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Quantitative equidistribution of horocycle push-forwards of transverse arcs

Davide RAVOTTI

Abstract. Let $M = \Gamma \backslash \mathrm{SL}(2, \mathbb{R})$ be a compact quotient of $\mathrm{SL}(2, \mathbb{R})$ equipped with the normalized Haar measure vol , and let $\{h_t\}_{t \in \mathbb{R}}$ denote the horocycle flow on M . Given $p \in M$ and $W \in \mathfrak{sl}_2(\mathbb{R}) \setminus \{0\}$ not parallel to the generator of the horocycle flow, let γ_p^W denote the probability measure uniformly distributed along the arc $s \mapsto p \exp(sW)$ for $0 \leq s \leq 1$. We establish quantitative estimates for the rate of convergence of $[(h_t)_* \gamma_p^W](f)$ to $\mathrm{vol}(f)$ for sufficiently smooth functions f . Our result is based on the work of Bufetov and Forni [BF], together with a crucial geometric observation. As a corollary, we provide an alternative proof of Ratner's theorem on quantitative mixing for the horocycle flow.

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1. Introduction

The horocycle flow $\{h_t\}_{t \in \mathbb{R}}$ on compact quotients of $\mathrm{SL}(2, \mathbb{R})$ is one of the fundamental examples of parabolic unipotent flows. Its dynamical and ergodic properties are well-understood: it has zero entropy [Gur], it is minimal [Hed], uniquely ergodic [Fur], mixing and mixing of all orders [Ma2], and has countable Lebesgue spectrum [Par, GF].

Quantitative versions of these results have also been investigated. Ratner [Rat] established optimal polynomial mixing rates for Hölder observables. Moreover, it follows from a general result by Björklund, Einsiedler, and Gorodnik [BEG] that, for all $k \geq 2$, mixing of order k is also polynomial. Regarding quantitative equidistribution, Flaminio and Forni [FF] proved precise results on the asymptotics of ergodic averages of smooth functions. The results in [FF] have been refined by Bufetov and Forni in [BF], where the authors construct a family of finitely-additive Hölder measures and associated Hölder functionals which control the asymptotics

of ergodic integrals. The main result of this paper, Theorem 2 below, is based on the properties of these *Bufetov–Forni functionals*, which we recall in §2.3.

The construction of Hölder functionals asymptotic to ergodic integrals and results analogous to [BF] were first obtained by Bufetov [Buf] for another class of parabolic flows, namely for translation flows on higher genus surfaces. Bufetov’s *Hölder cocycles*, introduced in [Buf], are also related to *limit shapes* for Interval Exchange Transformations, which were studied by Marmi, Moussa and Yoccoz [MMY]. More recently, these ideas were extended by Forni and Kanigowski [FK] to the case of Heisenberg nilflows which, similarly to horocycle flows and translation flows, are *renormalizable* parabolic flows.

1.1. Shearing properties of the horocycle flow. One of the key geometric properties of the horocycle flow is a form of *shearing* of transverse arcs. For example, let $\gamma(s) = p \exp(sX)$, $0 \leq s \leq 1$, be the geodesic segment starting at $p \in M$ of length 1. From the usual commutation relation between the geodesic and horocycle flow, it is easy to see that the curve $h_t \circ \gamma$ is *sheared* along the direction of the horocycle flow and, for t large enough, it approximates a long segment of an orbit of the horocycle flow. In particular, it becomes *equidistributed*; namely, given any continuous function f , the integral of $f \circ h_t$ along $\gamma(s)$ converges to the space average of f , when $t \rightarrow \infty$. This mechanism has been exploited in the proofs of several results; for example, by Marcus in [Mal] to prove mixing for horocycle flows in variable negative curvature, and by Forni and Ulcigrai in [FU] to establish Lebesgue spectrum of smooth time-changes of the standard horocycle flow.

Moreover, it follows from the work of Bufetov and Forni [BF] that such sheared geodesic arcs equidistribute at the same rate as horocycle orbits. It is also plain that the same phenomenon happens if one replace the initial geodesic arc with any homogeneous segment which lies in a single $\{X, U\}$ -leaf; that is, a single weak-stable leaf for the geodesic flow.

What happens if the arc $\gamma(s) = p \exp(sW)$ is not contained in a single $\{X, U\}$ -leaf (i.e., if the generator $W \in \mathfrak{sl}_2(\mathbb{R})$ has a non-zero component in the direction of the unstable horocycle flow) is less clear, since the curve $h_t \circ \gamma$ *does not* approximate a single orbit of the horocycle flow. By approximating it with *several* orbit segments, it is possible to show that also in this case the curve $h_t \circ \gamma$ equidistributes. However, the quantitative estimates one can prove with this approach are far from optimal. In this paper, following a different strategy which heavily relies on the properties of the Bufetov–Forni functionals (in particular, on the *unstable horocycle invariance*, see Lemma 9), we prove sharper estimates for the equidistribution of arbitrary sheared arcs, see Theorem 2. As a corollary,

we provide an alternative proof of Ratner's quantitative mixing estimates, see Corollary 3.

1.2. Definitions and notation. Before stating our main result, we recall some definitions and we fix the notation.

Let $\Gamma < \mathrm{SL}(2, \mathbb{R})$ be a co-compact lattice, let $M = \Gamma \backslash \mathrm{SL}(2, \mathbb{R})$, and let us denote by vol the smooth probability measure on M given locally by the Haar measure. The manifold M can be identified with the unit tangent bundle T^1S of the compact hyperbolic surface $S = \Gamma \backslash \mathbb{H}$. The spectrum of the *Laplace–Beltrami operator* Δ_S on S is pure point and discrete. In particular, if we denote by $(\mu_n)_{n \geq 0}$ the positive eigenvalues, there is a *spectral gap*, since the bottom μ_0 of the non-zero spectrum is strictly positive. Let us further define

$$\nu_0 := \begin{cases} \sqrt{1 - 4\mu_0} & \text{if } \mu_0 < 1/4, \\ 0 & \text{if } \mu_0 \geq 1/4, \end{cases} \quad \varepsilon_0 := \begin{cases} 0 & \text{if } \mu_0 \neq 1/4, \\ 1 & \text{if } \mu_0 = 1/4. \end{cases}$$

$$\text{and } \delta_0 := \begin{cases} 0 & \text{if } \mu_0 < 1/4, \\ 1 & \text{if } \mu_0 \geq 1/4. \end{cases}$$

Let us denote by $L^2(M, \mathrm{vol})$ the space of square-integrable functions and by $L_0^2(M, \mathrm{vol})$ the subspace of functions with zero integral. For any $r > 0$, let $W^r(M)$ be the Sobolev space of functions $f \in L^2(M, \mathrm{vol})$ such that $\Delta^{r/2} f \in L^2(M, \mathrm{vol})$, where Δ is a Laplacian on M (see §2.2 for definitions). Let us remark that, for $r > 3/2$, by Sobolev Embedding Theorem, we have that $W^r(M) \subset \mathcal{C}^\alpha(M)$, for $\alpha < r - 3/2$.

We denote the Lie algebra of $\mathrm{SL}(2, \mathbb{R})$ by $\mathfrak{sl}_2(\mathbb{R})$; it consists of 2×2 real matrices with zero trace. Each element $W \in \mathfrak{sl}_2(\mathbb{R}) \setminus \{0\}$ generates the homogeneous flow $\{\varphi_t^W\}_{t \in \mathbb{R}}$ on M given by

$$\varphi_t^W(\Gamma g) = \Gamma g \exp(tW).$$

We fix the basis $\mathcal{B} = \{V, X, U\}$ of $\mathfrak{sl}_2(\mathbb{R})$, where

$$V = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad X = \begin{pmatrix} 1/2 & 0 \\ 0 & -1/2 \end{pmatrix}, \quad U = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}.$$

Then, \mathcal{B} is a frame of the tangent bundle of M . The homogeneous flows generated by V, X, U are the unstable horocycle flow $\{h_t^u\}_{t \in \mathbb{R}}$, the geodesic flow $\{g_t\}_{t \in \mathbb{R}}$, and the stable horocycle flow $\{h_t\}_{t \in \mathbb{R}}$ respectively. Finally, let us denote by $\widehat{\mathcal{B}} = \{\widehat{V}, \widehat{X}, \widehat{U}\}$ the frame of the cotangent bundle of M dual to \mathcal{B} .

1.3. Statement of the main result. Let $f \in L_0^2(M, \mathrm{vol})$ be a sufficiently smooth function with zero average. We are interested in the asymptotics of the following integrals

$$\left| \int_0^\sigma f \circ h_t \circ \varphi_s^W(p) \, ds \right|,$$

for fixed $\sigma > 0$ and $W \in \mathfrak{sl}_2(\mathbb{R}) \setminus \{0\}$.

The case $W = X$ follows easily from the work of Bufetov and Forni; a more general statement for smooth time-changes of the horocycle flow can be found in [FU, Lemma 17].

Theorem 1. *Let $r > 11/2$. For any $\sigma > 0$, there exists a constant $C = C(r, \sigma) > 0$ such that for any $f \in W^r(M) \cap L_0^2(M, \text{vol})$, for all $0 \leq S \leq \sigma$, $t \geq 1$ and $p \in M$ we have*

$$\left| \int_0^S f \circ h_t \circ g_s(p) \, ds \right| \leq C \|f\|_r t^{-\frac{1-\nu_0}{2}} (\log t)^{\varepsilon_0}.$$

It is easy to see that the same estimates (up to constant) hold if the geodesic segment $\{g_s(p) : s \in [0, S]\}$ is replaced with any segment of the form $\{\varphi_s^W(p) : s \in [0, S]\}$, where $W = xX + uU$, with $x \neq 0$.

In this paper, we generalize the previous result to *arbitrary* $W \in \mathfrak{sl}_2(\mathbb{R}) \setminus \{0\}$, namely to arbitrary arcs $\{\varphi_s^W(p) : s \in [0, S]\}$ not tangent to the integrable distribution $\{X, U\}$. Our main result is the following.

Theorem 2. *Let $r > 11/2$ and let*

$$W = vV + xX + uU \in \mathfrak{sl}_2(\mathbb{R}), \quad \text{with } v \neq 0.$$

For any $\sigma > 0$, there exists a constant $C = C(r, \sigma, W)$ such that for any $f \in W^r(M) \cap L_0^2(M, \text{vol})$, for all $0 \leq S \leq \sigma$, $t \geq 1$ and $p \in M$, we have

$$\left| \int_0^S f \circ h_t \circ \varphi_s^W(p) \, ds \right| \leq C \|f\|_r t^{-(1-\nu_0)} (\log t)^{\delta_0}.$$

In particular, the result above shows that arcs with a non-zero component in the direction V equidistribute *faster* than geodesic arcs.

The proof of Theorem 2 follows two different strategies for functions supported on the principal and complementary series, and for functions supported on the discrete series, see Propositions 8 and 10. In the former case, treated in Proposition 8, we exploit the properties of the Hölder functionals constructed by Bufetov and Forni, together with a key geometrical observation, see Lemma 9. In the case of negative Casimir parameters, however, the Bufetov–Forni functionals are not defined. For functions supported on the discrete series, we then proceed by a standard approximation argument, analogous to the case of the push-forward of geodesic arcs. For this reason, the bounds achieved in this case, contained in Proposition 10, are not optimal. However, the resulting estimates for generic

functions obtained in Theorem 2 by combining Propositions 8 and 10 differ from the optimal ones only by a logarithmic factor, and only when $\mu_0 > 1/4$.

By a standard *mixing via shearing* argument, from equidistribution estimates for the push-forward of transverse arcs, it is possible to deduce quantitative mixing estimates. Optimal mixing rates for the geodesic and horocycle flows were obtained by Ratner in [Rat]; here, we provide an alternative geometric proof of her result¹ for smooth observables.

Corollary 3 (Ratner's quantitative mixing [Rat]). *Let $r > 11/2$. There exists a constant $C > 0$ such that for all $f \in W^r(M)$, for all $g \in L^2(M, \text{vol})$ such that $Vg \in L^2(M, \text{vol})$, and for all $t \geq 1$ we have*

$$\left| \int_M (f \circ h_t) g \, d\text{vol} - \int_M f \, d\text{vol} \int_M g \, d\text{vol} \right| \leq C \|f\|_r (\|g\|_2 + \|Vg\|_2) t^{-(1-\nu_0)} (\log t)^{\delta_0}.$$

2. Preliminaries

2.1. Shearing of transverse arcs. Let us recall that, if we denote by $L_g: \text{SL}(2, \mathbb{R}) \rightarrow \text{SL}(2, \mathbb{R})$ the left-multiplication by g and by $(L_g)_*$ its push-forward, any element $W \in \mathfrak{sl}_2(\mathbb{R})$ induces the vector field, which we still denote by W , defined by $W|_g = (L_g)_*(W)$.

We will need the following basic facts about the tangent vectors of curves in M .

Lemma 4. *Let $W = vV + xX + uU \in \mathfrak{sl}_2(\mathbb{R}) \setminus \{0\}$ and let*

$$\exp(sW) = \begin{pmatrix} a(s) & b(s) \\ c(s) & d(s) \end{pmatrix}.$$

Then

$$\begin{pmatrix} a'(s) & b'(s) \\ c'(s) & d'(s) \end{pmatrix} = \begin{pmatrix} a(s) & b(s) \\ c(s) & d(s) \end{pmatrix} \begin{pmatrix} x/2 & u \\ v & -x/2 \end{pmatrix}.$$

¹ As explained before, our estimates differ from Ratner's by a factor $\log t$ when $\mu_0 > 1/4$.

Proof. The claim follows by the chain rule: let $g = \exp(sW)$; then we have

$$\begin{aligned} \begin{pmatrix} a'(s) & b'(s) \\ c'(s) & d'(s) \end{pmatrix} &= \frac{d}{ds} \exp(sW) = (L_g)_* \left(\frac{d}{ds} \Big|_{s=0} \exp(sW) \right) = (L_g)_*(W) \\ &= \begin{pmatrix} a(s) & b(s) \\ c(s) & d(s) \end{pmatrix} \begin{pmatrix} x/2 & u \\ v & -x/2 \end{pmatrix}. \end{aligned}$$

□

For any $g \in \mathrm{SL}(2, \mathbb{R})$, the Adjoint $\mathrm{Ad}_g : \mathfrak{sl}_2(\mathbb{R}) \rightarrow \mathfrak{sl}_2(\mathbb{R})$ is the linear map defined by $\mathrm{Ad}_g(W) = g^{-1}Wg$. The Adjoint describes the action of a homogeneous flow on tangent vectors, namely we have the following result.

Lemma 5. *Let $W, Y \in \mathfrak{sl}_2(\mathbb{R}) \setminus \{0\}$. For all $t, s \in \mathbb{R}$ and for all $p \in M$, we have*

$$\frac{d}{ds} \varphi_t^Y \circ \varphi_s^W(p) = \mathrm{Ad}_{\exp(tY)}(W)|_{\varphi_t^Y \circ \varphi_s^W(p)}$$

Proof. We have

$$\begin{aligned} \frac{d}{ds} \varphi_t^Y \circ \varphi_s^W(p) &= \frac{d}{ds} (p \exp(sW) \exp(tY)) \\ &= (L_{p \exp(sW) \exp(tY)})_* \left(\frac{d}{ds} \Big|_{s=0} \exp(-tY) \exp(sW) \exp(tY) \right) \\ &= (L_{\varphi_t^Y \circ \varphi_s^W(p)})_* \mathrm{Ad}_{\exp(tY)}(W). \end{aligned}$$

□

2.2. Spectral theory of the horocycle flow. We now recall some results about the spectral theory of the horocycle flow. Let

$$\Theta = \begin{pmatrix} 0 & 1/2 \\ -1/2 & 0 \end{pmatrix}$$

be a generator of the maximal compact subgroup $K = \mathrm{SO}(2) \subset \mathrm{SL}(2, \mathbb{R})$. We define the *Casimir operator* \square by

$$\square = -X^2 - (V + \Theta)^2 + \Theta^2,$$

which is a generator of the centre of the universal enveloping algebra of $\mathfrak{sl}_2(\mathbb{R})$. By the classical theory of unitary representations of $\mathrm{SL}(2, \mathbb{R})$, we have the following orthogonal decomposition into irreducible components (listed with multiplicity)

$$(1) \quad L^2(M) = \bigoplus_{\mu \in \mathrm{Spec}(\square)} H_\mu,$$

and on each H_μ , the Casimir operator \square acts as the constant $\mu \in \mathbb{R}_{>0} \cup \{-n^2 + n : n \in \mathbb{Z}_{\geq 0}\}$. The component H_0 corresponds to the trivial representation and appears with multiplicity one. The positive Casimir parameters $\mu > 0$ coincide with the positive eigenvalues of the Laplace–Beltrami operator Δ_S on the hyperbolic surface $S = \Gamma \backslash \mathbb{H}$.

The irreducible components H_μ are divided into three series: the *principal series* for Casimir parameters $\mu \geq 1/4$, the *complementary series* for Casimir parameters $0 < \mu < 1/4$, and the *discrete series* for negative Casimir parameters $\mu \leq 0$.

Let Δ be the *Laplacian* defined by $\Delta = -(X^2 + U^2/2 + V^2/2)$. For any Hilbert space H on which $\mathrm{SL}(2, \mathbb{R})$ acts unitarily, we define the *Sobolev space* $W^r(H)$ of order $r > 0$ to be the maximal domain of the operator $(1 + \Delta)^{r/2}$, equipped with the inner product

$$\langle f, g \rangle_r = \langle (1 + \Delta)^r f, g \rangle_H.$$

We denote by $\|\cdot\|_r$ the norm in $W^r(H)$ defined by the inner product above. When $H = L^2(M, \mathrm{vol})$, we will simply write $W^r(M)$ instead of $W^r(L^2(M, \mathrm{vol}))$. The space $W^r(M)$ coincides with the completion of $\mathcal{C}^\infty(M)$ of infinitely differentiable functions on M with respect to the norm $\|\cdot\|_r$. The direct sum (1) induce a corresponding decomposition of the Sobolev spaces for any fixed $r > 0$; namely we have

$$(2) \quad W^r(M) = \bigoplus_{\mu \in \mathrm{Spec}(\square)} W^r(H_\mu).$$

2.3. Bufetov–Forni functionals. The proof of our main result is a consequence of the properties of the Bufetov–Forni functionals introduced in [BF]. We briefly recall the results we will use, and we refer the reader to [BF] and to the work of Flaminio and Forni [FF] for more details. The reader can also compare the results by Bufetov and Forni stated below to the previous work of Bufetov for translation flows [Buf], and to the work of Forni and Kanigowski for Heisenberg nilflows [FK].

For every positive Casimir parameter $\mu > 0$, let $\nu = \sqrt{1 - 4\mu} \in \mathbb{C}$. We remark that ν is purely imaginary if μ belongs to the principal series, and $0 < \nu < 1$ if μ belongs to the complementary series.

Theorem 6 ([BF, Theorem 1.1]). *For any positive Casimir parameter $\mu > 0$, there exist two independent normalized finitely additive measures $\widehat{\beta}_\mu^\pm$ such that for any rectifiable arc $\gamma \subset M$ the following properties hold:*

(1) *for any decomposition $\gamma = \gamma_1 + \gamma_2$ into subarcs,*

$$\widehat{\beta}_\mu^\pm(\gamma) = \widehat{\beta}_\mu^\pm(\gamma_1) + \widehat{\beta}_\mu^\pm(\gamma_2);$$

(2) for all $t \in \mathbb{R}$ and $\mu \neq 1/4$,

$$\widehat{\beta}_\mu^\pm(g_{-t}\gamma) = \exp\left(\frac{1 \mp \nu}{2}t\right) \widehat{\beta}_\mu^\pm(\gamma),$$

while, for $\mu = 1/4$,

$$\begin{pmatrix} \widehat{\beta}_{1/4}^+(g_{-t}\gamma) \\ \widehat{\beta}_{1/4}^-(g_{-t}\gamma) \end{pmatrix} = \exp\left(\frac{t}{2}\right) \begin{pmatrix} 1 & -\frac{t}{2} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \widehat{\beta}_{1/4}^+(\gamma) \\ \widehat{\beta}_{1/4}^-(\gamma) \end{pmatrix}$$

(3) for all $t \in \mathbb{R}$,

$$\widehat{\beta}_\mu^\pm(h_t^u\gamma) = \widehat{\beta}_\mu^\pm(\gamma),$$

(4) there exists a constant $C > 0$ such that for all $\mu \neq 1/4$,

$$\left| \widehat{\beta}_\mu^\pm(\gamma) \right| \leq C \left(1 + \int_\gamma |\widehat{X}| + \int_\gamma |\widehat{U}| \int_\gamma |\widehat{V}| \right) \left(\int_\gamma |\widehat{U}| \right)^{\frac{1 \mp \text{Re} \nu}{2}}$$

and, for $\mu = 1/4$,

$$\begin{aligned} \left| \widehat{\beta}_{1/4}^+(\gamma) \right| &\leq C \left(1 + \int_\gamma |\widehat{X}| + \int_\gamma |\widehat{U}| \int_\gamma |\widehat{V}| \right) \left(\int_\gamma |\widehat{U}| \right)^{\frac{1}{2}} \left(1 + \log \int_\gamma |\widehat{U}| \right) \\ \left| \widehat{\beta}_{1/4}^-(\gamma) \right| &\leq C \left(1 + \int_\gamma |\widehat{X}| + \int_\gamma |\widehat{U}| \int_\gamma |\widehat{V}| \right) \left(\int_\gamma |\widehat{U}| \right)^{\frac{1}{2}} \end{aligned}$$

For any $\mu > 0$, let us denote by D_μ^+, D_μ^- the two normalized invariant distributions introduced by Flaminio and Forni in [FF]. Let $r > 11/2$; for any function $f \in W^r(M)$ supported on irreducible components of the principal and complementary series and for any rectifiable arc $\gamma \subset M$, let us define

$$\widehat{\beta}_f(\gamma) = \sum_{\mu \in \text{Spec}(\square) \cap \mathbb{R}_{>0}} D_\mu^+(f) \widehat{\beta}_\mu^+(\gamma) + D_\mu^-(f) \widehat{\beta}_\mu^-(\gamma).$$

Theorem 7 ([BF, Theorem 1.3]). *For any $r > 11/2$ there exists a constant $C_r > 0$ such that for every rectifiable arc $\gamma \subset M$ and for all $f \in W^r(M)$ supported on irreducible components of the principal and complementary series we have*

$$\left| \int_\gamma f \widehat{U} - \widehat{\beta}_f(\gamma) \right| \leq C_r \|f\|_r \left(1 + \int_\gamma |\widehat{X}| + \int_\gamma |\widehat{V}| \right).$$

3. Proof of Theorem 2

Let $W = vV + xX + uU \in \mathfrak{sl}_2(\mathbb{R})$ with $v \neq 0$ be fixed. Let $\sigma > 0$ and $0 \leq S \leq \sigma$, and define the curve

$$\gamma_{p,t}^W(s) = h_t \circ \varphi_s^W(p), \quad \text{for } s \in [0, S].$$

Up to replacing σ and S by $c\sigma$ and cS , for some $c > 0$, we can assume that $\max\{|v|, |x|, |u|\} \leq 1$.

Let $r > 11/2$ and let $f \in W^r(M) \cap L_0^2(M, \text{vol})$. Using the orthogonal decomposition (2), we can write

$$f = \sum_{\mu \in \text{Spec}(\square)} f_\mu,$$

where $f_\mu \in W^r(H_\mu)$, and, for $\mu = 0$, the component $f_0 = 0$. Let us denote by

$$f_d = \sum_{\mu \in \text{Spec}(\square) \cap \mathbb{R}_{<0}} f_\mu, \quad \text{and} \quad f_d^\perp = \sum_{\mu \in \text{Spec}(\square) \cap \mathbb{R}_{>0}} f_\mu,$$

so that

$$\int_0^S f \circ \gamma_{p,t}^W(s) \, ds = \int_0^S f_d \circ \gamma_{p,t}^W(s) \, ds + \int_0^S f_d^\perp \circ \gamma_{p,t}^W(s) \, ds.$$

We will estimate the two integrals separately. The proof of Theorem 2 follows from Propositions 8 and 10 below applied to f_d^\perp and f_d respectively.

3.1. Positive Casimir parameters. This section is devoted to the proof of the following result.

Proposition 8. *Let $r > 11/2$ and let $f \in W^r(M) \cap L_0^2(M, \text{vol})$ be supported on irreducible components of the principal and complementary series. Then, there exists a constant $C = C(r, \sigma, W) > 0$ such that for all $0 \leq S \leq \sigma$, $t \geq 2$ and $p \in M$, we have*

$$\left| \int_0^S f \circ h_t \circ \varphi_s^W(p) \, ds \right| \leq C \|f\|_r t^{-(1-\nu_0)} (\log t)^{\varepsilon_0}.$$

By Lemma 5, we can compute the tangent vector of $\gamma_{p,t}^W(s)$ by

$$\begin{aligned} (3) \quad \frac{d}{ds} \gamma_{p,t}^W(s) &= \text{Ad}_{\exp(tU)}(W)|_{\gamma_{p,t}^W(s)} \\ &= (vV + (x - 2tv)X + (u + xt - vt^2)U)|_{\gamma_{p,t}^W(s)} \end{aligned}$$

so that

$$\int_0^S f \circ \gamma_{p,t}^W(s) \, ds = (u + xt - vt^2)^{-1} \int_{\gamma_{p,t}^W} f \widehat{U}.$$

For all $t \geq 2$ we have $|u + xt - vt^2| \geq |v|t^2(1 - t^{-1} - t^{-2}) \geq |v|t^2/4$. Therefore, by Theorem 7 and formula (3),

$$\begin{aligned}
(4) \quad \left| \int_0^S f \circ \gamma_{p,t}^W(s) \, ds \right| &\leq \frac{4}{|v|t^2} \left| \int_{\gamma_{p,t}^W} f \widehat{U} \right| \\
&\leq \frac{4}{|v|t^2} \left(\widehat{\beta}_f(\gamma_{p,t}^W) + C_r \|f\|_r S(1 + |v| + |x|t + 2|v|t) \right) \\
&\leq C_{r,W,\sigma} \left(\frac{1}{t^2} \widehat{\beta}_f(\gamma_{p,t}^W) + \frac{\|f\|_r}{t} \right),
\end{aligned}$$

where we can take $C_{r,W,\sigma} = \max\{4|v|^{-1}, 20C_r\sigma|v|^{-1}\}$.

Recall that we have

$$\widehat{\beta}_f(\gamma_{p,t}^W) = \sum_{\mu \in \text{Spec}(\square) \cap \mathbb{R}_{>0}} D_\mu^+(f) \widehat{\beta}_\mu^+(\gamma_{p,t}^W) + D_\mu^-(f) \widehat{\beta}_\mu^-(\gamma_{p,t}^W).$$

From properties (2) and (3) in Theorem 6, for any $\mu \neq 1/4$, we deduce that for all $T \in \mathbb{R}$,

$$\widehat{\beta}_\mu^\pm(\gamma_{p,t}^W) = t^{1 \mp v} \widehat{\beta}_\mu^\pm(g_{2 \log t} \circ \gamma_{p,t}^W) = t^{1 \mp v} \widehat{\beta}_\mu^\pm(h_T^u \circ g_{2 \log t} \circ \gamma_{p,t}^W),$$

while, for $\mu = 1/4$, we have

$$\begin{aligned}
\widehat{\beta}_\mu^+(\gamma_{p,t}^W) &= t \widehat{\beta}_\mu^+(g_{2 \log t} \circ \gamma_{p,t}^W) + t \log t \widehat{\beta}_\mu^-(g_{2 \log t} \circ \gamma_{p,t}^W) \\
&= t \widehat{\beta}_\mu^+(h_T^u \circ g_{2 \log t} \circ \gamma_{p,t}^W) + t \log t \widehat{\beta}_\mu^-(h_T^u \circ g_{2 \log t} \circ \gamma_{p,t}^W),
\end{aligned}$$

and

$$\widehat{\beta}_\mu^-(\gamma_{p,t}^W) = t \widehat{\beta}_\mu^-(g_{2 \log t} \circ \gamma_{p,t}^W) = t \widehat{\beta}_\mu^-(h_T^u \circ g_{2 \log t} \circ \gamma_{p,t}^W).$$

The following is our key geometrical observation: choosing $T = -t$ in the formulas above, the components of the tangent vectors of the curve $h_{-t}^u \circ g_{2 \log t} \circ \gamma_{p,t}^W$ with respect to the frame \mathcal{B} are *uniformly bounded* in t .

Lemma 9. *We have*

$$\frac{d}{ds}(h_{-t}^u \circ g_{2 \log t} \circ \gamma_{p,t}^W)(s) = -uV - \left(x + \frac{2u}{t}\right)X + \left(\frac{u}{t^2} + \frac{x}{t} - v\right)U,$$

in particular for all $t \geq 1$ we have

$$\int_{h_{-t}^u \circ g_{2 \log t} \circ \gamma_{p,t}^W} |\widehat{Y}| \leq 3S \leq 3\sigma, \quad \text{for any } \widehat{Y} \in \widehat{\mathcal{B}} = \{\widehat{V}, \widehat{X}, \widehat{U}\}.$$

Proof. By Lemma 5, from an easy computation it follows

$$\begin{aligned}
& \frac{d}{ds} (h_{-t}^u \circ g_{2 \log t} \circ \gamma_{p,t}^W)(s) \\
&= \text{Ad}_{\exp(-tV)} \circ \text{Ad}_{\exp(2 \log t X)} \circ \text{Ad}_{\exp(tU)}(W) \Big|_{h_{-t}^u \circ g_{2 \log t} \circ \gamma_{p,t}^W(s)} \\
&= \text{Ad}_{\exp(-tV)} \\
&\quad \circ \text{Ad}_{\exp(2 \log t X)}(vV + (x - 2vt)X + (-vt^2 + xt + u)U) \Big|_{h_{-t}^u \circ g_{2 \log t} \circ \gamma_{p,t}^W(s)} \\
&= \text{Ad}_{\exp(-tV)}(vt^2V + (x - 2vt)X + (-v + xt^{-1} + ut^{-2})U) \Big|_{h_{-t}^u \circ g_{2 \log t} \circ \gamma_{p,t}^W(s)} \\
&= (-uV - (x + 2ut^{-1})X + (-v + xt^{-1} + ut^{-2})U) \Big|_{h_{-t}^u \circ g_{2 \log t} \circ \gamma_{p,t}^W(s)}
\end{aligned}$$

In particular, for all $t \geq 1$, we have $\max\{|u|, |x| + 2|u|t^{-1}, |v| + |x|t^{-1} + |u|t^2\} \leq 3$, which concludes the proof. \square

By property (4) in Theorem 6 and by Lemma 9, there exists a constant $C_\sigma > 0$ such that for all $\mu \neq 1/4$ we have

$$|\widehat{\beta}_\mu^\pm(\gamma_{p,t}^W)| \leq C_\sigma t^{1 \mp \text{Rev}}.$$

and, for $\mu = 1/4$,

$$|\widehat{\beta}_{1/4}^\pm(\gamma_{p,t}^W)| \leq C_\sigma t(1 + \log t).$$

Up to enlarging C_σ , for all $t \geq 1$ we then deduce

$$|\widehat{\beta}_f(\gamma_{p,t}^W)| \leq C_\sigma \|f\|_r t^{1+\nu_0} (\log t)^{\varepsilon_0},$$

which, together with (4), concludes the proof of Proposition 8.

3.2. The discrete series. In this section we prove the following result.

Proposition 10. *Let $r > 11/2$ and let $f \in W^r(M) \cap L_0^2(M, \text{vol})$ be supported on irreducible components of the discrete series. Then, there exists a constant $C = C(r, \sigma, W) > 0$ such that for all $0 \leq S \leq \sigma$, $t \geq 2$ and $p \in M$, we have*

$$\left| \int_0^S f \circ h_t \circ \varphi_s^W(p) ds \right| \leq C \|f\|_r t^{-1} (\log t).$$

Using the same notation as in Lemma 4, let

$$\exp(sW) = \begin{pmatrix} a(s) & b(s) \\ c(s) & d(s) \end{pmatrix}.$$

Since the entries above are smooth functions, we can define

$$\ell = \ell(\sigma) = 4 \max\{|a'(s)|, |b'(s)|, |c'(s)|, |d'(s)| : s \in [0, \sigma]\}.$$

For negative Casimir parameters, the Bufetov–Forni functionals are not defined. Hence, we proceed in the following way: we partition the curve $\gamma_{p,t}^W$ into $O(t)$ curves which are at distance $O(t^{-1})$ from a leaf tangent to the integrable distribution $\{X, U\}$, and we approximate them by their projection onto the $\{X, U\}$ -leaf. The estimate for each projection can be then deduced from [BF], see, e.g., [FU, Lemma 17].

We decompose the integral as follows:

$$\begin{aligned} (5) \quad \left| \int_0^S f \circ h_t \circ \varphi_s^W(s) \, ds \right| &= \left| \int_0^S f \circ \gamma_{p,t}^W(s) \, ds \right| \\ &\leq \frac{\|f\|_\infty}{\ell t} + \sum_{k=0}^{\lfloor S\ell t \rfloor - 1} \left| \int_{\frac{k}{\ell t}}^{\frac{k+1}{\ell t}} f \circ \gamma_{p,t}^W(s) \, ds \right| \\ &= \frac{\|f\|_\infty}{\ell t} + \sum_{k=0}^{\lfloor S\ell t \rfloor - 1} \left| \int_0^{\frac{1}{\ell t}} f \circ \gamma_{p_k,t}^W(s) \, ds \right|, \end{aligned}$$

where

$$p_k = p \exp\left(\frac{k}{\ell t} W\right).$$

For any $s \in [0, (\ell t)^{-1}]$, we have

$$|d(s) + c(s)t| \geq |d(s)| - |c(s)|t \geq 1 - \frac{\ell}{4}s - \frac{\ell}{4}st \geq 1 - \frac{1}{4t} - \frac{1}{4} \geq \frac{1}{2}.$$

Therefore, the function

$$J_0(s) = -\frac{c(s)}{d(s) + c(s)t}$$

is well-defined for $s \in [0, (\ell t)^{-1}]$.

For any $0 \leq k \leq \lfloor S\ell t \rfloor - 1$, let us define the curve

$$\overline{\gamma}_k(s) = h_{J_0(s)}^- \circ \gamma_{p_k,t}^W(s), \quad \text{for } s \in [0, (\ell t)^{-1}].$$

Explicitly, we have

$$\begin{aligned} \overline{\gamma}_k(s) &= p_k \exp(sW) \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ J_0(s) & 1 \end{pmatrix} = p_k \begin{pmatrix} \frac{1}{d(s)+c(s)t} & b(s) + a(s)t \\ 0 & d(s) + c(s)t \end{pmatrix} \\ &= p_k \begin{pmatrix} 1 & \frac{b(s)+a(s)t}{d(s)+c(s)t} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \frac{1}{d(s)+c(s)t} & 0 \\ 0 & d(s) + c(s)t \end{pmatrix} \\ &= g_{-2\log(d(s)+c(s)t)} \circ h_{\frac{b(s)+a(s)t}{d(s)+c(s)t}}(p_k) \end{aligned}$$

Notice that the curve $\overline{\gamma}_k$ is contained in the $\{X, U\}$ -leaf passing through p_k . We also remark that there exists a constant $C > 0$ such that for any $s \in [0, (\ell t)^{-1}]$,

$$\text{dist}\left(\overline{\gamma}_k(s), \gamma_{p_k, t}^W(s)\right) \leq C |J_0(s)| \leq 2C |c(s)| \leq 2C \ell s \leq \frac{2C}{t},$$

so that

$$\begin{aligned} (6) \quad \sum_{k=0}^{\lfloor S\ell t \rfloor - 1} \left| \int_0^{\frac{1}{\ell t}} f \circ \gamma_{p_k, t}^W(s) \, ds \right| &\leq \sum_{k=0}^{\lfloor S\ell t \rfloor - 1} \left| \int_0^{\frac{1}{\ell t}} f \circ \overline{\gamma}_k(s) \, ds \right| + \sum_{k=0}^{\lfloor S\ell t \rfloor - 1} \left| \frac{2C \|f\|_r}{\ell t^2} \right| \\ &\leq \sum_{k=0}^{\lfloor S\ell t \rfloor - 1} \left| \int_0^{\frac{1}{\ell t}} f \circ \overline{\gamma}_k(s) \, ds \right| + \frac{2C \|f\|_r \sigma}{t}. \end{aligned}$$

Lemma 11. *We have*

$$\widehat{U} \left(\frac{d}{ds} \overline{\gamma}_k(s) \right) = -vt^2 + \frac{x}{2}t + u,$$

and moreover

$$\left| \widehat{X} \left(\frac{d}{ds} \overline{\gamma}_k(s) \right) \right| \leq 20t.$$

Proof. Let us denote

$$J_1(s) = -2 \log(d(s) + c(s)t), \quad \text{and} \quad J_2(s) = \frac{b(s) + a(s)t}{d(s) + c(s)t}.$$

We compute

$$\begin{aligned} &\frac{d}{ds} \overline{\gamma}_k(s) \\ &= \frac{d}{ds} (g_{J_1(s)} \circ h_{J_2(s)}(p_k)) \\ &= Dg_{J_1(s)} \left(\frac{d}{ds} \circ h_{J_2(s)}(p_k) \right) + \left(\frac{d}{ds} g_{J_1(s)} \right) \circ h_{J_2(s)}(p_k) \\ &= Dg_{J_1(s)} \left(\left(\frac{d}{ds} J_2(s) \right) U \circ h_{J_2(s)}(p_k) \right) + \left(\left(\frac{d}{ds} J_1(s) \right) X \circ g_{J_1(s)} \right) \circ h_{J_2(s)}(p_k) \\ &= \left(\frac{d}{ds} J_2(s) \right) e^{-J_1(s)} U \circ \overline{\gamma}_k(s) + \left(\frac{d}{ds} J_1(s) \right) X \circ \overline{\gamma}_k(s) \\ &= \left((b'(s) + a'(s)t)(d(s) + c(s)t) - (b(s) + a(s)t)(d'(s) + c'(s)t) \right) U \circ \overline{\gamma}_k(s) \\ &\quad - 2 \frac{c'(s)t + d'(s)}{c(s)t + d(s)} X \circ \overline{\gamma}_k(s). \end{aligned}$$

By Lemma 4, for any $s \in [0, (\ell t)^{-1}]$, we have

$$\max\{c'(s), d'(s)\} \leq 2 \max\{c(s), d(s)\} \leq 2 \left(1 + \frac{\ell}{4}s\right) \leq 3,$$

so that

$$\left| 2 \frac{c'(s)t + d'(s)}{c(s)t + d(s)} \right| \leq 4(3t + 3) \leq 20t.$$

Using the formulas in Lemma 4 and the fact that $a(s)d(s) - b(s)c(s) = 1$, we obtain that

$$(b'(s) + a'(s)t)(d(s) + c(s)t) - (b(s) + a(s)t)(d'(s) + c'(s)t) = -vt^2 + \frac{x}{2}t + u,$$

which concludes the proof. \square

With our assumptions, for $t \geq 2$,

$$\left| -vt^2 + \frac{x}{2}t + u \right| \geq |v|t^2(1 - t^{-1} - t^{-2}) \geq \frac{|v|t^2}{4}.$$

Thus, it follows that (see, e.g., the proof of [FU, Lemma 17])

$$\begin{aligned} (7) \quad \left| \int_0^{\frac{1}{\ell t}} f \circ \overline{\gamma_k}(s) \, ds \right| &= \left| \left(-vt^2 + \frac{x}{2}t + u \right)^{-1} \int_{\overline{\gamma_k}} f \widehat{U} \right| \\ &\leq \frac{4}{|v|t^2} \left| C_r \|f\|_r \left(1 + \int_{\overline{\gamma_k}} |\widehat{X}| \right) \log \left(1 + \int_{\overline{\gamma_k}} |\widehat{U}| \right) \right| \\ &\leq \frac{16}{|v|t^2} C_r \|f\|_r \left(1 + \frac{20}{\ell} \right) (\log t) \leq C_{r,W,\sigma} \|f\|_r \frac{\log t}{t^2}. \end{aligned}$$

From (5), (6) and (7), we conclude

$$\left| \int_0^S f \circ h_t \circ \varphi_s^W(p) \, ds \right| \leq C_{r,W,\sigma} \|f\|_r \frac{\log t}{t}.$$

4. Proof of Corollary 3

The proof is a straightforward consequence of a standard “mixing via shearing” argument, see, e.g., [FU]. Without loss of generality, let us assume that $f \in W^r(M) \cap L_0^2(M, \text{vol})$ and let $g \in L^2(M, \text{vol})$ such that $Vg \in L^2(M, \text{vol})$. By invariance of the Haar measure and integrating by parts, for all $\sigma > 0$ and $t \geq 1$ we have

$$\begin{aligned} \langle f \circ h_t, g \rangle &= \frac{1}{\sigma} \int_0^\sigma \langle f \circ h_t \circ h_s^u, g \circ h_s^u \rangle \, ds \\ &= \left\langle \frac{1}{\sigma} \int_0^\sigma f \circ h_t \circ h_s^u \, ds, g \right\rangle - \frac{1}{\sigma} \int_0^\sigma \left\langle \int_0^S f \circ h_t \circ h_s^u \, ds, Vg \circ h_s^u \right\rangle, \end{aligned}$$

so that

$$\begin{aligned} |\langle f \circ h_t, g \rangle| &\leq \frac{\|g\|_2}{\sigma} \left\| \int_0^\sigma f \circ h_t \circ h_s^u \, ds \right\|_2 + \frac{\|Vg\|_2}{\sigma} \sup_{S \in [0, \sigma]} \left\| \int_0^S f \circ h_t \circ h_s^u \, ds \right\|_2 \\ &\leq \frac{\|g\|_2 + \|Vg\|_2}{\sigma} \sup_{S \in [0, \sigma]} \sup_{p \in M} \left| \int_0^S f \circ h_t \circ h_s^u(p) \, ds \right|. \end{aligned}$$

The result follows from the estimates in Theorem 2.

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References

- [BEG] M. BJÖRKLUND, M. EINSIEDLER and A. GORODNIK, Quantitative multiple mixing. *J. Eur. Math. Soc. (JEMS)* **22** (2020), 1475–1529. Zbl 07198112 MR 4081727
- [Buf] A. BUFETOV, Limit theorems for translation flows. *Ann. of Math.* **179** (2014), 431–499. Zbl 1290.37023 MR 3152940
- [BF] A. BUFETOV and G. FORNI, Limit theorems for horocycle flows. *Ann. Sci. Éc. Norm. Supér.* **47** (2014), 851–903. Zbl 1360.37086 MR 3294619
- [FF] L. FLAMINIO and G. FORNI, Invariant distributions and time averages for horocycle flows. *Duke Math. J.* **119** (2003), 465–526. Zbl 1044.37017 MR 2003124
- [FK] G. FORNI and A. KANIGOWSKI, Time-changes of Heisenberg nilflows. Preprint arXiv:1711.05543 (2017).
- [FU] G. FORNI and C. ULCIGRAI, Time-changes of horocycle flows. *Journal of Modern Dynamics* **6** (2012), 251–273. Zbl 1254.37006 MR 2968956
- [Fur] H. FURSTENBERG, The unique ergodicity of the horocycle flow. In *Recent Advances in Topological Dynamics* (New Haven, Conn., 1972), Lecture Notes in Math. 318, Springer, Berlin, 1973, 95–115. Zbl 0256.58009 MR 0393339
- [GF] I. M. GELFAND and S. V. FOMIN, Geodesic flows on manifolds of constant negative curvature. *Uspehi Matem. Nauk (N.S.)* **7** (1952), 118–137. MR 0052701
- [Ma1] B. MARCUS, Ergodic properties of horocycle flows for surfaces of negative curvature, *Ann. of Math.* **105** (1977), 81–105. Zbl 0322.28012 MR 0458496
- [Ma2] ———, The horocycle flow is mixing of all degrees. *Invent. Math.* **46** (1978), 201–209. Zbl 0395.28012 MR 0488168

- [MMY] S. MARMI, P. MOUSSA and J.-C. YOCOZ, Affine interval exchange maps with a wandering interval. *Proc. Lond. Math. Soc.* **100** (2010), 639–669. Zbl 1196.37041 MR 2640286
- [Gur] B. M. GUREVIC, The entropy of horocycle flows. *Dokl. Akad. Nauk SSSR* **136** (1961), 768–770. MR 0132410
- [Hed] G. A. HEDLUND, Fuchsian groups and transitive horocycles. *Duke Math. J.* **2** (1936), 530–542. Zbl 0015.10201 MR 1545946
- [Par] O. S. PARASYUK, Flows of horocycles on surfaces of constant negative curvature (in Russian). *Uspekhi Mat. Nauk* **8** (1953), 125–126. MR 0058883
- [Rat] M. RATNER, The rate of mixing for geodesic and horocycle flows. *Ergodic Theory Dynam. Systems* **7** (1987), 267–288. Zbl 0623.22008 MR 0896798

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