# Bounded cohomology of amenable covers
# via classifying spaces

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**Abstract.** Gromov and Ivanov established an analogue of Leray's theorem on cohomology of contractible covers for bounded cohomology of amenable covers. We present an alternative proof of this fact, using classifying spaces of families of subgroups.

## 1. Introduction

The idea that the cohomology of a space can be computed as cohomology of the nerve of an open cover consisting of contractible subsets first appeared in a paper by Weil [Wei], which was preceded by a paper of Leray [Ler] with a similar idea.

Gromov and Ivanov established partial analogues for bounded cohomology in terms of covers that consist of amenable subsets; we will consider the following version of this phenomenon:

**Theorem 1.1.** *Let $X$ be a path-connected CW-complex, let $U$ be an amenable open cover of $X$, let $N$ be the nerve of $U$, and let $|N|$ be the geometric realisation of $N$. Let $c_X : H_b^*(X;\mathbb{R}) \to H^*(X;\mathbb{R})$ be the comparison map from bounded to ordinary cohomology. Then the following hold.*

(1) *If $U$ is convex, then $c_X$ factors through the nerve map $\nu \colon X \to |N|$: More precisely, there is an $\mathbb{R}$-linear map $\varphi \colon H_b^*(X;\mathbb{R}) \to H^*(|N|;\mathbb{R})$ with*

$$H^*(\nu;\mathbb{R}) \circ \varphi = c_X.$$

$$H_b^*(X;\mathbb{R}) \xrightarrow{\ c_X\ } H^*(X;\mathbb{R})$$

$$\varphi \qquad \qquad \uparrow H^*(\nu;\mathbb{R})$$

$$H^*(|N|;\mathbb{R})$$

(2) *If the multiplicity of $U$ is at most $m$, then the comparison map $c_X$ vanishes in all degrees $* \geq m$.*

Here, an *amenable cover* of $X$ is an open cover $U$ of $X$ by path-connected sets with the following property: For each $V$ in $U$ and each $x \in V$, the image of the homomorphism $\pi_1(V,x) \to \pi_1(X,x)$ induced by the inclusion $V \hookrightarrow X$ is an amenable subgroup of $\pi_1(X,x)$ (Definition 4.10). Such an amenable cover $U$ is *convex* if all finite intersections of members of $U$ are path-connected or empty (Definition 4.3).

The first statement of Theorem 1.1 was proved by Ivanov [Ival, Section 6] using sheaf cohomology and a spectral sequence computation. The assumption on convexity is missing in Ivanov's paper but needed (see Example 1.4). The second statement of Theorem 1.1 is Gromov's vanishing theorem [Gro, Section 3] whose proof is based on the theory of multicomplexes. Recently, Frigerio and Moraschini reworked Gromov's theory of multicomplexes and gave a proof of Theorem 1.1 [FM, Section 6].

In this note, we present an alternative proof of Theorem 1.1 that only uses standard facts about bounded cohomology (via strong relatively injective resolutions) and the classifying space for the family of amenable subgroups. More precisely, we separate the proof into a statement about admissibility of the family of amenable subgroups (Section 5.1) and generic properties of classifying spaces of families (Section 5.2). The abstract version of Theorem 1.1 is Theorem 5.3.

Furthermore, our approach also leads to a straightforward proof for the corresponding statement in $\ell^1$-homology (and of the corresponding abstract version: Theorem 6.3):

**Theorem 1.2.** *Let $X$ be a path-connected CW-complex, let $U$ be an amenable open cover of $X$, let $N$ be the nerve of $U$, and let $|N|$ be the geometric realisation of $N$. Let $c_X^{\ell^1} \colon H_*(X;\mathbb{R}) \to H_*^{\ell^1}(X;\mathbb{R})$ be the comparison map from ordinary to $\ell^1$-homology. Then the following hold.*

(1)  *If $U$ is convex, then $c_X^{\ell^1}$ factors through the nerve map $\nu\colon X \to |N|$:*
    *More precisely, there is an $\mathbb{R}$-linear map $\varphi\colon H_*(|N|;\mathbb{R}) \to H_*^{\ell^1}(X;\mathbb{R})$ with*

$$\varphi \circ H_*(\nu;\mathbb{R}) = c_X^{\ell^1}.$$

$$
\begin{array}{ccc}
H_*(X;\mathbb{R}) & \xrightarrow{\;c_X^{\ell^1}\;} & H_*^{\ell^1}(X;\mathbb{R}) \\[2pt]
{\scriptstyle H_*(\nu;\mathbb{R})}\Big\downarrow & \nearrow & \\[2pt]
H_*(|N|;\mathbb{R}) & {}_{\varphi} &
\end{array}
$$

(2)  *If the multiplicity of $U$ is at most $m$, then the comparison map $c_X^{\ell^1}$ vanishes in all degrees $* \geq m$.*

In particular, as a special case, we obtain the corresponding vanishing theorem for $\ell^1$-homology and $\ell^1$-invisibility results for amenable convex open covers on CW-complexes (Corollary 1.5), established recently by Frigerio [Fri2]. Using Lemma 4.1. in *loc. cit.*, Frigerio reduces the statement for topological spaces to the one for CW-complexes and thus can drop the assumption on the space being a CW-complex. Using the same lemma we may also drop the assumption on $X$ being a CW-complex in Theorems 1.1 (2) and Theorem 1.2 (2).

**1.1. Two non-examples.** We give two simple examples complementing the hypotheses and the conclusion in Theorem 1.1.

Both examples are based on the oriented closed connected surface $\Sigma$ of genus 2. Because $\Sigma$ admits a hyperbolic structure, the comparison map

$$H_b^2(\Sigma;\mathbb{R}) \to H^2(\Sigma;\mathbb{R}) \cong \mathbb{R}$$

is surjective (and thus non-trivial) [Gro, p. 9/17].

**Example 1.3.** In general, in Theorem 1.1, there is *no* such factorisation $\varphi$ that is bounded: We consider the open cover $U$ of $\Sigma$ depicted in Figure 1. All members of $U$ are contractible or homotopy equivalent to $S^1$. Therefore, $U$ is an amenable cover; moreover, one easily checks that the cover is convex (in the sense of Definition 4.3).

Then the nerve $N$ of $U$ satisfies $|N| \simeq S^1 \vee S^2 \vee S^1$ (the $S^2$ stems from the octahedron spanned by the sets $D_1$, $D_2$, $U_1$, ..., $U_4$).

We will now show that all non-trivial classes in $H^2(|N|;\mathbb{R})$ are unbounded: To this end, we consider the inclusion $i\colon S^2 \to S^2 \vee S^2 \vee S^1 \simeq |N|$. Then the induced map $H^2(i;\mathbb{R})\colon H^2(|N|;\mathbb{R}) \to H^2(S^2;\mathbb{R})$ is an isomorphism. However, all non-trivial classes in $H^2(S^2;\mathbb{R})$ are known to be unbounded [Gro, p. 8/17]. Therefore, also all non-trivial classes in $H^2(|N|;\mathbb{R})$ are unbounded.
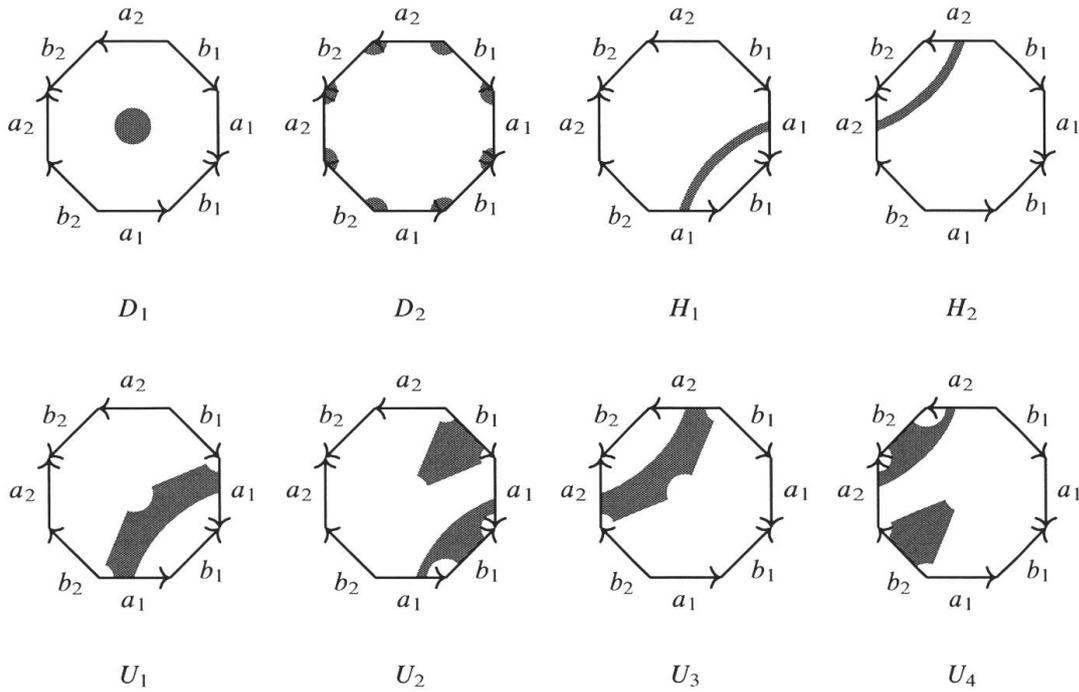
FIGURE 1
The open cover of $\Sigma$ in Example 1.3

In particular, the surjection $H_b^2(\Sigma;\mathbb{R}) \to H^2(\Sigma;\mathbb{R}) \cong \mathbb{R}$ does *not* admit a factorisation $H_b^2(\Sigma;\mathbb{R}) \to H^2(|N|;\mathbb{R})$ over the nerve map by a bounded linear map.

**Example 1.4.** In general, Theorem 1.1 does *not* hold without the convexity condition: We consider the following open cover $U$ of $\Sigma$ depicted in Figure 2.

Then the nerve $N$ of $U$ satisfies $|N| \cong \Delta^3 \simeq \bullet$. In particular, $H^2(|N|;\mathbb{R}) \cong 0$. Therefore, the comparison map $H_b^2(\Sigma;\mathbb{R}) \to H^2(\Sigma;\mathbb{R})$ (which is non-trivial) cannot factor over $H^2(|N|;\mathbb{R})$.
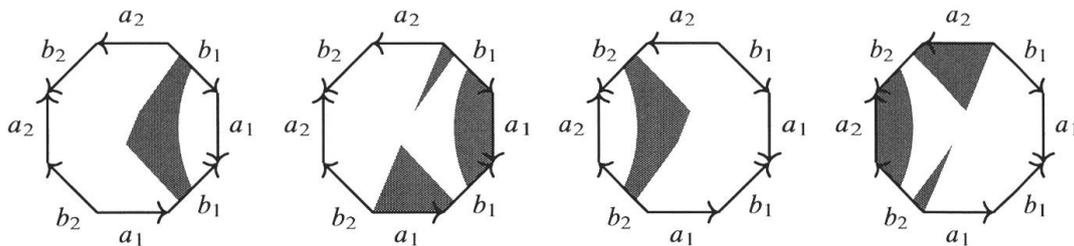


FIGURE 2
The open cover of $\Sigma$ in Example 1.4

**1.2. Applications.** The standard application of results of the type of Theorem 1.1 and Theorem 1.2 are vanishing theorems for the $\ell^1$-semi-norm on singular homology (and whence to simplicial volume):

**Corollary 1.5** (a vanishing theorem). *Let $X$ be a path-connected CW-complex, let $U$ be an amenable convex open cover of $X$, and let $N$ be the nerve of $U$. Furthermore, let $k \in \mathbb{N}$ with $H_k(N; \mathbb{R}) \cong 0$.*

(1) *Then the comparison maps $c_X \colon H_b^k(X; \mathbb{R}) \to H^k(X; \mathbb{R})$ in bounded cohomology and $c_X^{\ell^1} \colon H_k(X; \mathbb{R}) \to H_k^{\ell^1}(X; \mathbb{R})$ in $\ell^1$-homology are the zero maps.*

(2) *In particular, for all $\alpha \in H_k(X; \mathbb{R})$, we have*

$$\|\alpha\|_1 = 0.$$

*Proof. Ad 1.* We have $H_k(|N|; \mathbb{R}) \cong H_k(N; \mathbb{R}) \cong 0$ by assumption. Moreover, by the universal coefficient theorem, we also have $H^k(|N|; \mathbb{R}) \cong 0$. Therefore, Theorem 1.1 and Theorem 1.2 show that the comparison maps in bounded cohomology and $\ell^1$-homology factor over 0 and so are the zero maps.

*Ad 2.* The comparison map $c_X^{\ell^1}$ is isometric with respect to the $\ell^1$-semi-norm [Löh2, Proposition 2.5]. Applying the first part proves the claim.    □

The hypothesis is satisfied if the multiplicity of the cover is at most $k$ (and thus the nerve has dimension at most $k - 1$). Further examples are contained in Gromov's original article [Gro, Section 3.1].

**1.3. Notation.** In the rest of this paper, homology, cohomology as well as bounded cohomology of spaces, groups, or simplicial complexes is always taken with $\mathbb{R}$-coefficients (and we will mostly omit this from the notation).

**Organisation of this article.** We first recall basics on bounded cohomology (Section 2), classifying spaces of families of subgroups (Section 3), and nerves of covers (Section 4). It should be noted that all of this material is standard; we collected it here in one place for convenience and to introduce the notation used in the main proof.

The proof of Theorem 1.1 is given in Section 5; the proof of Theorem 1.2 is given in Section 6.

## 2. Preliminaries: Bounded cohomology

We first collect basic notation and basic facts on bounded cohomology and $\ell^1$-homology, as needed in the sequel; in fact, no other input from bounded cohomology will be needed for the proofs of Theorem 1.1 and Theorem 1.2. For details and further results, we refer the reader to the literature [Gro, Iva1, Iva2, Mon, Fri1, Bou, Löh1].

**2.1. Bounded cohomology.** Bounded cohomology of spaces and groups is defined as the cohomology of the topological dual of the standard chain complexes. Let $B(\,\cdot\,,\mathbb{R})$ be the contravariant endofunctor on the category of normed $\mathbb{R}$-vector spaces and continuous linear maps that is given by taking the topological dual. A *normed chain complex* is a chain complex consisting of normed $\mathbb{R}$-vector spaces and continuous boundary maps. Then $B(\,\cdot\,,\mathbb{R})$ induces a contravariant functor from the category of normed chain complexes (and degree-wise continuous chain maps) to the category of Banach cochain complexes (and degree-wise continuous cochain maps).

If $X$ is a topological space, then the singular chain complex $C_*(X;\mathbb{R})$ is a normed chain complex with respect to the $\ell^1$-norm $|\cdot|_1$ associated with the $\mathbb{R}$-bases given by the sets of all singular simplices of $X$. If $f\colon X \to Y$ is a continuous map, then the chain map $C_*(f;\mathbb{R})\colon C_*(X;\mathbb{R}) \to C_*(Y;\mathbb{R})$ is degree-wise of norm at most $1$.

**Definition 2.1** (bounded cohomology of spaces).

- The *bounded cochain complex functor* $C_b^*(\,\cdot\,;\mathbb{R})$ is the contravariant functor from the category of topological spaces to Banach cochain complexes given by the composition $B(C_*(\,\cdot\,;\mathbb{R}),\mathbb{R})$.

- The *bounded cohomology functor* $H_b^*(\,\cdot\,;\mathbb{R})$ is the contravariant functor from the category of topological spaces to (semi-normed) graded $\mathbb{R}$-vector spaces, given by the composition $H^*(C_b^*(\,\cdot\,;\mathbb{R}))$.

- The natural transformation $H_b^*(\,\cdot\,;\mathbb{R}) \to H^*(\,\cdot\,;\mathbb{R})$ induced by the natural inclusion $C_b^*(\,\cdot\,;\mathbb{R}) \to C^*(\,\cdot\,;\mathbb{R})$ is the *comparison map*.

One of the key properties of bounded cohomology is Gromov's mapping theorem [Gro, p. 40], [Iva1, Theorem 4.3], [Iva2], [Fri1, Corollary 5.11]:

**Theorem 2.2** (Gromov's mapping theorem). *Let $f\colon X \to Y$ be a continuous map of path-connected spaces such that $\pi_1(f)\colon \pi_1(X) \to \pi_1(Y)$ is surjective and has amenable kernel. Then $H_b^*(f;\mathbb{R})\colon H_b^*(Y;\mathbb{R}) \to H_b^*(X;\mathbb{R})$ is an (isometric) isomorphism.*

Furthermore, bounded cohomology admits a description in terms of injective resolutions [Iva1], [Fri1, Chapter 4]. We will need the following facts:

**Proposition 2.3** ([Fri1, Lemma 4.22]). *Let $\Gamma$ be a group and let $S$ be a $\Gamma$-set all of whose isotropy groups are amenable. Then $B(S, \mathbb{R})$ is a relatively injective $\Gamma$-module.*

**Theorem 2.4.** *Let $\Gamma$ be a group, let $\mathbb{R} \to C^*$ be a strong relatively injective resolution of $\mathbb{R}$ by Banach $\Gamma$-modules, and let $X$ be a path-connected topological space with fundamental group $\Gamma$. Then every degree-wise bounded $\Gamma$-cochain map $C^* \to C_b^*(\mathrm{E}\,\Gamma; \mathbb{R})$ that extends $\mathrm{id}_\mathbb{R}: \mathbb{R} \to \mathbb{R}$ induces an isomorphism $H^*((C^*)^\Gamma) \to H^*(C_b^*(\mathrm{E}\,\Gamma)^\Gamma; \mathbb{R})$.*

*Proof.* The cochain complex $C_b^*(\mathrm{E}\,\Gamma; \mathbb{R})$, together with the canonical augmentation $\mathbb{R} \to C_b^0(\mathrm{E}\,\Gamma; \mathbb{R})$, is a strong relatively injective resolution [Iva1, Section 4], [Fri1, Proposition 4.8]. Therefore, applying the fundamental theorem for this type of homological algebra [Iva1, Section 3], [Fri1, Theorem 4.5] completes the proof. $\square$

## 2.2. $\ell^1$-Homology.

Instead of taking the topological dual functor, one can also take the completion functor. Applying the completion functor to the singular chain complex $C_*( \cdot ; \mathbb{R})$ (with the $\ell^1$-norm) leads to the $\ell^1$-*chain complex* $C_*^{\ell^1}( \cdot ; \mathbb{R})$ and, after taking homology, to $\ell^1$-*homology* $H_*^{\ell^1}( \cdot ; \mathbb{R})$.

While the duality between $\ell^1$-homology and bounded cohomology is not as straightforward as in the case of singular (co)homology, we still have the following tools:

**Theorem 2.5** (Translation principle [Löh1, Theorem 1.1]). *If $f_*: C_* \to D_*$ is a morphism of Banach chain complexes, then $H^*(B(f_*, \mathbb{R})): H^*(B(D_*, \mathbb{R})) \to H^*(B(C_*, \mathbb{R}))$ is an isomorphism if and only if $H_*(f_*): H_*(C_*) \to H_*(D_*)$ is an isomorphism.*

**Corollary 2.6** (Mapping theorem for $\ell^1$-homology [Löh1, Corollary 5.2][Bou, Corollaire 5]). *Let $f: X \to Y$ be a continuous map between path-connected spaces such that $\pi_1(f): \pi_1(X) \to \pi_1(Y)$ is surjective and has amenable kernel. Then $H_*^{\ell^1}(f; \mathbb{R}): H_*^{\ell^1}(X; \mathbb{R}) \to H_*^{\ell^1}(Y; \mathbb{R})$ is an (isometric) isomorphism.*

# 3. Preliminaries: Classifying spaces of families of subgroups

**3.1. Classifying spaces.** We briefly recall basic terminology concerning classifying spaces of families of subgroups; further information can, e.g., be found in Lück's survey [Lüc2].

**Definition 3.1** (Subgroup family). Let $\Gamma$ be a group. A *subgroup family* of $\Gamma$ is a set $F$ of subgroups of $\Gamma$ with the following properties:

- The set $F$ is closed under conjugation.

- The set $F$ is closed under taking subgroups.

**Definition 3.2** (Classifying space of a subgroup family). Let $\Gamma$ be a group and let $F$ be a subgroup family of $\Gamma$.

- A $\Gamma$-CW-complex has $F$-*restricted isotropy* if all isotropy groups lie in $F$.

- A *model for* $\mathrm{E}_F\,\Gamma$ is a $\Gamma$-CW-complex $X$ with $F$-restricted isotropy and the following universal property: For every $\Gamma$-CW-complex $Y$ with $F$-restricted isotropy, there exists up to $\Gamma$-homotopy exactly one continuous $\Gamma$-map $Y \to X$.

  We will also abuse the symbol $\mathrm{E}_F\,\Gamma$ to denote a choice of a model for $\mathrm{E}_F\,\Gamma$ (this is well-defined up to canonical $\Gamma$-homotopy equivalence) and $f_{Y,\Gamma,F}\colon Y \to \mathrm{E}_F\,\Gamma$ for a choice of a ("the") continuous $\Gamma$-map.

If $F$ is the family that only contains the trivial subgroup of $\Gamma$, then $\mathrm{E}_F\,\Gamma = \mathrm{E}\,\Gamma$, and we abbreviate $f_{Y,\Gamma} := f_{Y,\Gamma,F}$.

**Theorem 3.3** (Alternative characterisation of classifying spaces [Lüc2, Theorem 1.9]). *Let $\Gamma$ be a group and let $F$ be a subgroup family of $\Gamma$.*

(1) *Then there exists a model for $\mathrm{E}_F\,\Gamma$.*

(2) *A $\Gamma$-CW-complex $Y$ is a model for $\mathrm{E}_F\,\Gamma$ if and only if the following conditions are satisfied:*

  - *The $\Gamma$-CW-complex $Y$ has $F$-restricted isotropy.*

  - *For every subgroup $H \in F$, the $H$-fixed point set $X^H$ is weakly contractible.*

**3.2. The family of amenable subgroups.** Classically, key examples are given by the family that consists solely of the trivial subgroup (leading to the ordinary classifying space) and the subgroup family of all finite subgroups (leading for discrete groups to the classifying space of proper actions). In the setting of bounded cohomology, it is natural to work with the extension of finite groups to amenable groups:

**Example 3.4.** Let $\Gamma$ be a group and let Am be the set of all amenable subgroups of $\Gamma$. Then Am is a subgroup family of $\Gamma$ in the sense of Definition 3.1.

**3.3. An example.** We explain an explicit model of $\mathsf{E}_{\mathsf{Am}}\,\Gamma$ for the free group $\Gamma = F_2$ of rank 2. In this case, $\mathsf{Am}$ is the family of cyclic subgroups of $F_2$. The following lemma holds, suitably modified, in the greater generality of word-hyperbolic groups [JPL, Remark 7].

**Lemma 3.5.** *Every cyclic subgroup of $F_2$ is contained in a maximal cyclic subgroup. The normaliser of a maximal cyclic subgroup $C$ of $F_2$ is $C$ itself.*

If $K, H < \Gamma$ are subgroups of a group $\Gamma$ and $X$ is an $H$-space, then the $K$-fixed points of the induced $\Gamma$-space $\Gamma \times_H X$ are

$$(1) \qquad \left(\Gamma \times_H X\right)^K = \left\{ [\gamma, x] \,\middle|\, \gamma^{-1} K \gamma \subset H, \ x \in X^{\gamma^{-1} K \gamma} \right\}.$$

**Example 3.6** (A model of $\mathsf{E}_{\mathsf{Am}}\,F_2$). Let $\mathsf{MCyc}$ be a complete set of representatives of conjugacy classes of maximal cyclic subgroups in $F_2$. Every subgroup $C \in \mathsf{MCyc}$ is isomorphic to $\mathbb{Z}$. Hence we can take $\mathbb{R}$ as a model of $\mathsf{E}\,C$ for every $C \in \mathsf{MCyc}$ on which $C \cong \mathbb{Z}$ acts by translations. We pick the 4-regular tree $T$ as a model of $\mathsf{E}\,F_2$. A 2-dimensional model $Y$ of the classifying space $\mathsf{E}_{\mathsf{Am}}\,F_2$ is given by the pushout of $F_2$-spaces:

$$
\begin{array}{ccc}
\coprod_{C \in \mathsf{MCyc}} F_2 \times_C \mathbb{R} & \xrightarrow{\ c\ } & T \\
\downarrow & & \downarrow \\
\coprod_{C \in \mathsf{MCyc}} F_2 \times_C \mathrm{cone}(\mathbb{R}) & \longrightarrow & Y
\end{array}
$$

Here $\mathrm{cone}(\mathbb{R})$ is the cone over the free $C$-space $\mathbb{R}$. The $C$-action on $\mathbb{R}$ naturally extends to the cone in such a way that the cone tip is a fixed point. The map $c$ is the classifying map for $T$ as a model of $\mathsf{E}\,F_2$. The left vertical map is induced by the inclusion of the bottom into the cone.

According to Theorem 3.3 we have to show that $Y^C$ is contractible for every cyclic subgroup $C < F_2$ and is empty for every non-cyclic subgroup $C < F_2$. Let $K < F_2$ be a non-trivial subgroup. We obviously have $T^K = \varnothing$ and by (1) also

$$
\left( \coprod_{C \in \mathsf{MCyc}} F_2 \times_C \mathbb{R} \right)^K = \coprod_{C \in \mathsf{MCyc}} \left( F_2 \times_C \mathbb{R} \right)^K = \varnothing.
$$

Since taking $K$-fixed points respects the pushout property [tDie, (1.17) exercise 5 on p. 103] we obtain that

$$
Y^K \cong \left( \coprod_{C \in \mathsf{MCyc}} F_2 \times_C \mathrm{cone}(\mathbb{R}) \right)^K = \coprod_{C \in \mathsf{MCyc}} \left( F_2 \times_C \mathrm{cone}(\mathbb{R}) \right)^K.
$$

If $K$ is not cyclic, then it follows from (1) that $Y^K$ is empty. Let $K$ be cyclic. Let $C_0 \in \mathrm{MCyc}$ be the unique element such that a conjugate $\gamma_0 C_0 \gamma_0^{-1}$, $\gamma_0 \in F_2$, is the unique maximal cyclic subgroup containing $K$ (Lemma 3.5). Further, $\gamma_0$ is uniquely determined up to multiplication with elements in the normaliser of $C_0$, which equals $C_0$. Then (1) implies that

$$Y^K = \left( F_2 \times_{C_0} \mathrm{cone}(\mathbb{R}) \right)^K = \left\{ [\gamma_0, \mathrm{cone\ tip}] \right\}$$

consists of a single point. It remains to show that $Y^K = Y$ is contractible for $K = \{1\}$. We only sketch the argument: The three spaces defining $Y$ in the above pushout have contractible path components. Hence $Y$ is acyclic. Using the van Kampen theorem one verifies that $Y$ is simply connected. Whitehead's theorem then implies that $Y$ is contractible.

The above example can be generalized to word-hyperbolic groups [JPL].

## 4. Preliminaries: Nerves of covers

In the following, we discuss nerves of open covers as well as their lifts to universal coverings. In order to keep the notation simple, we will view open covers as *sets* of open subsets of the given ambient space, not as families of subsets.

**Setup 4.1.** Let $X$ be a path-connected CW-complex with universal covering $\pi \colon \widetilde{X} \to X$, let $x_0 \in X$, and let $\Gamma := \pi_1(X, x_0)$ be the fundamental group of $X$. Moreover, let $U$ be an open cover of $X$ by path-connected sets and let

$$\widetilde{U} := \left\{ V \subset \widetilde{X} \mid \text{there exists a } W \in U \text{ such that} \right.$$
$$\left. V \text{ is a path-connected component of } \pi^{-1}(W) \right\}$$

be the associated cover of $\widetilde{X}$.

**Remark 4.2.** In the situation of Setup 4.1, let $V \in \widetilde{U}$. Then $V$ is open (as $X$ is locally path-connected and standard lifting properties in coverings apply).

Moreover, $\pi(V) \in U$: By construction, there is a $W \in U$ such that $V$ is a path-connected component of $\pi^{-1}(W)$. In particular, $\pi(V) \subset W$. In fact, we have $\pi(V) = W$. Indeed, let $x \in W$. Because $V$ is non-empty and $W$ is path-connected, there is a continuous path $w \colon [0,1] \to W$ with $w(0) \in \pi(V)$ and $w(1) = x$. Then the lifting properties of the covering $\pi|_{\pi^{-1}(W)} \colon \pi^{-1}(W) \to W$ show that there is a continuous $\pi$-lift $\widetilde{w} \colon [0,1] \to \pi^{-1}(W)$ with $\widetilde{w}(0) \in V$.

Because $\widetilde{w}\,([0,1])$ is path-connected and $V$ is a path-connected component of $\pi^{-1}(W)$, we obtain $\widetilde{w}\,(1) \in V$. In particular,

$$x = w(1) = \pi \circ \widetilde{w}\,(1) \in \pi(V).$$

**Definition 4.3** (Convex cover). An open cover $U$ of a topological space $X$ is called *convex* if for every finite set $U' \subset U$, the intersection $\bigcap U'$ is path-connected (or empty).

### 4.1. Nerves and equivariance.

**Definition 4.4** (Nerve). In the situation of Setup 4.1, the *nerve of $U$* is the (abstract) simplicial complex $N$ given by the following data: For each $n \in \mathbb{N}$, the set of $n$-simplices of $N$ is

$$\left\{ \{V_0, \ldots, V_n\} \;\middle|\; V_0, \ldots, V_n \in U, \; \bigcap_{j=0}^{n} V_j \neq \varnothing, \; \forall_{j,k \in \{0,\ldots,n\}, j \neq k} \;\; V_j \neq V_k \right\}.$$

**Lemma 4.5** (Actions on nerves). *In the situation of Setup 4.1, let $N$ be the nerve of $U$ and let $\widetilde{N}$ be the nerve of $\widetilde{U}$. Then:*

(1) *The deck transformation action of $\Gamma$ on $\widetilde{X}$ turns the nerve $\widetilde{N}$ of the induced cover $\widetilde{U}$ into a $\Gamma$-simplicial complex.*

(2) *The universal covering map $\pi \colon \widetilde{X} \to X$ induces a well-defined simplicial map $p \colon \widetilde{N} \to N$.*

(3) *If the open cover $U$ is convex and $n \in \mathbb{N}$, then $p$ induces a bijection between $\Gamma \backslash (\widetilde{N})_n$ and $N_n$. Here, $(\widetilde{N})_n$ carries the diagonal $\Gamma$-action.*

*Proof. Ad 1.* By construction, the set $\widetilde{U}$ is closed under the deck transformation action of $\Gamma$. Moreover, this $\Gamma$-action is compatible with the simplicial structure (because homeomorphisms preserve intersections).

*Ad 2.* For every $V \in \widetilde{U}$, we have $\pi(V) \in U$ (Remark 4.2).

Let $n \in \mathbb{N}$ and let $\{V_0, \ldots, V_n\}$ be an $n$-simplex of $\widetilde{N}$. Then, the projections $\pi(V_0), \ldots, \pi(V_n)$ are pairwise different (because $V_0 \cap \cdots \cap V_n \neq \varnothing$ and elements of $\widetilde{U}$ that lie over the same set in $U$ have empty intersection). Moreover,

$$\bigcap_{j=0}^{n} \pi(V_j) \supset \pi\left( \bigcap_{j=0}^{n} V_j \right) \neq \varnothing.$$

Hence, $n$-simplices are mapped to $n$-simplices.

*Ad 3.* Let $n \in \mathbb{N}$ and let $\{W_0, \ldots, W_n\}$ be an $n$-simplex of $N$.

- Then there exists an $n$-simplex $\{V_0, \ldots, V_n\}$ of $\widetilde{N}$ with

$$p(\{V_0, \ldots, V_n\}) = \{W_0, \ldots, W_n\}.$$

  This can be seen as follows: Let $x \in \bigcap_{j=0}^{n} W_j$ and let $\widetilde{x} \in \pi^{-1}(\{x\})$. Then, for each $j \in \mathbb{N}$, we choose the element $V_j \in \widetilde{U}$ with $\widetilde{x} \in V_j$ and $\pi(V_j) = W_j$. By construction, the intersection $\bigcap_{j=0}^{n} V_j$ contains $\widetilde{x}$ and thus is non-empty. Therefore, $\{V_0, \ldots, V_n\}$ is an $n$-simplex of $\widetilde{N}$.

- If $\{V_0', \ldots, V_n'\}$ is another $n$-simplex of $\widetilde{N}$ with $p(\{V_0', \ldots, V_n'\}) = \{W_0, \ldots, W_n\}$, then there exists a $\gamma \in \Gamma$ with

$$\forall_{j \in \{0, \ldots, n\}} \quad V_j = \gamma \cdot V_j',$$

  because: Let $x \in \bigcap_{j=0}^{n} V_j$ and $y \in \bigcap_{j=0}^{n} V_j'$. As $U$ is a convex open cover, the intersection $\bigcap_{j=0}^{n} W_j$ is path-connected. Let $w \colon [0,1] \to \bigcap_{j=0}^{n} W_j$ be a path from $\pi(x)$ to $\pi(y)$ and let $\widetilde{w} \colon [0,1] \to \widetilde{X}$ be a $\pi$-lift of $w$. Then $\widetilde{w}([0,1]) \subset V_j$ for each $j \in \{0, \ldots, n\}$ and there exists a $\gamma \in \Gamma$ with

$$\gamma \cdot y = \widetilde{w}(1).$$

  By construction, $\widetilde{w}(1) \in \bigcap_{j=0}^{n} \gamma \cdot V_j'$. Therefore, for all $j \in \{0, \ldots, n\}$, we have $V_j \cap \gamma \cdot V_j' \neq \varnothing$ and so $V_j = \gamma \cdot V_j'$. $\qquad\square$

**Proposition 4.6.** *In the situation of Setup 4.1, let $U$ be convex, let $N$ be the nerve of $U$, and let $\widetilde{N}$ be the nerve of $\widetilde{U}$. Then the map $p \colon \widetilde{N} \to N$ induced by $\pi$ (Lemma 4.5) induces a chain homotopy equivalence*

$$C_*(|p|)_\Gamma \colon C_*(|\widetilde{N}|)_\Gamma \to C_*(|N|).$$

*Proof.* By the previous lemma (Lemma 4.5), the simplicial map $p$ induces a chain isomorphism $C_*(\widetilde{N})_\Gamma \to C_*(N)$ between the *simplicial* chain complexes.

Moreover, the canonical inclusion $i \colon C_*(N) \to C_*(|N|)$ is a chain homotopy equivalence and the canonical inclusion $\widetilde{i} \colon C_*(\widetilde{N}) \to C_*(|\widetilde{N}|)$ is a $\Gamma$-chain homotopy equivalence [Lüc1, Proposition 13.10 b) on p. 264]. It should be noted that this is the step in the proof of Theorem 1.1, where we lose control over the norms.

Therefore, the commutative diagram

$$
\begin{array}{ccc}
C_*(|\widetilde{N}|)_\Gamma & \xrightarrow{\ C_*(|p|)_\Gamma\ } & C_*(|N|) \\
{\scriptstyle \widetilde{i}_\Gamma} \Big\uparrow {\scriptstyle \simeq} & & {\scriptstyle \simeq} \Big\uparrow {\scriptstyle i} \\
C_*(\widetilde{N})_\Gamma & \xrightarrow[\ C_*(p)_\Gamma\ ]{\simeq} & C_*(N)
\end{array}
$$

proves the claim. $\qquad\square$

## 4.2. The nerve map.

**Remark 4.7** (Nerve map [Dol, p. 355]). In the situation of Setup 4.1, the space $X$ admits a partition of unity subordinate to the open cover $U$ (because CW-complexes are Hausdorff and paracompact). Every partition of unity $(\varphi_V)_{V \in U}$ of $X$ that is subordinate to $U$ gives rise to a continuous map

$$\nu \colon X \to |N|$$

into the geometric realisation $|N|$ of the nerve $N$ of $U$: For $x \in X$, we just set (in barycentric coordinates)

$$\nu(x) := \sum_{V \in U} \varphi_V(x) \cdot V.$$

Different choices of partitions of unity lead to homotopic maps. We therefore speak of *the nerve map* $X \to |N|$.

**Lemma 4.8** (Lifting the nerve map). *In the situation of Setup* 4.1, *let* $N$ *be the nerve of* $U$, *let* $\widetilde{N}$ *be the nerve of* $\widetilde{U}$, *let* $|p| \colon |\widetilde{N}| \to |N|$ *be the map induced by* $\pi$ (*Lemma* 4.5), *and let* $\nu \colon X \to |N|$ *be a nerve map. Then there exists a continuous* $\Gamma$-*map* $\widetilde{\nu} \colon \widetilde{X} \to |\widetilde{N}|$ *with*

$$|p| \circ \widetilde{\nu} = \nu \circ \pi.$$

$$
\begin{array}{ccc}
\widetilde{X} & \overset{\widetilde{\nu}}{\dashrightarrow} & |\widetilde{N}| \\
{\scriptstyle \pi}\big\downarrow & & \big\downarrow{\scriptstyle |p|} \\
X & \underset{\nu}{\longrightarrow} & |N|
\end{array}
$$

*Proof.* Let $(\varphi_W)_{W \in U}$ be a partition of unity of $X$ that is subordinate to $U$ (which induces the nerve map $\nu$). We construct $\widetilde{\nu}$ as nerve map of the lift of this partition of unity to $\widetilde{U}$: For $V \in \widetilde{U}$, we define

$$\widetilde{\varphi}_V := \chi_V \cdot \varphi_{\pi(V)} \circ \pi \colon \widetilde{X} \to [0,1],$$

where $\chi_V \colon \widetilde{X} \to \{0,1\}$ denotes the characteristic function of the subset $V \subset \widetilde{X}$. We will now establish the following properties of these functions:

(1)  The function $\widetilde{\varphi}_V \colon \widetilde{X} \to [0,1]$ is continuous.

   (Because $X$ is locally path-connected, the standard lifting properties show that $\pi(\partial V) = \partial(\pi(V))$. Now continuity of $\widetilde{\varphi}_V$ easily follows from the continuity of $\varphi_{\pi(V)}$.)

(2)  For each $\gamma \in \Gamma$, we have $\widetilde{\varphi}_V(\gamma \cdot x) = \widetilde{\varphi}_{\gamma^{-1} \cdot V}(x)$.

(This equivariance property follows directly from the construction.)

(3)  The family $(\widetilde{\varphi}_V)_{V \in \widetilde{U}}$ is a partition of unity on $\widetilde{X}$, subordinate to $\widetilde{U}$.

(This is a straightforward computation.)

If $W \in U$, then we abbreviate $\widetilde{U}|_W := \{V \in \widetilde{U} \mid \pi(V) = W\}$. Then

$$\widetilde{v}: \widetilde{X} \to |\widetilde{N}|$$

$$x \mapsto \sum_{W \in U} \sum_{V \in \widetilde{U}|_W} \widetilde{\varphi}_V(x) \cdot V$$

is the nerve map associated with this partition of unity $(\widetilde{\varphi}_V)_{V \in \widetilde{U}}$. The second property shows that

$$\widetilde{v}(\gamma \cdot x) = \sum_{W \in U} \sum_{V \in \widetilde{U}|_W} \widetilde{\varphi}_V(\gamma \cdot x) \cdot V$$

$$= \sum_{W \in U} \sum_{V \in \widetilde{U}|_W} \widetilde{\varphi}_{\gamma^{-1} \cdot V}(x) \cdot V$$

$$= \sum_{W \in U} \sum_{V \in \widetilde{U}|_W} \widetilde{\varphi}_V(x) \cdot (\gamma \cdot V)$$

$$= \gamma \cdot \widetilde{v}(x)$$

holds for all $x \in \widetilde{X}$ and all $\gamma \in \Gamma$. Hence, $\widetilde{v}$ is a continuous $\Gamma$-map.

Moreover, by construction, we have that $|p| \circ \widetilde{v} = v \circ \pi$.  □

In summary, we can pass from the nerve $N$ to the $\Gamma$-equivariant setting (and whence to the right context for classifying spaces) as follows:

**Proposition 4.9.** *In the situation of Setup* 4.1, *let* $U$ *be convex, let* $N$ *be the nerve of* $U$, *let* $\widetilde{N}$ *be the nerve of* $\widetilde{U}$, *let* $p: \widetilde{N} \to N$ *be the map induced by* $\pi$ *(Lemma* 4.5*), let* $v: X \to |N|$ *be a nerve map, and let* $\widetilde{v}: \widetilde{X} \to |\widetilde{N}|$ *be as in Lemma* 4.8. *Then there exists a chain map* $\tau_p: C_*(|N|) \to C_*(|\widetilde{N}|)_\Gamma$ *such that*

$$\tau_p \circ C_*(v) \circ C_*(\pi)_\Gamma \simeq C_*(\widetilde{v})_\Gamma: C_*(\widetilde{X})_\Gamma \to C_*(|\widetilde{N}|)_\Gamma.$$

*Proof.* By Lemma 4.8, we have $|p| \circ \widetilde{v} = v \circ \pi$, whence

$$C_*(|p|)_\Gamma \circ C_*(\widetilde{v})_\Gamma = C_*(v) \circ C_*(\pi)_\Gamma.$$

In view of Proposition 4.6, we can now take a chain homotopy inverse of $C_*(|p|)_\Gamma: C_*(|\widetilde{N}|)_\Gamma \to C_*(|N|)$ for $\tau_p$.  □

## 4.3. Families and covers.

**Definition 4.10** (Amenable cover). In the situation of Setup 4.1 (which includes that all sets in $U$ are path-connected), let $F$ be a subgroup family of $\Gamma$. We call $U$ an $F$-*cover* of $X$ if for each $V \in U$ and each $x \in V$, the subgroup

$$\mathrm{im}\big(\pi_1(V, x) \to \pi_1(X, x)\big) \subset \pi_1(X, x)$$

induced by the inclusion $V \hookrightarrow X$ lies in $F$ under an isomorphism $\pi_1(X, x) \cong \pi_1(X, x_0) = \Gamma$ induced by conjugation with a path between $x$ and $x_0$ (because $F$ is closed under conjugation in $\Gamma$, this property does *not* depend on the chosen paths).

We call $U$ an *amenable cover* if $U$ is an Am-cover.

**Lemma 4.11** (Nerves of amenable covers). *In the situation of Setup* 4.1, *let $F$ be a subgroup family of $\Gamma$ and let $U$ be an $F$-cover. Then the isotropy groups of the corresponding $\Gamma$-space $|\widetilde{N}|$ all lie in $F$.*

*In particular, if $F = \mathrm{Am}$, then these isotropy groups are amenable.*

*Proof.* Because the $\Gamma$-action on $|\widetilde{N}|$ (Lemma 4.5) is obtained from the simplicial $\Gamma$-action on $\widetilde{N}$ by affine extension, it suffices to show that the isotropy groups of the vertices in the barycentric subdivision $S$ of $\widetilde{N}$ with the induced simplicial $\Gamma$-action all lie in $F$.

By definition of the barycentric subdivision, the vertex set of $S$ is the set of simplices of $N$. Let $v$ be a vertex of $S$; i.e., there exist $n \in \mathbb{N}$, $V_0, \ldots, V_n \in \widetilde{U}$ with $V_0 \cap \cdots \cap V_n \neq \varnothing$ and $v = \{V_0, \ldots, V_n\}$. Then the stabiliser $\Gamma_v$ of $v$ consists precisely of those $\gamma \in \Gamma$ with

$$\{\gamma \cdot V_0, \ldots, \gamma \cdot V_n\} = \{V_0, \ldots, V_n\}.$$

We distinguish the following cases:

- If $n = 0$, then the stabiliser of $v$ is

$$\{\gamma \in \Gamma \mid \gamma \cdot V_0 = V_0\},$$

  which is (conjugate to) a subgroup of $\mathrm{im}(\pi_1(\pi(V_0) \hookrightarrow X))$. Because $\pi(V_0) \in U$ (Remark 4.2) and $U$ is an $F$-cover, the stabiliser of $v$ lies in $F$.

- Let $n > 0$. If $\gamma \in \Gamma$ is in the stabiliser of $v$ and $j \in \{0, \ldots, n\}$ with $\gamma \cdot V_j = V_k$, then $j = k$, which can be seen as follows: Because of $\gamma \cdot V_j = V_k$, we have $\pi(V_j) = \pi(V_k)$. Therefore, $V_j$ and $V_k$ are path-connected components of the $\pi$-preimage of the same element of $U$

(Remark 4.2). On the other hand, $V_j \cap V_k \neq \varnothing$. Therefore, $V_j = V_k$, and so $j = k$.

In particular, the stabiliser $\Gamma_v$ is a subgroup of $\Gamma_{V_0} \cap \cdots \cap \Gamma_{V_n}$. The first case shows that $\Gamma_{V_0} \in F$. As $F$ is a subgroup family, also the subgroup $\Gamma_v$ lies in $F$.                                                                    $\square$


## 5. Proof of Theorem 1.1

For the proof of Theorem 1.1, we will first recall that the family of amenable subgroups can be used to compute bounded cohomology; more generally, we introduce the notion of $H_b^*$-admissible subgroup families (and then show that Am is such a family).

As second step, we will combine $H_b^*$-admissibility with the universal property of classifying spaces of families to obtain the factorisation over the cohomological nerve map.

### 5.1. Admissible families of subgroups.

**Definition 5.1.** Let $\Gamma$ be a group, let $F$ be a subgroup family of $\Gamma$. We consider the induced map

$$H^*\big(C_b^*(f_{\mathrm{E}\,\Gamma,\Gamma,F})^\Gamma\big)\colon H^*\big(C_b^*(\mathrm{E}_F\,\Gamma)^\Gamma\big) \to H_b^*\big(C_b^*(\mathrm{E}\,\Gamma)^\Gamma\big) =: H_b^*(\Gamma).$$

The notation $f_{\mathrm{E}\,\Gamma,\Gamma,F}$ is introduced in Definition 3.2. The family $F$ is

- $H_b^*$-*admissible* if $H^*\big(C_b^*(f_{\mathrm{E}\,\Gamma,\Gamma,F})^\Gamma\big)$ is surjective.

- strongly $H_b^*$-*admissible* if $H^*\big(C_b^*(f_{\mathrm{E}\,\Gamma,\Gamma,F})^\Gamma\big)$ is bijective.

Clearly, every subfamily of an $H_b^*$-admissible family is also $H_b^*$-admissible. Moreover, if $\Gamma$ is a group with $H_b^*(\Gamma) \cong H_b^*(1)$, then every subgroup family of $\Gamma$ is $H_b^*$-admissible (for trivial reasons); examples of such groups are all mitotic groups [Löh3].

**Proposition 5.2.** *Let $\Gamma$ be a group. Then the family* Am *of amenable subgroups of $\Gamma$ is strongly $H_b^*$-admissible.*

*Proof.* The isotropy groups of the set of singular $n$-simplices of $\mathrm{E}_{\mathsf{Am}}\,\Gamma$ are amenable since the isotropy group of $\sigma\colon \Delta^n \to \mathrm{E}_{\mathsf{Am}}\,\Gamma$ is the intersection of isotropy groups of points in the image of $\sigma$. Hence $C_b^n(\mathrm{E}_{\mathsf{Am}}\,\Gamma)$ is a relatively injective Banach $\Gamma$-module for every $n \geq 0$ (Proposition 2.3). Since $\mathrm{E}_{\mathsf{Am}}\,\Gamma$ is also contractible (Theorem 3.3), $C_b^*(\mathrm{E}_{\mathsf{Am}}\,\Gamma)$, together with the canonical

augmentation $\mathbb{R} \to C_b^0(E_{Am} \Gamma)$ by constant functions, is a strong relatively injective resolution of $\mathbb{R}$ by Banach $\Gamma$-modules.

Therefore, the fundamental theorem for this type of homological algebra (Theorem 2.4) shows that the bounded $\Gamma$-cochain map

$$C_b^*(f_{E\Gamma,\Gamma,F})^\Gamma \colon C_b^*(E_{Am} \Gamma)^\Gamma \to C_b^*(E \Gamma)^\Gamma$$

induces an isomorphism in bounded cohomology. In fact, this isomorphism is even isometric [Fri1, Theorem 4.23]. $\square$

This proposition is the "group-theoretic essence" of Theorem 1.1.

### 5.2. Bounded cohomology of admissible covers.

Using the notion of $H_b^*$-admissibility, we have the following version of Theorem 1.1, leading to a more conceptual understanding of this phenomenon:

**Theorem 5.3.** *Let $\Gamma$ be a group and let $F$ be an $H_b^*$-admissible family of subgroups of $\Gamma$. Let $X$ be a path-connected CW-complex with $\pi_1(X) \cong \Gamma$, and let $U$ be an open $F$-cover of $X$. Then the following hold:*

(1) *If $U$ is convex, there exists a factorisation*

$$
\begin{array}{ccc}
H_b^*(X;\mathbb{R}) & \xrightarrow{\; c_X \;} & H^*(X;\mathbb{R}) \\
& \searrow & \big\uparrow {\scriptstyle H^*(v;\mathbb{R})} \\
& & H^*(|N|;\mathbb{R})
\end{array}
$$

*of the comparison map $c_X$ through the nerve map $v \colon X \to |N|$ of $U$.*

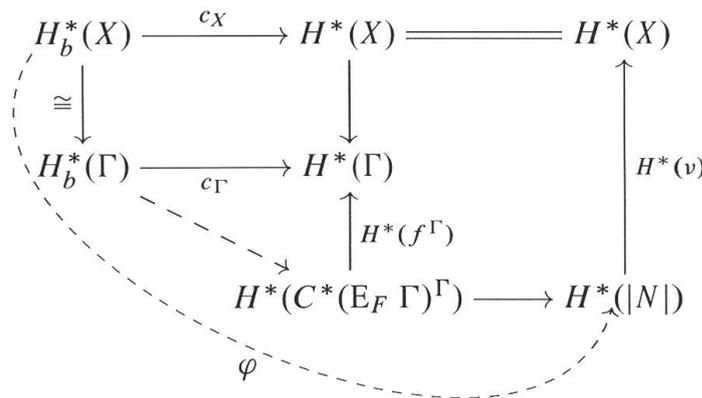(2) *If the multiplicity of $U$ is at most $m$, then $c_X$ vanishes in all degrees $* \geq m$.*

*Proof.* We pick a basepoint $x_0 \in X$ and consider the group $\Gamma := \pi_1(X, x_0)$. Now we can invoke the classifying spaces in the following way:

- As the universal covering $\widetilde{X}$ of $X$ is a $\Gamma$-CW-complex with free $\Gamma$-action (via the deck transformation action), there is a continuous $\Gamma$-map $f_{\widetilde{X},\Gamma} \colon \widetilde{X} \to E\Gamma$.

- Because $U$ is an open $F$-cover , the geometric realisation $|\widetilde{N}|$ of the associated open cover of $\widetilde{X}$ is a $\Gamma$-CW-complex with $F$-restricted isotropy (Lemma 4.11). Therefore, we obtain a corresponding continuous $\Gamma$-map $f_{|\widetilde{N}|,\Gamma,F} \colon |\widetilde{N}| \to E_F \Gamma$.

- Moreover, we have the continuous $\Gamma$-map $f_{E\Gamma,\Gamma,F} \colon E\Gamma \to E_F \Gamma$.

*Ad 1.* To this end, we consider the diagram in Figure 3 and explain why it commutes up to cochain homotopy. The squares on the left-hand side clearly commute.

The universal property of $E_F \Gamma$ implies that the rectangle in the middle commutes up to cochain homotopy: Both $f_{E \Gamma, \Gamma, F} \circ f_{\widetilde{X}, \Gamma}$ and $f_{|\widetilde{N}|, \Gamma, F} \circ \widetilde{\nu}$ are $\Gamma$-equivariant continuous maps $\widetilde{X} \to E_F \Gamma$, which thus have to be $\Gamma$-homotopic. The right polygon is commutative by (the algebraic dual of) Proposition 4.9.

Taking cohomology leads to the solid part of the following commutative diagram:

$$
\begin{array}{ccccc}
H_b^*(X) & \xrightarrow{c_X} & H^*(X) & == & H^*(X) \\
{\scriptstyle \cong}\downarrow & & \downarrow & & \uparrow {\scriptstyle H^*(\nu)} \\
H_b^*(\Gamma) & \xrightarrow{c_\Gamma} & H^*(\Gamma) & & \\
& & \uparrow {\scriptstyle H^*(f^\Gamma)} & & \\
& & H^*(C^*(E_F \Gamma)^\Gamma) & \longrightarrow & H^*(|N|) \\
& \varphi & & &
\end{array}
$$

The left vertical arrow exists and is an isomorphism by Gromov's mapping theorem (Theorem 2.2) (because $f_{\widetilde{X}, \Gamma}$ induces an isomorphism on bounded cohomology). Moreover, because $F$ is $H_b^*$-admissible, we can easily fill in the dashed diagonal arrow to obtain a commutative triangle. Hence, taking the dotted composition gives us the claimed factorisation $\varphi$.

*Ad 2.* Without the convexity assumption we still have a similar commutative diagram as above with the right vertical map $H^*(\nu)$ replaced by

$$
H^*(\pi)^{-1} \circ H^*(\widetilde{\nu}) \colon H^*\big(C^*(|\widetilde{N}|)^\Gamma\big) \to H^*(X).
$$

Under the assumption on multiplicity, the dimension of the simplicial complex $\widetilde{N}$ satisfies

$$
\dim \widetilde{N} \le \dim N < m.
$$

Furthermore, the simplicial chain complex $C_*(\widetilde{N})$ and the singular chain complex $C_*(|\widetilde{N}|)$ are equivariantly chain homotopic [Lüc1, Proposition 13.10 b) on p. 264]. Hence $H^*(C^*(|\widetilde{N}|)^\Gamma)$ vanishes in degrees $* \ge m$ and statement (2) follows. $\qquad\square$

The proof shows that we need even less than $H_b^*$-admissibility: It suffices that the composition $H_b^*(\Gamma; F) \to H_b^*(\Gamma) \to H^*(\Gamma)$ of the canonical map with the comparison map hits all bounded classes in $H^*(\Gamma)$.

$$
\begin{array}{ccccc}
C_b^*(X) & \lhook\joinrel\longrightarrow & C^*(X) & =\!=\!=\!=\!=\!= & C^*(X) \\
\pi \downarrow \cong & & \cong \downarrow \pi & & \uparrow \\
C_b^*(\widetilde{X})^\Gamma & \lhook\joinrel\longrightarrow & C^*(\widetilde{X})^\Gamma & =\!=\!= & C^*(\widetilde{X})^\Gamma \\
\uparrow f_{\widetilde{X},\Gamma} & & \uparrow f_{\widetilde{X},\Gamma} & & \uparrow \nu \\
C_b^*(\mathrm{E}\,\Gamma)^\Gamma & \lhook\joinrel\longrightarrow & C^*(\mathrm{E}\,\Gamma)^\Gamma & \widetilde{\nu} \uparrow & \\
& & \uparrow f_{\mathrm{E}\,\Gamma,\Gamma,F} & & \\
& & C^*(\mathrm{E}_F\,\Gamma)^\Gamma \xrightarrow[f_{\widetilde{|N|},\Gamma,F}]{} C^*(|\widetilde{N}|)^\Gamma_{\mathrm{Hom}_\mathbb{R}(\tau_p,\mathbb{R})} \xrightarrow{\simeq} C^*(|N|)
\end{array}
$$

FIGURE 3

Proof of Theorem 5.3; for improved readability, cochain maps induced by continuous maps only carry the underlying continuous map as label.

**5.3. Proof of Theorem 1.1.** As a special case, we obtain Theorem 1.1: By Proposition 5.2, the family Am of amenable subgroups is $H_b^*$-admissible. Therefore, Theorem 5.3 is applicable.

# 6. The case of $\ell^1$-homology

We now explain how to derive the $\ell^1$-analogues of Proposition 5.2 and Theorem 5.3, and whence prove Theorem 1.2.

## 6.1. $\ell^1$-admissiblity.

**Definition 6.1.** Let $\Gamma$ be a group, let $F$ be a subgroup family of $\Gamma$, and let $f_{\mathrm{E}\,\Gamma,\Gamma,F}\colon \mathrm{E}\,\Gamma \to \mathrm{E}_F\,\Gamma$ be the canonical map; we consider the induced map

$$
H_*\big(C_*^{\ell^1}(f_{\mathrm{E}\,\Gamma,\Gamma,F})_\Gamma\big)\colon H_*^{\ell^1}(\Gamma) := H_*\big(C_*^{\ell^1}(\mathrm{E}\,\Gamma)_\Gamma\big) \to H_*\big(C_*^{\ell^1}(\mathrm{E}_F\,\Gamma)_\Gamma\big).
$$

The family $F$ is

- $H_*^{\ell^1}$-admissible if $H_*(C_*^{\ell^1}(f_{\mathrm{E}\,\Gamma,\Gamma,F})_\Gamma)$ is injective.

- strongly $H_*^{\ell^1}$-admissible if $H_*(C_*^{\ell^1}(f_{\mathrm{E}\,\Gamma,\Gamma,F})_\Gamma)$ is bijective.

**Proposition 6.2.**

(1) *Let $\Gamma$ be a group and let $F$ be a subgroup family. If $F$ is strongly $H_b^*$-admissible, then $F$ is strongly $H_*^{\ell^1}$-admissible.*

(2) *In particular, the family* Am *is* $H_*^{\ell^1}$*-admissible.*

*Proof.* The first part follows from the fact that the topological dual of the topological coinvariants are the invariants of the topological dual and the translation principle (Theorem 2.5). The second part is then a consequence of the first part and Proposition 5.2.                                                    □

## 6.2. $\ell^1$-Homology of admissible covers.

**Theorem 6.3.** *Let* $\Gamma$ *be a group and let* $F$ *be an* $H_*^{\ell^1}$*-admissible subgroup family of* $\Gamma$. *Let* $X$ *be a path-connected CW-complex with* $\pi_1(X) \cong \Gamma$, *and let* $U$ *be an open* $F$*-cover of* $X$. *Then the following hold:*

(1) *If* $U$ *is convex, then there exists a factorisation*

$$
\begin{array}{ccc}
H_*(X;\mathbb{R}) & \xrightarrow{\ c_X^{\ell^1}\ } & H_*^{\ell^1}(X;\mathbb{R}) \\
{\scriptstyle H_*(v;\mathbb{R})}\Big\downarrow & \nearrow & \\
H_*(|N|;\mathbb{R}) & &
\end{array}
$$

*of the comparison map* $c_X^{\ell^1}$ *through the nerve map* $v\colon X \to |N|$ *of* $U$.

(2) *If the multiplicity of* $U$ *is at most* $m$, *then* $c_X^{\ell^1}$ *vanishes in all degrees* $* \geq m$.

*Proof.* We can argue exactly as in the proof of Theorem 5.3, by working on the chain level instead of the cochain level; instead of the mapping theorem in bounded cohomology, we use the corresponding mapping theorem in $\ell^1$-homology (Corollary 2.6).                                                    □

## 6.3. Proof of Theorem 1.2.
As in the case of bounded cohomology, Theorem 1.2 now follows from Theorem 6.3 and Proposition 6.2.

# References

[Bou] A. Bouarich, Théorèmes de Zilber–Eilemberg et de Brown en homologie $\ell^1$. *Proyecciones* **23** (2004), 151–186. MR 2142264

[Dol] A. Dold, *Lectures on Algebraic Topology.* Springer, 1980. Zbl 0434.55001 MR 0606196

[FM] R. Frigerio and M. Moraschini, Gromov's theory of multicomplexes with applications to bounded cohomology and simplicial volume. To appear in *Mem. Amer. Math. Soc.*. arXiv:1808.07307 [math.GT] (2018).

[Fri1] R. Frigerio, *Bounded Cohomology of Discrete Groups.* Mathematical Surveys and Monographs, 227, AMS, 2017. Zbl 1398.57003 MR 3726870

[Fri2] R. Frigerio. Amenable covers and $\ell^1$-invisibility. *J. Topol. Anal.* (in print) (2020).

[Gro] M. Gromov, Volume and bounded cohomology. *Publ. Math. IHES* **56** (1992), 5–99. Zbl 0516.53046 MR 0686042

[Iva1] N. V. Ivanov, Foundations of the theory of bounded cohomology. *J. Soviet Math.* **37** (1987), 1090–1114. Zbl 0612.55006 MR 0806562

[Iva2] —— Notes on the bounded cohomology theory. arXiv:1708.05150 [math.AT] (2017).

[JPL] D. Juan-Pineda and I. Leary, On classifying spaces for the family of virtually cyclic subgroups. *Recent Developments in Algebraic Topology, Contemp. Math.* **407** (2006), 135–145. Zbl 1107.19001 MR 2248975

[Ler] J. Leray, Sur la forme des espaces topologiques et sur les points fixes des représentations. *J. Math. Pures Appl.* **24** (1945), 95–167. Zbl 0060.40703 MR 0015786

[Löh1] C. Löh, Isomorphisms in $\ell^1$-homology. *Münster J. Math.* **1** (2008), 237–266. Zbl 1158.55007 MR 2502500

[Löh2] —— $\ell^1$-*Homology and Simplicial Volume.* PhD thesis, WWU Münster, 2007. Zbl 1152.57304

[Löh3] —— A note on bounded-cohomological dimension of discrete groups. *J. Math. Soc. Japan* **69** (2017), 715–734. Zbl 1376.55005 MR 3638282

[Lüc1] W. Lück, *Transformation Groups and Algebraic K-Theory.* Volume 1408 of *Lecture Notes in Mathematics*, Springer, 1989. Zbl 0679.57022 MR 1027600

[Lüc2] —— Survey on classifying spaces for families of subgroups. *Infinite Groups: Geometric, Combinatorial and Dynamical Aspects*, 269–322, *Progress in Mathematics*, 248, Birkhäuser, 2005. Zbl 1117.55013 MR 2195456

[Mon] N. Monod, *Continuous Bounded Cohomology of Locally Compact Groups.* Volume 1758 of *Lecture Notes in Mathematics*, Springer, 2001. Zbl 0967.22006 MR 1840942

[tDie] T. tom Dieck, *Transformation Groups.* De Gruyter Studies in Mathematics, 8. Walter de Gruyter & Co., 1987. Zbl 0611.57002 MR 0889050

[Wei]     A. Weil, Sur les théorèmes de de Rham. *Comment. Math. Helv.* **26** (1952), 119–145. Zbl 0047.16702 MR 0050280

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