1$.

7. Summary of Heegaard Floer theory

Heegaard Floer theory associates to each knot $K \subset S^3$ a chain complex $\mathrm{CFK}^\infty(K)$ and to each three-manifold Y, a collection of chain complexes $\mathrm{CF}^\infty(Y, \mathfrak{s})$ (see [OS2]). Here $\mathfrak{s} \in \mathrm{Spin}^c(Y)$, the set of Spin^c-structures on Y. We will leave the definition of $\mathrm{Spin}^c(Y)$ to the references; the key fact that we will be using is that in general there is a correspondence between $\mathrm{Spin}^c(Y)$ and $H^2(Y) \cong H_1(Y)$ and in the case of $Y = S^3_m(K)$, there is a natural choice for that correspondence. In particular, invariants associated to a given Spin^c-structure, such as $d(S^3_m(K), \mathfrak{s})$, can be written as $d(S^3_m(K), i)$, where $i \in \mathbb{Z}$ satisfies $(-|m| + 1)/2 \leq i \leq |m|/2$ and thus uniquely represents an element in $\mathbb{Z}_{|m|}$. In this section, we will summarize some of the invariants and their properties.

7.1. Heegaard Floer knot invariants $V_k(K)$. These are integer-valued invariants defined for $k \geq 0$, defined in [NW]. (We provide further discussion in Appendix B.) They satisfy the following properties.

- $V_k(K) \geq V_{k+1}(K) \geq V_k(K) - 1$ for all $k \geq 0$.
- $V_k(K) = 0$ for all $k \geq g(K)$.

In general, these are difficult to compute. There are two cases in which they are accessible.

Example 7.1 (Alternating knots). The Heegaard Floer complex for an alternating knot is determined entirely by the knot's signature, as follows. If K is alternating and $\sigma(K) \geq 0$, then $V_k(K) = 0$ for all $k \geq 0$. If $\sigma(K) < 0$, then $V_k(K) = \max\{\lfloor \frac{-\sigma(K)+2(1-k)}{4} \rfloor, 0\}$ for $k \geq 0$.

Example 7.2 (Torus knots). The $V_k(T(p,q))$ are determined by the Alexander polynomial. See, for example, [BL].

7.2. Heegaard Floer knot invariants $\nu^+(K)$. This invariant has a simple definition in terms of the $V_k(K)$:

$$\nu^+(K) = \min\{n \mid V_n(K) = 0\}.$$

We have $g(K) \geq \nu^+(K)$ for all knots K.

7.3. The Upsilon invariant $\Upsilon_K(t)$. The Upsilon function $\Upsilon_K(t)$ is a piecewise linear function defined for $0 \le t \le 2$. Some of its key properties are the following.

- For all $t \in [0,2]$ and for all knots K and J, $\Upsilon_{K\#J}(t) = \Upsilon_K(t) + \Upsilon_J(t)$.
- $\Upsilon_{-K}(t) = -\Upsilon_K(t)$.
- For all nonsingular points t, the derivative satisfies $|\Upsilon_K'(t)| \le g(K)$.

In general, for a particular knot K, the invariants $V_k(K)$ offer stronger constraints than does Υ_K. However, the additivity of Υ_K makes it computable in cases in which computing the V_k might be difficult. The proof of the following theorem is left to Appendix B since it calls on some details of Heegaard Floer theory that are not presented in the body of this paper.

Proposition 7.3. *Let K be a knot and $g = g(K)$ be the genus of K. Then for $t \in [0,2]$ and $s \ge 0$,*

$$-st - 2V_s(K) \le \Upsilon_t(K) \le \begin{cases} -gt - 2V_s(K) - 2s + 2g + 2 & t \le 1 - \frac{s}{g}, \\ gt - 2V_s(K) + 2 & t \ge 1 - \frac{s}{g}. \end{cases}$$

7.4. The Heegaard Floer correction term, $d(Y, \mathfrak{s})$. Heegaard Floer theory associates to each three-manifold Y with Spin^c-structure $\mathfrak{s}$, a rational invariant denoted $d(Y, \mathfrak{s})$, first defined in [OS1]. We will need these invariants in the case that Y is surgery on a knot. In this case, an immediate consequence of [NW, Remark 2.10] is the following result, showing that the relevant d-invariants of surgery on K are determined by the invariants $V_i(K)$.

Theorem 7.4. *For $n > 0$ and $0 \le i \le n/2$,*

$$d\big(S^3_n(K), i\big) = \frac{(2i-n)^2 - n}{4n} - 2V_i(K).$$

The main theorems from [OS1] that we will use concerning the d-invariants are as follows.

Theorem 7.5. *Suppose that $|H_1(Y)| = m$ and a given Spin^c-structure $\mathfrak{s}$ on Y extends to a rational homology ball W having boundary Y. Then $d(Y, \mathfrak{s}) = 0$.*

There are some subtleties about determining which Spin^c-structures extend, but if we appropriately choose identifications of Spin^c with $H_1(Y)$ or $H^2(W)$, the key results are easily summarized. In the background we have that a Spin^c-structure on Y extends to W if and only if the corresponding element in $H^2(Y)$ is in the image of the restriction map from $H^2(W)$.

Theorem 7.6. *Suppose that* $Y = \partial W$, *where* $H_*(W, \mathbb{Q}) \cong H_*(B^4, \mathbb{Q})$. *Then* $|H_1(Y)| = l^2$ *for some integer* $l > 0$ *and there exists a coset* H *of an index* l *subgroup of* $H_1(Y)$ *such that* $d(Y, i) = 0$ *for all* $i \in H$.

Corollary 7.7. *If* $S^3_n(K)$ *bounds a rational homology ball, then* $n = \pm l^2$ *for some* l. *If* l *is odd, then* $d(S^3_n(K), kl) = 0$ *for* $0 \le k \le l - 1$. *If* l *is even, then* $d(S^3_n(K), (k + \frac{1}{2})l) = 0$ *for* $0 \le k \le l - 1$.

Proof. A duality argument shows that if a rational homology three-sphere M bounds a rational homology four-ball X, then $|H_1(M)| = |\ker(H_1(M) \to H_1(X))|^2$; see, for instance, [CG2]. Thus, we write $n = l^2$. It follows from Theorem 7.5 that for l values of i, $d(S^3_n(K), i) = 0$. It is now an exercise in arithmetic, using Theorem 7.4, to show that the only integer values occur at $i = kl$ for l odd and at $i = (k + \frac{1}{2})l$ for l even. □

8. Single twist unknotting: Heegaard Floer obstructions

In [BL], the Heegaard Floer d-invariants of three-manifolds manifolds of the form $S^3_{k^2}(K)$ are studied in the case of algebraic knots. Aceto and Golla [AG] expanded on this, undertaking an extensive study of the question of, for a given knot K, which of the manifolds $S^3_{p/q}(K)$ bound rational balls. Many of the results of this section are built from special cases of what appears there: for instance, their theorem that if $S^3_{l^2}(K)$ and $S^3_{m^2}(K)$ both bound rational homology balls, then l and m are consecutive. Our Theorem 8.2 follows immediately, showing that if $0 < l < m$ and $\{l^-, m^-\} \subset \mathcal{U}(K)$, then $l - m = 1$. We will include proofs of the results we need for two reasons: in our setting the arguments are fairly straightforward and accessible, and the arguments provide access to stronger results in the case of the unknotting problem.

We begin with the following, which follows readily from [BL] and is stated explicitly in the context of unknotting twists by Sato [Sat].

Theorem 8.1. *Suppose that* K *can be unknotted with a negative twist of linking number* $l > 0$.

- *If* $l = 2\alpha + 1$, *then for all* $0 \le k \le \alpha$,

$$V_{kl}(K) = (\alpha - k)(\alpha - k + 1)/2.$$

- *If* $l = 2\beta + 2$, *then for all* $0 \le k \le \beta$,

$$V_{(k+\frac{1}{2})l}(K) = (\beta - k)(\beta - k + 1)/2.$$

Proof. Suppose that K can be unknotted with a negative twist of linking number $l > 0$. Then $S^3_{l^2}(K)$ bounds a rational homology ball and we can apply Corollary 7.7. If l is odd, then $l = 2\alpha + 1$ for some $\alpha \geq 0$. Theorems 7.4 and 7.7 imply that for $0 \leq k \leq \alpha$,

$$\begin{aligned} V_{kl}(K) &= \frac{(2kl - l^2)^2 - l^2}{8l^2} \\ &= \frac{(2k - 2\alpha - 1)^2 - 1}{8} \\ &= \frac{(\alpha - k)(\alpha - k + 1)}{2}, \end{aligned}$$

as desired.

Similarly, if l is even, then $l = 2\beta + 2$ for some $\beta \geq 0$. For $0 \leq k \leq \beta$, we have

$$\begin{aligned} V_{(k+\frac{1}{2})l}(K) &= \frac{(2\left(k + \frac{1}{2}\right)l - l^2)^2 - l^2}{8l^2} \\ &= \frac{(2\left(k + \frac{1}{2}\right) - 2\beta - 2)^2 - 1}{8} \\ &= \frac{(2k - 2\beta - 1)^2 - 1}{8} \\ &= \frac{(\beta - k)(\beta - k + 1)}{2}, \end{aligned}$$

as desired. □

This theorem places unexpectedly strong constraints on the possible values of l. Recall $\nu^+ = \nu^+(K) = \min\{n \mid V_n(K) = 0\}$.

Theorem 8.2. *For a knot K, there are at most two positive values of l for which K can be unknotted by a negative twist of linking number l. If K can be unknotted using negative twists of two different linking numbers, then $\nu^+(K) = \gamma(\gamma+1)/2$ for some γ and the two values of l are $l_1 = (1 + \sqrt{1 + 8\nu^+(K)})/2)$ and $l_2 = l_1 + 1$. If $\nu^+(K)$ is not of this form, there is at most one possible value for l, and it is given by the ceiling, $\lceil(1 + \sqrt{1 + 8\nu^+(K)})/2)\rceil$.*

Proof. In order to simplify the notation in this proof, we are abbreviating $V_i(K)$ by V_i and similarly are writing $\nu^+(K)$ as ν^+.

For odd l, if we let $k = \alpha - 1$ we see that $V_{(\alpha-1)(2\alpha+1)} = 1$. Letting $k = \alpha$, we have $V_{\alpha(2\alpha+1)} = 0$. Thus, for l odd,

$$(\alpha - 1)(2\alpha + 1) < \nu^+ \leq \alpha(2\alpha + 1).$$

For even l, if we let $k = \beta - 1$ we see that $V_{(\beta-\frac{1}{2})(2\beta+2)} = 1$. If we let $k = \beta$ we see that $V_{(\beta+\frac{1}{2})(2\beta+2)} = 0$. Thus, we arrive at the inequalities

$$\left(\beta - \frac{1}{2}\right)(2\beta + 2) < \nu^+ \leq \left(\beta + \frac{1}{2}\right)(2\beta + 2).$$

In either case, these are quadratic in α or β and the bounds on each are determined using the quadratic formula. Expressing either in terms of l (and recalling that $\alpha, \beta \geq 0$) yields the same inequality:

$$\frac{1 + \sqrt{1 + 8\nu^+}}{2} \leq l < \frac{3 + \sqrt{9 + 8\nu^+}}{2}.$$

For $\nu^+ > 0$, the difference of these bounds is strictly between 1 and 2. If $\nu^+ = 0$, then the difference of these bounds is exactly 2. In either case, the interval can contain at most two integers. The left endpoint is an integer exactly when $\nu^+ = \gamma(\gamma + 1)/2$ for some integer γ. In this case the interval contains two integers. If ν^+ is reduced by 1, then the right endpoint becomes an integer. In this case, since the right endpoint is not included in the interval, there is only one integer in the interval. □

Example 8.3. Let $K = T(7, 8)$. Then K can be unknotted with a negative twist of linking number 7, and we have $V_0(K) = 6, V_7(K) = 3, V_{14}(K) = 1$, and $V_{21}(K) = 0$.

We also have that K can be unknotted with a negative twist of linking number 8, and $V_4(K) = 6$, $V_{12}(K) = 3$, $V_{20}(K) = 1$, and $V_{28}(K) = 0$.

Corollary 8.4. *For any knot K, there are at most three values of l such that K can be unknotted with a single positive twist of linking number l. Similarly, there are at most three values of l such that K can be unknotted with a single negative twist of linking number l.*

Proof. There are two possible positive linking numbers for negative twists. Considering mirror images, we see there are at most two possible positive linking numbers for negative twists. Finally, there is the possibility of unknotting with a linking number 0 twist. □

Thus, for a given knot K, $|\mathcal{U}(K)| \leq 6$. This combined with Corollary 5.6 implies the following result.

Corollary 8.5. *If $|\mathcal{U}(K)| = 6$, then $\mathcal{U}(K) = \{2^-, 1^-, 0^-, 0^+, 1^+, 2^+\}$.*

Example 8.6. The unknot realizes the unknotting set $\{2^-, 1^-, 0^-, 0^+, 1^+, 2^+\}$. The smallest nontrivial example we know of which realizes this unknotting set is the knot 8_9. Figure 4 presents two diagrams of 8_9. In the first, drawn instead as a long knot in $\mathbb{R}^3$, illustrates that it is amphicheiral (consider the rotation around the origin) and thus we can restrict to positive twists. A crossing change at the point marked with a square is an unknotting operation, so this provides $\{2^-, 0^-, 0^+, 2^+\} \subset \mathcal{U}(K)$. The second diagram provides a linking number 1 twist that unknots 8_9. The knot 8_9 will reappear in Section 13.

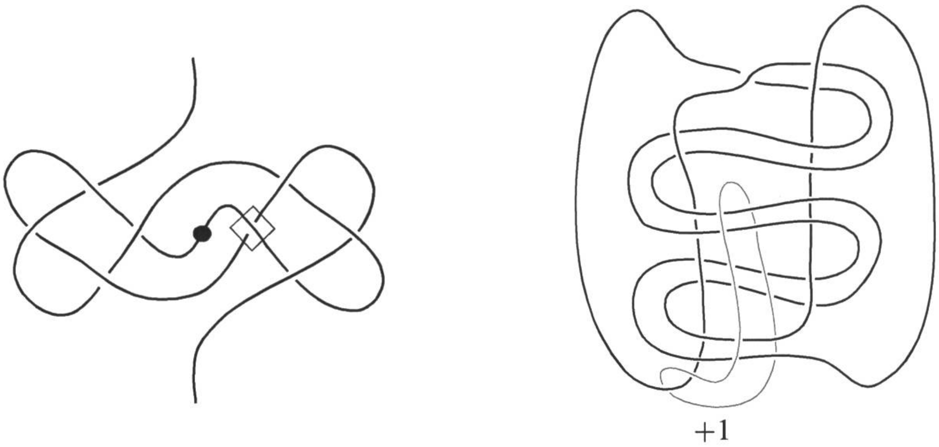

FIGURE 4
On the left, the knot 8_9 drawn as a long knot in $\mathbb{R}^3$. On the right, a ribbon diagram of 8_9 which can be unknotted with a positive twist on the strands captured by the linking circle.

Addenda. Mohamed Aït Nouh has discovered an alternative proof of the portion of Theorem 8.2 that states that if K can be unknotted with two positive (or negative) twists of linking numbers $l_2 > l_1 > 0$, then $l_2 = l_1 + 1$. This alternative is based on Donaldson's work in [Don]; as far as we can determine, it does not yield estimates of the possible values of l_1. Here is an outline of the argument. The knot $K\# -K$ is slice, so it bounds a disk $D_1 \subset B^4$. Appropriately adding two two-handles to B^4 with framings $+1$ and -1 constructs a manifold $W \cong (\mathbb{CP}^2 \# -\mathbb{CP}^2) \setminus B^4$, with $K\# -K$ becoming unknotted in the boundary, and thus bounding a disk $D_2 \subset B^4$. In the union $W \cup B^4 \cong \mathbb{CP}^2 \# -\mathbb{CP}^2$, the union $D_1 \cup D_2$ forms a smoothly embedded 2-sphere representing the class $(l_1, l_2) \in H_2(\mathbb{CP}^2 \# -\mathbb{CP}^2)$. Luo [Luo] applied Donaldson's results to prove that this in turn implies the l_1 and l_2 are consecutive.

9. Further Heegaard Floer obstructions

Standard modifications of surgery diagrams of three-manifolds, as described, for instance, in [Rol, Chapter 9H], permit us to find two contrasting surgery descriptions of a specific manifold built from a knot K that can be unknotted with a single negative twist of linking number l. That is, $Y = S^3_{l^2+1}(K)$ has a surgery description as surgery on a link (U, K^*), where -1-surgery is performed on U which is unknotted and has linking number l with K^* and 1-surgery is performed on K^*, which is also unknotted. (This operation is sometimes referred to as *blowing-up* a -1.)

We can modify the surgery description of Y again by *blowing-down* the $+1$-framed K^*: as a consequence, Y can be described by $(-l^2-1)$-surgery on a second knot, which we denote by J. The following lemma is proved by keeping track of the meridian of K as the blow-up and blow-down are performed.

Lemma 9.1. *If K can be unknotted with a single negative twist of linking number l, then there is a knot J and an orientation-preserving homeomorphism from $S^3_{l^2+1}(K)$ to $S^3_{-l^2-1}(J)$. On homology, this homeomorphism carries the first homology class represented by the meridian of K to l times the first homology class represented by the meridian of J.*

To compute d-invariants, we will want to reduce integers modulo l^2+1 appropriately.

Definition 9.2. For $a, n \in \mathbb{Z}$ with $n > 1$, we define a_n to be the least nonnegative number for which $a - a_n$ is divisible by n. We define

$$[a]_n = \left| \left(a + \frac{n-1}{2} \right)_n - \frac{n-1}{2} \right|.$$

We leave it to the reader to check that this can be thought of as a *cyclic distance* in the following sense. If the metric on the unit circle in $\mathbb{C}$ is scaled so that the circumference is n, then $[a]_n$ is the geodesic distance from 1 to $e^{a2\pi/n}$.

Example 9.3.

- $[0]_4 = 0$ $[1]_4 = 1$ $[2]_4 = 2$ $[3]_4 = 1$.
- $[0]_5 = 0$ $[1]_5 = 1$ $[2]_5 = 2$ $[3]_5 = 2$ $[4]_5 = 1$.

Theorem 9.4. *If K can be unknotted with a single negative twist of linking number l, then there exists a knot J such that*

$$d\big(S^3_{l^2+1}(K), i\big) = d\big(S^3_{-l^2-1}(J), [li + \beta]_{l^2+1}\big)$$

for all integers i *satisfying* $0 \leq i < \frac{l^2+1}{2}$. *Here* $\beta = 0$ *if* l *is even and* $\beta = \frac{l^2+1}{2}$ *if* l *is odd.*

Proof. The only issue that requires proof is the term β that appears. The issue arises because of how the Spin^c-structures are parameterized with integers. In the case that $n = l^2 + 1$ is odd there is a unique Spin-structure on $S^3_{-l^2-1}(K)$ and this determines which Spin^c-structure is denoted $\mathfrak{s}_0$. However, if n is even, there are two Spin structures, one of which corresponds to $\mathfrak{s}_0$ and the other to $\mathfrak{s}_k$, where $k = \frac{l^2+1}{2}$. There is a simple means to rule out one of the possibilities: if $n = l^2 + 1$ is even, and $\beta = 0$, then

$$d\big(S^3_{l^2+1}(K), i\big) \not\equiv d\big(S^3_{-l^2-1}(J), [li + \beta]_{l^2+1}\big) \mod \mathbb{Z}. \qquad \square$$

It is simpler to have both surgery coefficients positive, so we consider the mirror image of J and use the symmetry of the d-invariants under conjugation to conclude the following.

Theorem 9.5. *If* K *can be unknotted with a single negative twist of linking number* l, *then there exists a knot* J' *such that*

$$d\big(S^3_{l^2+1}(K), i\big) = -d\big(S^3_{l^2+1}(J'), [li + \beta]_{l^2+1}\big)$$

for all integers i *satisfying* $0 \leq i < \frac{l^2+1}{2}$. *Here* $\beta = 0$ *if* l *is even and* $\beta = \frac{l^2+1}{2}$ *if* l *is odd.*

We can now apply Theorem 7.4 to get the following result.

Theorem 9.6. *Suppose that* K *can be unknotted with a single negative twist of linking number* l *and* $n = l^2 + 1$. *Then there exists a knot* J' *such that for all* i *satisfying* $0 \leq i \leq n/2$,

$$\frac{(2i-n)^2 - n}{4n} - 2V_i(K) = -\frac{(2[li+\beta]_n - n)^2 - n}{4n} + 2V_{[li+\beta]_n}(J').$$

Here $\beta = 0$ *if* l *is even and* $\beta = n/2$ *if* l *is odd.*

Rearranging the terms of this expression, we have:

Corollary 9.7. *Suppose that* K *can be unknotted with a single negative twist of linking number* l *and* $n = l^2 + 1$. *Then there exists a knot* J' *such that for all* i *satisfying* $0 \leq i \leq n/2$,

$$V_{[li+\beta]_n}(J') = -\frac{1}{4} - \frac{i}{2} - \frac{[li+\beta]_n}{2} + \frac{i^2}{2n} + \frac{[li+\beta]_n^2}{2n} + \frac{n}{4} - V_i(K).$$

Here $\beta = 0$ *if* l *is even and* $\beta = n/2$ *if* l *is odd.*

To apply this corollary, we use one of the basic properties of the V_i invariants described in Section 7.1 to conclude:

$$0 \le V_i(J') - V_{i+1}(J') \le 1. \tag{5}$$

Let $n = l^2 + 1$ and define

$$j(i) = [li + \beta]_n \text{ and } s(i) = -\frac{1}{4} - \frac{i}{2} - \frac{j(i)}{2} + \frac{i^2}{2n} + \frac{j(i)^2}{2n} + \frac{n}{4}$$

so that, for each $i \in \{0, \ldots, n/2\}$, we have

$$V_{j(i)}(J') = s(i) - V_i(K). \tag{6}$$

If $j(i) < n/2$, we also have

$$V_{j(i)+1}(J') = s(i') - V_{i'}(K) \tag{7}$$

for some i'.

Substituting Equations (6) and (7) into Equation (5) and rearranging, we get:

$$s(i') - s(i) \le V_{i'}(K) - V_i(K) \le s(i') - s(i) + 1.$$

This process yields $n/2$ inequalities. Due to Equation (5), some of these inequalities will be redundant.

Example 9.8. Consider the case of $l = 0$ and $i = 0$. Then Corollary 9.7 implies that $V_0(J') = -V_0(K)$. Since these are non-negative, we have the following theorem, first proved by Sato [Sat].

Theorem 9.9. *If K can be unknotted with single negative twist of linking number $l = 0$, then $\nu^+(K) = 0$.*

Example 9.10. Consider the case of $l = 4$. We have the following table of values.

i	0	1	2	3	4	5	6	7	8
$j(i)$	0	4	8	5	1	3	7	6	2
$s(i)$	4	2	1	1	2	1	0	0	1

We conclude that if a knot K can be unknotted with a single negative twist of linking number $l = 4$, then the following inequalities must be satisfied. (The redundant inequalities have been removed.)

$$1 \leq V_0(K) - V_4(K) \leq 2,$$
$$V_4(K) - V_8(K) \leq 1,$$
$$1 \leq V_1(K) - V_5(K) \leq 2,$$
$$V_1(K) - V_3(K) \leq 1,$$
$$V_3(K) - V_7(K) \leq 1,$$
$$V_2(K) - V_6(K) \leq 1.$$

Example 9.11. In the case of $l = 7$, a similar computation yields 23 inequalities after 6 redundant ones have been removed. (This is a tedious computation which we omit.) We now compare this to the values given by Theorem 8.1: if a knot K can be unknotted with a negative twist of linking number $l = 7$, then

$$V_0(K) = 6, V_7(K) = 3, V_{14}(K) = 1, V_{21}(K) = 0.$$

Note that this implies that $V_i(K) = 0$ for all $i \geq 21$. Imposing these restrictions reduces our initial list of inequalities to a list of 14 inequalities (5 of which consist of a single sub-inequality). This indicates that, for a fixed linking number l, the construction may yield finer information than Theorem 8.1. For example, the inequality

$$1 \leq V_9(K) - V_{16}(K) \leq 2$$

remains, while Theorem 8.1 tells us that

$$1 \leq V_9(K) \leq 3 \quad \text{and} \quad 0 \leq V_{16}(K) \leq 1,$$

implying that

$$0 \leq V_9(K) - V_{16}(K) \leq 3,$$

a broader range.

Consider the knot $K = T(3, 17)$. One can compute the following table of V_i invariants.

i	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
$V_i(K)$	6	5	5	5	4	4	4	3	3	3	2	2	2	1	1	1	0

We see that Theorem 8.1 cannot rule out a negative twist of linking number 7, while our work above does:

$$V_9(K) - V_{16}(K) = 3 > 2.$$

Furthermore, a negative twist of linking number 7 is not ruled out by Corollary 5.2 (signature obstruction) or Theorem 6.1 (Arf obstruction).

10. Heegaard Floer obstructions related to the Upsilon invariant

As we have seen, the invariants $V_i(K)$ provide strong obstructions for a given integer l to satisfy $l \in \mathcal{U}(K)$. However, these invariants can be difficult to compute; for instance, they do not behave additively under connected sums of knots. In this section, we will apply Theorems 8.1 and 8.2 along with Proposition 7.3 to determine bounds on l for which the specific computation of the V_k would be difficult.

Example 10.1. Consider the knot $K = T(2,25) - T(3,8)$. This knot has $\tau(K) = 5$ and $g_4(K) = 7$ (see [Fel]). Since $\tau(K) \leq \nu^+(K) \leq g_4(K)$ (see [HW]), Theorem 8.2 implies

$$\frac{1+\sqrt{1+8\tau(K)}}{2} \leq l < \frac{3+\sqrt{9+8g_4(K)}}{2}$$

and we have $4 \leq l \leq 5$.

From Theorem 8.1, we know the following:

$$\text{If } l = 4, \text{ then } V_2 = 1, V_6 = 0. \tag{8}$$

$$\text{If } l = 5, \text{ then } V_0 = 3, V_5 = 1, V_{10} = 0. \tag{9}$$

Proposition 7.3 yields a list of restrictions on the Upsilon function of K:

- If $l = 4$, then $\Upsilon_K(t) \geq \max\{-2t-2, -6t\} = \begin{cases} -6t & t \leq 1/2, \\ -2t-2 & t \geq 1/2. \end{cases}$
- If $l = 5$, then $\Upsilon_K(t) \geq \max\{-6, -5t-2, -10t\} = \begin{cases} -10t & t \leq 2/5, \\ -5t-2 & 2/5 \leq t \leq 4/5, \\ -6 & t \geq 4/5. \end{cases}$

On the other hand, Upsilon functions of torus knots are easily computed [OSS]. We have that

$$\Upsilon_{T(2,25)}(t) = \begin{cases} -12t & 0 \leq t \leq 1, \\ 12t-24 & 1 \leq t \leq 2, \end{cases} \quad \text{and} \quad \Upsilon_{T(3,8)}(t) = \begin{cases} -7t & 0 \leq t \leq 2/3, \\ -t-4 & 2/3 \leq t \leq 1, \\ t-6 & 1 \leq t \leq 4/3, \\ 7t-14 & 4/3 \leq t \leq 2, \end{cases}$$

and so

$$\Upsilon_{T(2,25)-T(3,8)}(t) = \begin{cases} -5t & 0 \leq t \leq 2/3, \\ -11t+4 & 2/3 \leq t \leq 1, \\ 11t-18 & 1 \leq t \leq 4/3, \\ 5t-10 & 4/3 \leq t \leq 2. \end{cases}$$

Comparing this to the restrictions above, we have an obstruction when $t = 1$ for both $l = 4$ and $l = 5$. We conclude that the knot $K = T(2,25) - T(3,8)$ cannot be unknotted with a negative twist of linking number $l > 0$.

11. Obstructions from the Heegaard Floer homology of double branched covers

In Section 4 we explored how the linking form on the two-fold branched cover of a knot K provides obstructions to unknotting with a single twist. For a rational homology sphere M, the Heegaard Floer correction term $d(M, \mathfrak{s})$ can be thought of as a $\mathbb{Q}$-valued lift of the value of the linking form $\mathrm{lk}(x,x) \in \mathbb{Q}/\mathbb{Z}$ for $x \in H_1(M)$ after an appropriate identification of Spin^c-structures on M with $H_1(M)$. Thus, as we now describe, when the linking form obstructions vanish, it is possible for the lifted invariants to provide non-trivial obstructions.

The needed result from Heegaard Floer theory is the following.

Theorem 11.1 ([OS1, Owe]). *Let Y be a rational homology three-sphere which is the boundary of a simply-connected positive definite four-manifold X with $|H^2(Y;\mathbb{Z})|$ odd. Let the intersection pairing of X be represented in a basis by the matrix Q. Define a function*

$$m_Q : \mathbb{Z}^r / Q(\mathbb{Z}^r) \to \mathbb{Q}$$

by

$$m_Q(g) = \min \left\{ \frac{\xi^T Q^{-1} \xi - r}{4} \,\middle|\, \xi \in Char(Q), [\xi] = g \right\}$$

where $Char(Q)$ is the set of characteristic covectors for Q. Then there exists a group isomorphism

$$\phi : \mathbb{Z}^r / Q(\mathbb{Z}^r) \to Spin^c(Y)$$

with

$$\begin{aligned} & m_Q(g) \geq d\big(Y, \phi(g)\big) \\ \text{and} \quad & m_Q(g) \equiv d\big(Y, \phi(g)\big) \pmod 2 \end{aligned}$$

for all $g \in \mathbb{Z}^r / Q(\mathbb{Z}^r)$.

Note that $\mathrm{Char}(Q)$ corresponds to the set of first Chern classes of Spin^c-structures for X and we are using the identification of Spin^c-structures on Y with $H^2(Y)$. We will also use the fact that

$$\mathrm{Char}(Q) = \big\{ \xi = (\xi_1, \xi_2, \ldots, \xi_r) \in \mathbb{Z}^r \mid \xi_i \equiv Q_{ii} \big\}.$$

According to [Owe], to compute m_Q it suffices to consider characteristic covectors such that

$$-Q_{ii} \le \xi_i \le Q_{ii} - 2,$$

which, for forms of rank two, makes the computation fast, even for relatively large values of the Q_{ii}.

To illustrate the application of these results to the untwisting problem, we begin with a basic example.

Example 11.2. Suppose a knot K satisfies $\sigma(K) = -2$ and $\det(K) = 3$; for instance the trefoil knot or any knot with the same Seifert form. Suppose that, like the trefoil, K can be unknotted with a negative twist of linking number 2. Corollary 4.3 implies that $M_2(-K)$ bounds a simply-connected, positive definite four-manifold N with $b_2(N) = 2$ and intersection pairing

$$Q = \begin{pmatrix} a & b \\ b & a \end{pmatrix}$$

with determinant 3. There are only two such matrices:

$$\begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix} \text{ and } \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}.$$

These differ by a change of basis, so we consider only $Q = \left(\begin{smallmatrix} 2 & 1 \\ 1 & 2 \end{smallmatrix}\right)$. Then we have the quotient map $\phi\colon \mathbb{Z}^2 \to \mathbb{Z}^2/Q(\mathbb{Z}^2) \cong \mathbb{Z}/3\mathbb{Z} \cong H^2(Y)$; the cosets of the kernel have representatives $g_0 = \left(\begin{smallmatrix} 0 \\ 0 \end{smallmatrix}\right)$, $g_1 = \left(\begin{smallmatrix} 0 \\ 1 \end{smallmatrix}\right)$, and $g_2 = \left(\begin{smallmatrix} 0 \\ 2 \end{smallmatrix}\right)$. To compute m_Q we consider the subset of characteristic covectors

$$\left\{ \begin{pmatrix} -2 \\ -2 \end{pmatrix}, \begin{pmatrix} -2 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ -2 \end{pmatrix} \right\}.$$

A quick computation shows that only $\left(\begin{smallmatrix} 0 \\ 0 \end{smallmatrix}\right)$ is in the coset of $\left(\begin{smallmatrix} 0 \\ 0 \end{smallmatrix}\right)$, only $\left(\begin{smallmatrix} -2 \\ -2 \end{smallmatrix}\right)$ is in the coset of $\left(\begin{smallmatrix} 0 \\ 2 \end{smallmatrix}\right)$, and $\left(\begin{smallmatrix} -2 \\ 0 \end{smallmatrix}\right)$ and $\left(\begin{smallmatrix} 0 \\ -2 \end{smallmatrix}\right)$ are both in the coset of $\left(\begin{smallmatrix} 0 \\ 1 \end{smallmatrix}\right)$. We compute that

$$m_Q(g_0) = -\frac{1}{2} \quad \text{and} \quad m_Q(g_1) = m_Q(g_2) = \frac{1}{6}.$$

Thus Theorem 11.1 implies that ϕ satisfies

$$-\frac{1}{2} \ge d\big(\Sigma(-K), \phi(g_0)\big),$$

$$\frac{1}{6} \ge d\big(\Sigma(-K), \phi(g_1)\big),$$

$$\frac{1}{6} \ge d\big(\Sigma(-K), \phi(g_2)\big).$$

Note that $\phi(g_0)$ necessarily represents the Spin-structure on the two-fold branched cover. In the case where K is the trefoil, $\Sigma(-K) = -L(3,1)$ and the three bounds are sharp.

Manolescu and Owens [MO] computed that for the untwisted right-handed Whitehead double of the trefoil, $J = \mathrm{Wh}^+(T(2,3),0)$, the d-invariant of the Spin-structure on its two-fold branched cover is -4. This knot also satisfies $\Delta_J(t) = 1$, and thus $\det(J) = 1$ and $\sigma(J) = 0$. Thus, the calculation shows that $T(2,3) \# J$ cannot be unknotted with a negative twist of linking number 2, and this cannot be obstructed by any classical knot invariant.

Example 11.3. Consider the knot $K = 9_5$. This is a two-bridge knot with $\sigma(K) = -2$, $\det(K) = 23$, and $\Sigma(K) = L(23,17)$. We will show that $2^- \notin \mathcal{U}(K)$. Suppose that K could be unknotted with a single negative twist of linking number 2. Then Corollary 4.3 implies that $\Sigma(-K)$ bounds a simply-connected, positive definite four-manifold N with $b_2(N) = 2$ and intersection pairing

$$Q = \begin{pmatrix} a & b \\ b & a \end{pmatrix}$$

with determinant 23. Up to change of basis, there is only one such matrix:

$$Q = \begin{pmatrix} 12 & 11 \\ 11 & 12 \end{pmatrix}.$$

Note that this matrix is not ruled out by the methods of Section 4. We have the quotient map $\psi : \mathbb{Z}^2 \to \mathbb{Z}^2/Q(\mathbb{Z}^2) \cong \mathbb{Z}/23\mathbb{Z} \cong H^2(\Sigma(-K))$; the cosets of the kernel have representatives $g_i = \binom{0}{i}$ for $0 \le i \le 22$. We compute that, in particular,

$$m_Q(g_4) = -\frac{19}{46}.$$

In [OS1], Ozsváth and Szabó gave a formula for computing the d-invariants for $-L(p,q)$. We find that the set of d-invariants for $Y = \Sigma(-K) = -L(23,17)$ are

$$\Big\{\frac{29}{46}, \frac{1}{46}, -\frac{11}{46}, -\frac{7}{46}, \frac{13}{46}, \frac{49}{46}, \frac{9}{46}, -\frac{15}{46}, -\frac{1}{2}, -\frac{15}{46}, \frac{9}{46}, \frac{49}{46}, \frac{13}{46}, -\frac{7}{46}, -\frac{11}{46}, \frac{1}{46}, \frac{29}{46}, \frac{73}{46}, \frac{41}{46}, \frac{25}{46}, \frac{25}{46}, \frac{41}{46}, \frac{73}{46}\Big\}.$$

Among these, the only value which is congruent to $-\frac{19}{46}$ modulo $2\mathbb{Z}$ is $\frac{73}{46}$. Thus any isomorphism ϕ such that

$$d\big(Y, \phi(g_4)\big) \equiv m_Q(g_4) \pmod 2$$

would not satisfy

$$d\big(Y, \phi(g_4)\big) \leq m_Q(g_4).$$

By Theorem 11.1, we have reached a contradiction. Therefore, $K = 9_5$ cannot be unknotted with a single negative twist of linking number 2.

12. Obstructions for alternating knots

A knot is called (Floer homologically) thin if its knot Floer homology is supported on a single diagonal. A knot K is called σ-thin if its knot Floer homology is supported on the $-\sigma(K)/2$ diagonal. Alternating and quasi-alternating knots are σ-thin. In [Pet], Petkova showed that the minus version of the Heegaard Floer knot complex, $\mathrm{CFK}^-(K)$, of a thin knot K is completely determined by its Ozsváth-Szabó tau invariant and its Alexander polynomial. For a σ-thin knot, $\mathrm{CFK}^-(K)$ is determined by $\sigma(K)$ and its Alexander polynomial. It follows that for each thin knot K, its $V_i(K)$ invariants are equal to those of some $T(2,n)$ torus knot, determined by $\tau(K)$ (or $\sigma(K)$ in the case of σ-thin).

Lemma 12.1. *If $K = T(2, 2k+1)$, then*

$$V_i(K) = \begin{cases} \frac{k}{2} - \lfloor \frac{i}{2} \rfloor & \text{if } k \text{ is even and } 0 \leq i < k, \\ \frac{k+1}{2} - \lceil \frac{i}{2} \rceil & \text{if } k \text{ is odd and } 0 \leq i < k, \\ 0 & \text{if } i \geq k\ . \end{cases}$$

Proposition 12.2. *Suppose that K is a thin knot. If K can be unknotted with a negative twist of linking number $l > 0$, then $l \in \{1, 2, 3, 4\}$.*

Notes.

(1) A similar result appears in [AG, Proposition 5.6] using methods related to those of Section 8. Ruling out the case of $l = 5$ requires the methods of Section 9.

(2) Thanks are due to Linh Truong for pointing out that Proposition 12.2 applies generally to thin knots, rather than just the subset of alternating knots.

Proof. The proof follows from the following observation: Lemma 12.1 implies that for any knot K, the values of $V_i(K)$ decrease linearly as a function of i, while Theorem 8.1 implies that values of $V_i(K)$ decrease quadratically. More precisely, fix a linking number $l > 0$. From Lemma 12.1 we have that, for $i \geq 0$ and $i + l \leq k$,

$$V_i(K) - V_{i+l}(K) \in \left\{ \left\lfloor \frac{l}{2} \right\rfloor, \left\lfloor \frac{l}{2} \right\rfloor + 1 \right\}. \tag{10}$$

From Theorem 8.1, we have that if l is odd, $j \geq 0$, and $j+1 \leq \frac{l-1}{2}$, then

$$V_{jl}(K) - V_{jl+l}(K) = \frac{l-1}{2} - j.$$

If, in addition, $jl + l \leq k$, then we can apply Equation (10) to see that $j = 0$. This implies that Theorem 8.1 can determine at most 2 nonzero $V_i(K)$ values. Thus $l \in \{1, 3, 5\}$. We repeat this process when l is even. From Theorem 8.1, if l is even, $j \geq 0$, and $j+1 \leq \frac{l-2}{2}$, then

$$V_{(j+\frac{1}{2})l}(K) - V_{(j+\frac{1}{2})l+l}(K) = \frac{l-2}{2} - j.$$

If, in addition, $\left(j + \frac{1}{2}\right)l + l \leq k$, then we can apply Equation (10) and reach a contradiction for all values of l.

Thus, if Theorem 8.1 determines at least two nonzero $V_i(K)$ values, or if it determines one nonzero value in addition to determining that $V_k(K) = 0$, then it must be that $l = 1, 3,$ or 5. The remaining cases are when Theorem 8.1 determines

(1) one nonzero value and one $V_i(K) = 0$ where $i > k$.

(2) only one value.

In the notation of Theorem 8.1, these are the cases where $\alpha, \beta = 1$ or 0 respectively, or $l = 1, 2, 3,$ or 4. Therefore we must have that $l \in \{1, 2, 3, 4, 5\}$.

Note that if $l = 5$, then Theorem 8.1 implies that $V_0(K) = 3$ and $V_5(K) = 1$. Comparing this to Lemma 12.1, we conclude that $V_i(K) = V_i(T(2, 13))$. In this situation, a computation similar to that in Example 9.11 yields a contradiction. Thus a thin knot cannot be unknotted with a negative twist of linking number 5. $\square$

Combining Proposition 12.2 with the results of the previous sections, we can draw conclusions about the set $\mathcal{U}(K)$. The following two results are stated for alternating knots but hold (with identical proof) for σ-thin knots, as well.

Theorem 12.3. *Suppose that K is an alternating knot. Then $\mathcal{U}(K)$ is a subset of one of the following, determined by $\sigma(K)$.*

$\sigma(K)$	
0	$\{2^-, 1^-, 0^-, 0^+, 1^+, 2^+\}$
± 2	$\{1^\mp, 0^\mp, 2^\pm, 3^\pm\}$
± 4	$\{1^\mp, 3^\pm\}$
$\pm 6, \pm 8$	$\{1^\mp, 4^\pm\}$
> 8	$\{1^-\}$
< -8	$\{1^+\}$

Example 12.4. Here we show that $K = 12a_{369}$ cannot be unknotted with a single twist. We first note that K is alternating with $\sigma(K) = 6$ so that $\mathcal{U}(K) \subseteq \{1^-, 4^+\}$. Because $\mathrm{Arf}(K) = 1$, Theorem 6.1 implies that $1^- \notin \mathcal{U}(K)$. If $4^+ \in \mathcal{U}(K)$, then Corollary 5.2 implies that $|\sigma_{1/4}(K) - 5| \leq 1$. However, $\sigma_{1/4}(K) = 2$ and thus $\mathcal{U}(K) = \varnothing$.

Proof of Theorem 12.3. Suppose that K is an alternating knot. Let $k = -\sigma(K)/2$. Then for each i, we have that $V_i(K) = V_i(T(2, 2k+1))$ and, because $-K$ is also alternating, $V_i(-K) = V_i(-T(2, 2k+1)) = V_i(T(2, 2(-k-1)+1))$. Note that if K can be unknotted with a twist of linking number l, then $-K$ can be unknotted with an opposite twist of the same linking number. From Lemma 12.1, we know that

$$\nu^+\big(T(2, 2k+1)\big) = \begin{cases} k & \text{if } k \geq 0, \\ 0 & \text{if } k < 0\,. \end{cases}$$

In the proof of Theorem 8.2, it is shown that if K can be unknotted with a negative twist of positive linking number l, then

$$\frac{1 + \sqrt{1 + 8\nu^+(K)}}{2} \leq l < \frac{3 + \sqrt{9 + 8\nu^+(K)}}{2}.$$

In particular, this implies that if $\nu^+(K) \geq 7$, then $l > 4$. This contradicts Proposition 12.2 and so we conclude that $k \leq 6$. Similarly, by considering $\nu^+(-K)$, we conclude that $k \geq -6$ for K to be unknotted with a positive twist of positive linking number. Applying this bound for each value of k yields a short list of possible values for both positive and negative twists. Factoring in the signature function obstructions from Section 5 (and recalling that, with our conventions, $\sigma(K) = \sigma_{1/2}(K)$), further restricts the lists. In particular, if $\sigma(K) > 2$ or $\sigma(K) < -2$, then $0^\pm \notin \mathcal{U}(K)$. We are left with the following possibilities:

- If $k = 0$, then $\mathcal{U}(K) \subseteq \{2^-, 1^-, 0^-, 0^+, 1^+, 2^+\}$.
- If $k = \mp 1$, then $\mathcal{U}(K) \subseteq \{1^\mp, 0^\mp, 2^\pm, 3^\pm\}$.
- If $k = \mp 2$, then $\mathcal{U}(K) \subseteq \{1^\mp, 3^\pm\}$.
- If $k = \mp 3$, then $\mathcal{U}(K) \subseteq \{1^\mp, 3^\pm, 4^\pm\}$.
- If $k = \mp 4$, then $\mathcal{U}(K) \subseteq \{1^\mp, 4^\pm\}$.
- If $k = \mp 5$, then $\mathcal{U}(K) \subseteq \{1^\mp, 4^\pm\}$.
- If $k = \mp 6$, then $\mathcal{U}(K) \subseteq \{1^\mp, 4^\pm\}$.
- If $k < -6$, then $\mathcal{U}(K) \subset \{1^-\}$.
- If $k > 6$, then $\mathcal{U}(K) \subset \{1^+\}$.

Finally, combining Theorem 8.1 with Lemma 12.1, we can rule out $3^{\pm}$ for $k = \mp 3$ and $4^{\pm}$ for $k = \mp 5, \mp 6$. □

For large signature, Theorem 6.1 implies a slightly stronger result.

Corollary 12.5. *If K is an alternating knot with $|\sigma(K)| > 8$ and $\operatorname{Arf}(K) = 1$, then K cannot be unknotted with a single twist.*

13. Aït Nouh's linking numbers ±1 examples

The following construction was discovered by Aït Nouh. Figure 5 presents two illustrations of the knot 6_1. Blowing down the -1 in the diagram on the left or the $+1$ in the diagram on the right has the effect of removing the clasp in the evident ribbon disk (while adding a twist to the ribbon). In both cases, the effect of removing that clasp is to unknot 6_1. Thus, we see that 6_1 can be unknotted with either a positive or negative twist of linking number 1.

A similar approach applies for any ribbon knot which is unknotted by removing a single clasp in a ribbon disk. An examination of ribbon diagrams for ribbon knots presented in [Kaw] reveals that this can be successfully applied for the knots $6_1, 8_{20}, 9_{46}, 10_{140}$, and 10_{153}. This method can be expanded to some other ribbon knots with nice diagrams. A special case of this was discussed in Example 8.6 and illustrated in Figure 4 (cf. Page 564) where we showed that the knot 8_9 can also be unknotted with either a positive or negative twist of linking number 1. This resolves the remaining unknown cases for the ribbon knot 8_9 and implies that $\mathcal{U}(8_9) = \{2^-, 1^-, 0^-, 0^+, 1^+, 2^+\}$.

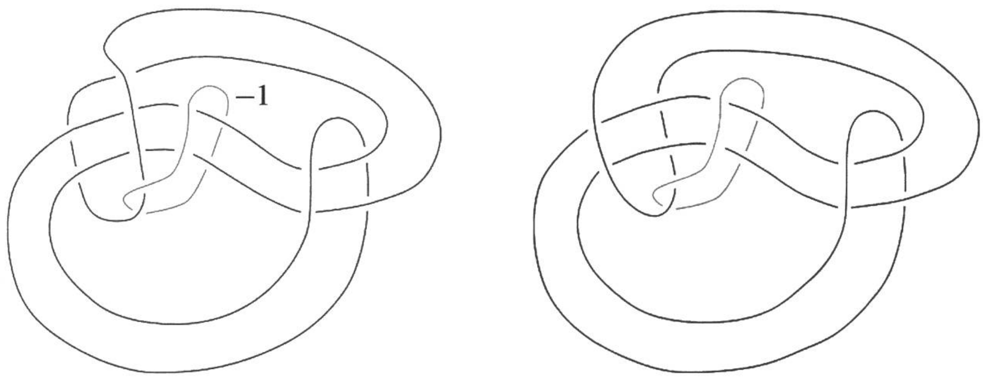

FIGURE 5
Ribbon untwisting

14. Unknotting with a single twist

We now return to the original question posed in this paper. Given a knot K, is it possible to unknot it using a single twist? In applying our results to any particular knot, the analysis of the signature constraints and Arf invariant is a straightforward computation. For an alternating knot, computing the Heegaard Floer invariants is also straightforward, while in the non-alternating case it can be more difficult. There is no all-encompassing statement that provides a simply stated single test that implies $\mathcal{U}(K) = \varnothing$. In this section we make three observations: (1) For low-crossing number knots, it is difficult to rule out $\mathcal{U}(K) = \varnothing$; (2) it is fairly easy to construct large families of knots for which $\mathcal{U}(K) = \varnothing$; and (3) in a non-formal sense, it is our expectation that generically most knots have $\mathcal{U}(K) = \varnothing$. Stronger statements in the case of alternating knots are possible; these represent a fairly special case that can be addressed by the methods of Section 12 (see, for instance, Corollary 12.5).

14.1. Low-crossing number knots. The results summarized in Table 1 concerning prime knots with 8 or fewer crossings are not encouraging with respect to finding reasonable criteria that imply $\mathcal{U}(K) = \varnothing$: of the 35 knots, 22 can be unknotted with a single twist and the remaining 13 are unknown. On close examination, this lack of examples is not so surprising and is specifically the result of the knots having so few crossings. First, for low-crossing number knots, having unknotting number 1 is fairly likely: in the case of 8 and fewer crossings, 18 are unknotting number 1. As the crossing number increases, the proportion of unknotting number 1 knots decreases. Second, for low-crossing number knots the genus is also small, which implies that the signature function and Heegaard Floer complex have fairly simple structure. The obstructions we have developed become more effective as these become more complicated.

To further indicate the limitations of working with low-crossing number knots, we note that among prime knots of 12 or fewer crossings, there is only one example that can readily be shown to have $\mathcal{U}(K) = \varnothing$, and that is the the torus knot $T(2,11)$, for which the signature and Arf invariant rule out all possible linking numbers. If in addition we consider obstructions arising from Heegaard Floer theory, then we can show that $\mathcal{U}(K) = \varnothing$ for only three additional prime knots of 12 or fewer crossings: $12a_{369}$, $12a_{716}$, and $12a_{1220}$.

14.2. Building examples. To illustrate the simplicity of building examples for which $\mathcal{U}(K) = \varnothing$, we provide one example, $K = T(2,7) \# 3T(2,-5)$, which can be addressed using the Arf invariant and the conditions on the signature function. We leave the explicit calculation of the invariants and details of the use of Theorem 5.1 to the reader.

- We have that $\mathrm{Arf}(T(2,7)) = 0$ and $\mathrm{Arf}(T(2,-5))$ =1, so $\mathrm{Arf}(K) = 1 \in \mathbb{Z}_2$, ruling out linking numbers $l = \pm 1$.
- $\sigma_{1/2}(K) = 6$, ruling out all cases of l even with the exception of $l = 4$.
- $\sigma_{1/4}(K) = 2$, ruling out $l = 4$.
- $\sigma_{1/3}(K) = 8$, ruling out $l = \pm 3$.
- $\sigma_{1/5}(K) = 8$, ruling out $l = \pm 5$.
- For all l, Theorem 5.1 implies that if K can be unknotted with a linking number l twist, then $|\sigma_{1/l}(K)| \geq 2l - 2$. For the knot in question, this is easily shown to be false for $l \geq 6$.

14.3. Generically, $\mathcal{U}(K) = \varnothing$. Our remarks here are speculative, but it is worth observing that all of our results point to the likelihood that by any reasonable measure, for most large crossing number knots one has $\mathcal{U}(K) = \varnothing$.

First, if one considers all knots K of crossing number N, the expected value of $\nu^+(K)$ and of $\sigma(K)$ most certainly grows as a function of N. By considering Seifert's algorithm to analyze the genus, the growth rate probably cannot be greater that N^ϵ for some positive $\epsilon < 1$, and perhaps it is even logarithmic, but one can safely predict that it goes to infinity.

Given that $\sigma(K)$ is large, in general one does not expect to have a linking number $l = 0$ unknotting twist, which requires $|\sigma(K)| \leq 2$ by Corollary 4.4. Given that $\nu^+(K)$ is large, the possible values of a linking number for any unknotting twist are restricted to at most two possible values for each direction twist by Theorem 8.2.

Finally, for a given value of l, Theorem 8.1 places very strict constraints on $V_r(K)$ for l values of r. Similarly, by Corollary 5.2 the value of $\sigma_{r/l}(K)$ is highly constrained for $l-1$ values of r. To say the least, the expectation is that it would be very rare for all the constraints to be satisfied simultaneously.

15. Comments and Questions

(1) A census of prime knots with up to eight crossings, summarized in Table 1, reveals there are only ten such knots K for which $\mathcal{U}(K)$ is completely known. The first example with unknown values is $\mathcal{U}(6_1)$.

(2) Our results are based primarily on knot invariants that are related to four-manifolds in some way. As of yet, three-manifold techniques have provided

little access to solving the problem of determining $\mathcal{U}(K)$ for individual knots. On the other hand, they seem well-suited for addressing more geometric questions, for instance related to primeness, and for working with families of knots; some examples of this are included in [ANMM1, ANMM2, GoS, Mot].

(3) As is evident from our work here, the case of linking number one is especially challenging. This challenge is related to the difficulty of finding invariants related to homology three-spheres, as opposed to rational homology spheres. Recent years have seen seen the development of a host of powerful new approaches to studying homology spheres and knot concordance. As of yet, the applicability of these methods to the unknotting problem is speculative, but we mention three of these notable advances: Manolescu's Pin(2) concordance invariants [Man], Manolescu–Hendricks's involutive invariants [HM], and Hom–Levine–Lidman's Heegaard Floer invariants of knots in homology spheres [HLL]. We expect that a continued study of the linking number one problem will bring new focus on particular problems related to knots in homology three-spheres.

(4) Ohyama's theorem [Ohy] states that any knot can be unknotted with two twists. A closer look at his construction shows that the linking numbers are consecutive integers. With more care it can be seen that Ohyama's proof yields the following.

Theorem 15.1. *For every integer $l \geq 0$ and knot K, it is possible to unknot K with a pair of oppositely signed twists of linking numbers l and $l+1$.*

The results concerning signatures presented in this paper can be generalized to show that if the $l+1$ in the statement of the theorem is replaced with $l+k$ for any $k > 1$, then it is no longer true. On the other hand, for a fixed pair (l_1, l_2), we are unable to either offer a generalization or find an obstruction. For instance, the following statement is possibly true: *Every knot K can by unknotted by a pair of oppositely signed twists of linking numbers* 3 *and* 5. (Here, 3 and 5 could be replaced by any relatively prime pair.)

(5) The problem of determining whether a given knot has unknotting number one has been resolved for all prime knots of 10 or fewer crossings. There are 27 knots of 11 or fewer crossings, out of a total of 801 knots, for which it is unknown. Note that saying that a knot K has unknotting number one implies that $\{0^{\pm}, 2^{\mp}\} \in \mathcal{U}(K)$ (with one of the two choices of sign) but not conversely. A good but lengthy project is to review the calculations that went into determining the unknotting numbers to identify knots of low crossing number for which the question of whether $0 \in \mathcal{U}(K)$ is unresolved.

A. Signatures of torus knots

In [Lit], Litherland presented a formula for the signature function of a torus knot, generalizing a formula of Goldsmith [Gol] for the Murasugi signature. Figure 6 illustrates Litherland's description of the signature function of $T(p,q)$ in the case of $p=5, q=7$. We have changed the scale; Litherland worked with a unit square, and we have modified his coordinates so that we can work with points in the integer lattice. In a rectangle with vertices $(0,0)$, $(q,0)$, $(0,p)$, and (q,p), line segments are drawn: one from $(qx,0)$ to $(0,px)$, and the other from (qx,p) to (q,px). According to [Lit], the signature at $\omega = e^{2\pi i x}$ is given by counting the number of lattice points interior to the two triangular regions (C_1 and C_2) and subtracting the number of lattice points in the interior of the remaining region.

In the illustration we have $x = 0.6$ and find $\sigma_x(T(p,q)) = 3+1-20 = -16$. By symmetry, we can focus on the range $0 \le x \le \frac{1}{2}$.

It will be simpler to work with negative signature, which we denote

$$\overline{\sigma}_{p,q}(x) = -\sigma_x\big(T(p,q)\big).$$

Fix a choice of p, q. To avoid the case that lattice points lie on the side of the triangles, we let $\mathcal{S}_{p,q}$ denote the *singular set* $\{\frac{i}{pq}\}_{i\in\mathbb{Z}}$. We call the lower and upper triangle counts $\#C_1$ and $\#C_2$, and the remaining count by $\#C_3$, so that for $x \notin \mathcal{S}_{p,q}$, $\#C_1 + \#C_2 + \#C_3 = (p-1)(q-1)$. We then have

Theorem A.1 (Exact count). *For $x \notin \mathcal{S}_{p,q}$,*

$$\overline{\sigma}_{p,q}(x) = (p-1)(q-1) - 2(\#C_1 + \#C_2). \tag{11}$$

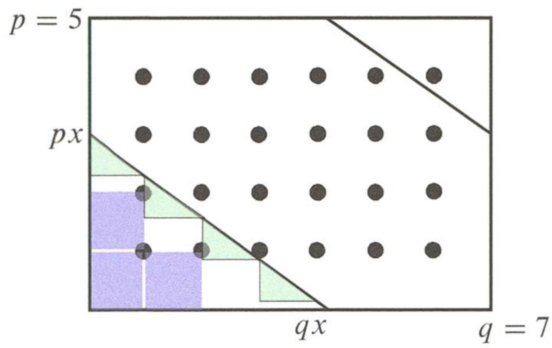

FIGURE 6
Signature count

The simplest estimate for the signature function results from a consideration of areas. The three (blue) shaded squares in Figure 6 illustrate that the count $\#C_1$ is the sum of the areas of squares, and this sum approximates the area of the lower triangle. In general, if we approximate the counts, $\#C_1$ and $\#C_2$, as well as the count of the lattice points in the complementary region, by the areas of the regions, we arrive at the following.

Theorem A.2 (Approximation). *For $x \notin \mathcal{S}_{p,q}$,*

$$\overline{\sigma}_{p,q}(x) \approx pq - pqx^2 - pq(1-x)(1-x) = 2pqx(1-x). \tag{12}$$

We need to improve this to find a precise lower bound for the $\overline{\sigma}_{p,q}(x)$. To do so, we can subtract the areas of the two triangles from the total number of lattice points, $(p-1)(q-1)$, yielding the next result.

Theorem A.3 (Lower approximation). *For $x \notin \mathcal{S}_{p,q}$,*

$$\overline{\sigma}_{p,q}(x) > (p-1)(q-1) - pqx^2 - pq(1-x)(1-x).$$

Theorem A.4. *For all $x \in [0,1]$, if $0 \leq x < \frac{1}{pq}$ then $\overline{\sigma}_{p,q}(x) = 0$. If $\frac{1}{pq} < x < \frac{1}{2}$ then $\overline{\sigma}_{p,q}(x) > 0$.*

Proof. Notice that to prove this theorem, we need to only verify it for points $x \notin \mathcal{S}_{p,q}$. The value of $\overline{\sigma}_{p,q}(x)$ for $x \in \mathcal{S}_{p,q}$ is given as the two-sided average at nearby points. If an *integer-valued* function is positive at almost all points in an open interval, the two-sided averaged function is positive everywhere on the interval.

Denote the hypotenuses of triangles C_1 and C_2 by l_1 and l_2, both of which depend on the choice of x. As x increases from 0 to $\frac{1}{2}$, the value of $\overline{\sigma}_{p,q}(x)$ increases when a lattice point lies on l_2 and it decreases when a lattice point lies on l_1.

Step 1: The first positive jump in $\overline{\sigma}_{p,q}(x)$. The function on the plane ϕ: $(i,j) \to ip + jq$ is constant on each line l_2, taking value $pq(1+x)$ on the general line and, in particular, taking value pq on the line when $x = 0$. To find the first positive jump, we must find the smallest $x > 0$ such that the line l_2 contains a lattice point (i,j) in the rectangle for which $\phi(i,j) > pq$. We show that occurs at $x = \frac{1}{pq}$.

We begin by using the fact that p and q are relatively prime: there exists an r satisfying $0 < r < q$ such that $rp + sq = 1$ for some s. A simple algebraic argument shows that $-p < s < 0$.

Consider the point $(i, j) = (r, p + s)$. Notice that this is in the interior of the rectangle. We have $\phi(i, j) = rp + (p + s)q = pq + 1$. This is clearly the least possible integer value of $ip + jq$ that is greater than pq. Writing $pq + 1 = pq(1 + \frac{1}{pq})$ shows that the corresponding value of x is $\frac{1}{pq}$, as desired.

Step 2: The first negative jump in $\overline{\sigma}_{p,q}(x)$. This case is simpler. It is evident that the first negative jump corresponds the value of x for which the line l_1 contains the lattice point $(1, 1)$. It is a simple algebra exercise to show that the value of x is $\frac{1}{p} + \frac{1}{q}$.

Step 3: $\overline{\sigma}_{p,q}(x) > 0$ for $x \geq \frac{1}{p} + \frac{1}{q}$. The proof of the theorem is completed by showing that $\overline{\sigma}_{p,q}(x) > 0$ for all x satisfying $\frac{1}{p} + \frac{1}{q} \leq x < \frac{1}{2}$. The lower bound given in Lemma A.3 is quadratic, increasing on $[0, \frac{1}{2}]$. If we denote that lower bound by $\beta_{p,q}(x)$, we need to check that $\delta = \beta_{p,q}(\frac{1}{p} + \frac{1}{q}) > 0$. A direct substitution and simplification yields

$$\delta = p + q - 2\Big(\frac{p}{q} + \frac{q}{p}\Big) - 3.$$

If we assume that $p \geq 3$ and $q \geq 5$, then we have

$$\delta \geq p + q - 2\Big(\frac{p}{5} + \frac{q}{3}\Big) - 3 = \frac{3}{5}p + \frac{1}{3}q - 3 \geq \frac{9}{5} + \frac{5}{3} - 3 = \frac{7}{25}.$$

The remaining cases $T(2, k)$ and $T(3, 4)$ can be computed explicitly. □

B. Basic definitions related to Upsilon, $\Upsilon(K)$, and proof of Proposition 7.3

Let K be a knot and let $C = \mathrm{CFK}^-(K)$ be the Heegaard Floer knot complex for K. This is a graded chain complex; the grading called the *Maslov grading*. The Maslow grading of an element is denoted $\mathrm{gr}(x)$. The complex has two increasing filtrations. One is called the *Alexander filtration*, with the filtration level of an element x denoted $\mathrm{Alex}(x)$. The other is called the *algebraic filtration*, with the filtration level of an element x denoted $\mathrm{Alg}(x)$.

The invariant V_s can be defined to be

$$V_s(K) := -\frac{1}{2}\max\Big\{\mathrm{gr}(x) \mid x \in H_*\big(C\{i \leq 0, j \leq s\}\big) \text{ and } \quad U^k x \neq 0 \in H_*(C) \text{ for all } k\Big\}$$

where $C\{i \leq 0, j \leq s\}$ is the subcomplex of C consisting of elements of Alexander filtration at most s and algebraic filtration at most 0. This definition is equivalent to that given in [NW].

In [OSS], Ozsváth, Stipsicz, and Szabó defined the knot invariant Upsilon $\Upsilon_K(t)$ for $t \in [0,2]$. Suppose that $\mathcal{B}$ is a bifiltered basis of $\mathrm{CFK}^\infty(K)$. In [Livl], it is shown that

$$\Upsilon_K(t) = -2 \cdot \min\bigl\{r \mid H_0(\mathcal{F}_{t,r}) \longrightarrow H_0\bigl(\mathrm{CFK}^\infty(K)\bigr) \text{ is surjective}\bigr\}$$

where $\mathcal{F}_{t,r}$ is the subcomplex generated by the set

$$\left\{x \in \mathcal{B} \,\middle|\, \left(\frac{t}{2}\mathrm{Alex}(x) + \left(1 - \frac{t}{2}\right)\mathrm{Alg}(x)\right) \leq r\right\}.$$

Diagrammatically, the subcomplex $\mathcal{F}_{t,r}$ is represented as the lower half-space with boundary line

$$\frac{t}{2}j + \left(1 - \frac{t}{2}\right)i = r.$$

Note that sums of elements in this half-space are in $\mathcal{F}_{t,r}$, but might not have bifiltration levels satisfying the given constraint.

B.1. Relating $\Upsilon(K)$ to $V_i(K)$.

Proposition B.1. *Let K be a knot and $g = g(K)$ be the genus of K. Then for $t \in [0,2]$ and $s \geq 0$,*

$$-st - 2V_s(K) \leq \Upsilon_t(K) \leq \begin{cases} -gt - 2V_s(K) - 2s + 2g + 2 & t \leq 1 - \frac{s}{g}, \\ gt - 2V_s(K) + 2 & t \geq 1 - \frac{s}{g}. \end{cases}$$

Proof. Fix $s \geq 0$ and abbreviate $V_s(K)$ by V_s. The maximum grading of a generator of homology in $C\{i \leq 0, j \leq s\}$ is $-2V_s$. Thus the maximum grading of a generator in $C\{i \leq V_s - 1, j \leq s + V_s - 1\}$ is -2 and there is a generator of grading 0 in $C\{i \leq V_s, j \leq s + V_s\}$.

In particular, if the complex $C\{i \leq V_s, j \leq s + V_s\}$ contains a grading 0 generator and the value of r is

$$r = \left(\frac{t}{2}(s + V_s) + \left(1 - \frac{t}{2}\right)V_s\right),$$

then $\mathcal{F}_{t,r}$ contains a generator of grading 0 and the map $H_0(\mathcal{F}_{t,r}) \longrightarrow H_0(\mathrm{CFK}^\infty(K))$ is surjective. Thus,

$$\Upsilon_K(t) \geq -2\left(\frac{t}{2}(s + V_s) + \left(1 - \frac{t}{2}\right)V_s\right) = -2V_s - ts.$$

On the other hand, since $C\{i \leq V_s - 1, j \leq s + V_s - 1\}$ does not contain a generator of grading 0 (the maximum grading here is -2), for each $t \in [0,2]$, the minimum r-value in the definition of $\Upsilon_K(t)$ is such that the following system of inequalities has a (nonempty) solution

$$
\begin{cases}
\frac{t}{2}j + \left(1 - \frac{t}{2}\right)i \leq r, & \text{(A)} \\
-g \leq j - i \leq g, & \text{(B)} \\
i > V_s - 1 \text{ or } j > s + V_s - 1, & \text{(C)}
\end{cases}
\tag{13}
$$

where $g = g(K)$ is the genus of the knot K. Combining inequalities (13)(B) and (13)(C), we have that if $i > V_s - 1$, then

$$\frac{t}{2}j + \left(1 - \frac{t}{2}\right)i = \frac{t}{2}(j - i) + i > -\frac{t}{2}g + V_s - 1,$$

and if $j > s + V_s - 1$, then

$$\frac{t}{2}j + \left(1 - \frac{t}{2}\right)i = \left(\frac{t}{2} - 1\right)(j - i) + j > \left(\frac{t}{2} - 1\right)g + s + V_s - 1.$$

Therefore, if the system of inequalities is to have a solution, either

$$r > -\frac{t}{2}g + V_s - 1 \tag{14}$$

or

$$r > \left(\frac{t}{2} - 1\right)g + s + V_s - 1. \tag{15}$$

This implies that for all $t \in [0,2]$,

$$r > \min\left\{-\frac{t}{2}g + V_s - 1, \left(\frac{t}{2} - 1\right)g + s + V_s - 1\right\}.$$

The two lower bounds agree when $t = 1 - \frac{s}{g}$. When $t \geq 1 - \frac{s}{g}$, Inequality (14) gives the weaker lower bound on r and we have

$$\Upsilon_K(t) = -2r < tg - 2V_s + 2.$$

When $t \leq 1 - \frac{s}{g}$, Inequality (15) gives a weaker lower bound on r and we have

$$\Upsilon_K(t) = -2r < -tg + 2g - 2s - 2V_s + 2. \qquad \square$$

C. Results for knots with up to eight crossings

In Table 1, we summarize our result for knots with up to eight crossings. We find ten knots for which there are no remaining unknown values and 13 knots for which there are no known values.

TABLE 1

Known and unknown values for prime knots with up to 8 crossings. Here, "known values" are those which are confirmed to be in the set of unknotting indices for the given knot. The "unknown values" are those which cannot yet be ruled out, but for which realizability is unknown.

Knot	Known values	Unknown values
3_1	$3^-, 2^-, 0^+$	
4_1	$2^-, 0^-, 0^+, 2^+$	
5_1	3^-	
5_2	$2^-, 0^+, 1^+$	
6_1	$1^-, 0^-, 1^+, 2^+$	$2^-, 0^+$
6_2	$2^-, 0^+$	3^-
6_3	$2^-, 0^-, 0^+, 2^+$	
7_1	4^-	
7_2	$2^-, 0^+$	3^-
7_3		3^-
7_4		$2^-, 0^+, 1^+$
7_5		1^+
7_6	$2^-, 0^+$	3^-
7_7	$2^-, 0^+$	2^+
8_1	$0^-, 2^+$	$2^-, 0^+$
8_2		1^+
8_3		$2^-, 1^-, 0^-, 0^+, 1^+, 2^+$
8_4		$0^-, 2^+, 3^+$
8_5		3^-
8_6		$2^-, 0^+, 1^+$
8_7	$0^-, 2^+$	3^+
8_8	$1^-, 1^+$	$2^-, 2^+$
8_9	$2^-, 1^-, 0^-, 0^+, 1^+, 2^+$	
8_{10}		$0^-, 2^+, 3^+$
8_{11}	$2^-, 0^+$	3^-
8_{12}		$2^-, 0^-, 0^+, 2^+$
8_{13}	$2^-, 0^-, 0^+, 2^+$	
8_{14}	$2^-, 0^+$	1^+
8_{15}		1^+
8_{16}		$0^-, 2^+, 3^+$
8_{17}	$2^-, 0^-, 0^+, 2^+$	
8_{18}		$2^-, 2^+$
8_{19}	$4^-, 3^-$	
8_{20}	$1^-, 0^-, 1^+, 2^+$	$2^-, 0^+$
8_{21}	$2^-, 0^+$	1^+

Most of the known values are a result of the observation that if K has unknotting number 1, then either $\{0^-, 2^+\} \subset U(K)$ or $\{0^+, 2^-\} \subset U(K)$, depending on the sign of the crossing change needed to unknot K. For the two knots 3_1 and 8_{19} (the torus knots $T(2,3)$ and $T(3,4)$, respectively), knowledge of their diagrams contributes to the known values. Using the method of Section 13, we have that $\{1^-, 1^+\}$ is a subset of $\mathcal{U}(6_1)$, $\mathcal{U}(8_8)$, $\mathcal{U}(8_9)$, and $\mathcal{U}(8_{20})$. That the knots 5_1 and 7_1 can be unknotted with negative twists of linking numbers 3 and 4, respectively, is shown in [Nou, Figure 6] and [Nou, Figure 5], respectively. Finally, in [Nou] Aït Nouh shows that the knot 7_1 is not slice in $\mathbb{C}P^2$. This rules out all remaining values for 7_1.

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