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Surface groups in the group of germs of analytic diffeomorphisms in one variable

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Abstract. We construct embeddings of surface groups into the group of germs of analytic diffeomorphisms in one variable.

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Contents

1. Introduction

1.1. The main result. Let $\mathbf{C}$ be the field of complex numbers and $\mathsf{Diff}(\mathbf{C}, 0)$ the group of germs of analytic diffeomorphisms at the origin $0 \in \mathbf{C}$. Choosing a local coordinate z near the origin, every element $f \in \mathsf{Diff}(\mathbf{C}, 0)$ is determined by a unique power series

$$f(z) = a_1 z + a_2 z^2 + a_3 z^3 + \ldots + a_n z^n + \ldots$$

with $f'(0) = a_1 \neq 0$ and with a positive radius of convergence

$$\text{rad}(f) = \left(\limsup_{n \to +\infty} |a_n|^{1/n} \right)^{-1}. \tag{1.1}$$

We denote by $\mathsf{Diff}(\mathbf{R}, 0) \subset \mathsf{Diff}(\mathbf{C}, 0)$ the subgroup of real germs in this chart, i.e., with $a_i \in \mathbf{R}$ for all $i \in \mathbf{N}$ (this inclusion depends on the choice of the coordinate z). The main goal of this note is the following result, that answers a question raised by E. Ghys (see [Cer], §3.3, or also [Bru1], Problem 4.15).

Theorem A. *Let Γ be the fundamental group of a closed orientable surface, or of a closed non-orientable surface of genus ≥ 4. Then Γ embeds in the group $\mathsf{Diff}(\mathbf{R}, 0)$ and in particular in $\mathsf{Diff}(\mathbf{C}, 0)$.*

We shall present three proofs of Theorem A. For simplicity, in this introduction, we restrict to the case where Γ is is the fundamental group of an orientable surface of genus 2, and we consider the presentation

$$\Gamma_2 = \langle a_1, b_1, a_2, b_2 \mid [a_1, b_1] = [a_2, b_2] \rangle. \tag{1.2}$$

Our proofs of theorem A are inspired by [BGSS], where it is proved that a compact topological group or a connected Lie group which contains a dense free group of rank 2 contains a dense subgroup isomorphic to Γ_2.

The surface groups considered in Theorem A are examples of limit groups. Recently, and independently, A. Brudnyi proved a related embedding theorem: limit groups embed into the group of (non converging) formal germs of diffeomorphisms (see [Bru2])

1.2. Compact groups. Let us describe the argument used in [BGSS] to prove the following result.

Theorem 1.1 ([BGSS]). *If a compact group G contains a free group F of rank 2, then there is an embedding $\rho \colon \Gamma_2 \to G$ such that $F \subset \rho(\Gamma_2)$.*

Proof. Denote by $\mathbf{F}_m$ the free group on m generators. The first ingredient is a result by Baumslag [Bau] saying that Γ_2 is *fully residually free*; this means that there exists a sequence of morphisms $p_N : \Gamma_2 \to \mathbf{F}_2$ which is asymptotically injective: for every $g \in \Gamma_2 \setminus \{1\}$, $p_N(g) \neq 1$ if N is large enough.

To be more explicit, we use the presentation (1.2) of Γ_2, and we note that the subgroup $\langle a_1, b_1 \rangle$ of Γ_2 is a free group $\mathbf{F}_2 = \langle a_1, b_1 \rangle$. Let $p : \Gamma_2 \to \langle a_1, b_1 \rangle$ be the morphism fixing a_1 and b_1 and sending a_2 and b_2 to a_1 and b_1

respectively. Let $\tau : \Gamma_2 \to \Gamma_2$ be the Dehn twist around the curve $c = [a_1, b_1]$, i.e. the automorphism that fixes a_1 and b_1 and sends a_2 and b_2 to ca_2c^{-1} and cb_2c^{-1} respectively.

Proposition 1.2 (see [BGSS, Corollary 2.2]). *Given any $g \in \Gamma_2 \setminus \{1\}$, there exists a positive integer n_0 such that $p \circ \tau^N(g) \neq 1$ for all $N \geq n_0$.*

Now, fix an embedding $\iota \colon \langle a_1, b_1 \rangle \to G$ such that $\iota(\langle a_1, b_1 \rangle) = F$. Composing $p \circ \tau^N$ with ι, we get a sequence of points $p_N := \iota \circ p \circ \tau^N$ in $\mathrm{Hom}(\Gamma_2, G)$. Now, consider the element $h = \iota(p(c))$ of G, and let T be the closure of the cyclic group $\langle h \rangle$ in the compact group G. For $t \in T$, define a morphism $\rho_t \colon \Gamma_2 \to G$ by

$$\rho_t(a_1) = \iota\big(p(a_1)\big), \qquad \rho_t(a_2) = t \circ \Big(\iota\big(p(a_1)\big)\Big) \circ t^{-1} \tag{1.3}$$

$$\rho_t(b_1) = \iota\big(p(b_1)\big), \qquad \rho_t(b_2) = t \circ \Big(\iota\big(p(b_1)\big)\Big) \circ t^{-1} \tag{1.4}$$

these representations are well defined and satisfy $\rho_t = \iota \circ p \circ \tau^N$ when $t = h^N$. Moreover, on the subgroup $\langle a_1, b_1 \rangle$, ρ_t coincides with $\iota \circ p$, so $F \subset \rho_t(\Gamma_2)$. Thus, $(\rho_t)_{t \in T}$ is a compact subset $\mathcal{R}(T) \subset \mathrm{Hom}(\Gamma_2, G)$ that contains the sequence of points p_N. For every g in $\Gamma_2 \setminus \{1\}$, the subset $\mathcal{R}(T)_g = \{\rho_t \mid \rho_t(g) \neq 1\}$ is open, and Proposition 1.2 shows that it is dense because $\{h^n \mid n \geq n_0\}$ is dense in T for every integer n_0. By the Baire theorem, the subset of injective representations ρ_t is a dense G_δ in $\mathcal{R}(T)$, and this proves Theorem 1.1. □

The group $\mathsf{Diff}(\mathbf{R}, 0)$ contains non-abelian free groups (this is well known, see Section 3.3), and one may want to copy the above argument for $G = \mathsf{Diff}(\mathbf{R}, 0)$ instead of a compact group. The Koenigs linearization theorem says that if $f \in \mathsf{Diff}(\mathbf{R}, 0)$ satisfies $f'(0) > 1$, then f is conjugate to the homothety $z \mapsto f'(0)z$; in particular, there is a flow of diffeomorphisms $(\varphi^t)_{t \in \mathbf{R}}$ for which $\varphi^1 = f$. In our argument, the compact group T introduced to prove Theorem 1.1 will be replaced by such a flow, hence by a group isomorphic to $(\mathbf{R}, +)$. Also, in that proof, $h = \iota(p(c))$ was a commutator, and the derivative of any commutator in $\mathsf{Diff}(\mathbf{R}, 0)$ is equal to 1 at the origin, so that Koenigs theorem can not be applied to a commutator. Thus, we need to change p_N into a different sequence of morphisms: the Dehn twist τ will be replaced by another automorphism of Γ_2, twisting along three non-separating curves.

This argument will be described in details in Sections 2 and 3; the reader who wants the simplest proof of Theorem A in the case of orientable surfaces only needs to read these sections. Non orientable surfaces are dealt with in Section 4.

1.3. Lie groups. Now, let us look at representations in a linear algebraic subgroup G of $\mathsf{GL}_m(\mathbf{R})$. Assuming that there is a faithful representation $\iota\colon \mathbf{F}_2 \to G$ with dense image, we shall construct a faithful representation $\Gamma_2 \to G$.

The representation variety $\mathrm{Hom}(\Gamma_2, G)$ is an algebraic subset of G^4. Let $\mathcal{R}$ be the irreducible component containing the trivial representation. Let $p_N : \Gamma_2 \to \mathbf{F}_2$ be an asymptotically injective sequence of morphisms, as given by Baumslag's proposition. When the image of ρ is dense, one can prove that $\iota \circ p_N$ is in $\mathcal{R}$ for arbitrarily large values of N. For $g \in \Gamma_2 \setminus \{1\}$, the subset $\mathcal{R}_g \subset \mathcal{R}$ of homomorphisms killing g is algebraic, and it is a proper subset because it does not contain $\iota \circ p_N$ for some large N. Then, a Baire category argument in $\mathcal{R}$ implies that a generic choice of $\rho \in \mathcal{R}$ is faithful.

To apply this argument to $G = \mathsf{Diff}(\mathbf{R}, 0)$, one needs a good topology on $\mathsf{Diff}(\mathbf{R}, 0)$, and a good "irreducible variety" $\mathcal{R} \subset \mathrm{Hom}(\Gamma_2, G)$ containing $\iota \circ p_N$, in which a Baire category argument can be used. This approach may seem difficult because $\mathrm{Hom}(\Gamma_2, G)$ is a priori far from being an irreducible analytic variety but, again, the Koenigs linearization theorem will provide the key ingredient.

First, we shall adapt an idea introduced by Leslie in [Les] to define a useful group topology on $\mathsf{Diff}(\mathbf{R}, 0)$ (see Section 5). With this topology, $\mathsf{Diff}(\mathbf{R}, 0)$ is an increasing union of Baire spaces, which will be enough for our purpose. Denote by $\mathsf{Cont}(\mathbf{R}, 0) \subset \mathsf{Diff}(\mathbf{R}, 0)$ the set of elements $f \in \mathsf{Diff}(\mathbf{R}, 0)$ with $|f'(0)| < 1$; Cont stands for "contractions". Consider the set $\mathcal{R}$ of representations $\rho\colon \Gamma_2 \to \mathsf{Diff}(\mathbf{R}, 0)$ with $\rho(a_1)$ tangent to the identity, and $\rho(b_1) \in \mathsf{Cont}(\mathbf{R}, 0)$. Then, the key fact is that the map

$$\begin{aligned} \Psi : \mathcal{R} &\to \mathsf{Cont}(\mathbf{R}, 0) \times \mathsf{Diff}(\mathbf{R}, 0) \times \mathsf{Diff}(\mathbf{R}, 0) \\ \rho &\mapsto (\rho(b_1), \rho(a_2), \rho(b_2)) \end{aligned}$$

is a continuous bijection. Indeed, the defining relation of Γ is equivalent to $a_1 b_1 a_1^{-1} = [a_2, b_2] b_1$. Given $(g_1, f_2, g_2) \in \mathsf{Cont}(\mathbf{R}, 0) \times \mathsf{Diff}(\mathbf{R}, 0) \times \mathsf{Diff}(\mathbf{R}, 0)$, the germs g_1 and $[f_2, g_2] g_1$ have the same derivative at the origin and, from the Koenigs linearization theorem, there is a unique $f_1 \in \mathsf{Diff}(\mathbf{R}, 0)$ tangent to the identity solving the equation $f_1 g_1 f_1^{-1} = [f_2, g_2] g_1$: by construction there is a unique morphism $\rho\colon \Gamma_2 \to \mathsf{Diff}(\mathbf{R}, 0)$ that maps the a_i to the f_i, and the b_i to the g_i, and this representation satisfies $\Psi(\rho) = (g_1, f_2, g_2)$. With this bijection Ψ and the topology of Leslie, we can identify $\mathcal{R}$ with a union of Baire spaces, in which the Baire category argument applies.

1.4. Other fields. Let $\mathbf{k}$ be a finite field with p elements. The group $\mathsf{Diff}^1(\mathbf{k}, 0)$, also known as the Nottingham group, is the group of power series tangent to the identity and with coefficients in the finite field $\mathbf{k}$. It is a compact group containing a free group (see [Sze]). Thus, by [BGSS], it contains a surface group.

Now, let p be a prime number, and let $\mathbf{Q}_p$ be the field of p-adic numbers. Consider the subgroup $\mathsf{Diff}^1(\mathbf{Z}_p, 0) \subset \mathsf{Diff}(\mathbf{Q}_p, 0)$ of formal power series tangent to the identity and with coefficients in $\mathbf{Z}_p$. First, note that all elements f of $\mathsf{Diff}^1(\mathbf{Z}_p, 0)$ satisfy $\mathrm{rad}(f) \geq 1$, so that $\mathsf{Diff}^1(\mathbf{Z}_p, 0)$ acts faithfully as a group of (p-adic analytic) homeomorphisms on $\{z \in \mathbf{Z}_p \ ; \ |z| < 1\}$. So, in that respect, $\mathsf{Diff}^1(\mathbf{Z}_p, 0)$ is much better than the group of germs of diffeomorphisms $\mathsf{Diff}(\mathbf{C}, 0)$. Moreover, with the topology given by the product topology on the coefficients $a_n \in \mathbf{Z}_p$ of the power series, the group $\mathsf{Diff}^1(\mathbf{Z}_p, 0)$ becomes a compact group. And this compact group contains a free group. By the result of [BGSS] described in Section 1.2, it contains a copy of the surface group Γ_2. So, we get a surface group acting faithfully as a group of p-adic analytic homeomorphisms on $\{z \in \mathbf{Z}_p \ ; \ |z| < 1\}$. In Section 7 we give a third proof of Theorem A that starts with the case of p-adic coefficients.

1.5. Organisation. The article is split in four parts.

I. Sections 2 to 4 give a first proof of Theorem A; Section 4 is the only place where we deal with non-orientable surfaces. We refer to Theorem B in Section 4.3 for a stronger result, in which the field $\mathbf{R}$ is replaced by any non-discrete, complete valued field $\mathbf{k}$.

II. Section 5 and 6 present our second proof, based on the construction of a group topology on $\mathsf{Diff}(\mathbf{C}, 0)$.

III. Then, our p-adic proof is presented in Section 7.

IV. Section 8 draws some consequences and list a few open problems, while the appendix shows how to construct free groups in $\mathsf{Diff}(\mathbf{C}, 0)$, or $\mathsf{Diff}(\mathbf{k}, 0)$ for any non-discrete and complete valued field.

Part I

2. Germs of diffeomorphisms and the Koenigs Linearization Theorem

2.1. Formal diffeomorphisms. Let $\mathbf{k}$ be a field (of arbitrary characteristic). Denote by $\mathbf{k}[\![z]\!]$ the ring of formal power series in one variable with coefficients in $\mathbf{k}$. For every integer $n \geq 0$, let $A_n \colon \mathbf{k}[\![z]\!] \to \mathbf{k}$ denote the n-th coefficient function:

$$A_n \colon f = \sum a_n z^n \ \mapsto \ A_n(f) = a_n. \tag{2.1}$$

A *formal diffeomorphism* is a formal power series $f \in \mathbf{k}[\![z]\!]$ such that $A_0(f) = 0$ and $A_1(f) \neq 0$. The composition $f \circ g$ determines a group law on the set

$$\widehat{\mathsf{Diff}}\,(\mathbf{k},0) = \{f \in \mathbf{k}[\![z]\!] \mid A_0(f) = 0 \text{ and } A_1(f) \neq 0\} \tag{2.2}$$

of all formal diffeomorphisms.

For each $n \geq 1$, there is a polynomial $P_n \in \mathbf{Z}[A_1, B_1, \dots, A_n, B_n]$ such that if $f = \sum a_n z^n$ and $g = \sum b_n z^n$ then $f \circ g = \sum_{n \geq 1} P_n(a_1, b_1, \dots, a_n, b_n) z^n$. Similarly, there are polynomials $Q_n \in \mathbf{Z}[A_1, \dots, A_n][A_1^{-1}]$ such that $f^{-1} = \sum_{n \geq 1} Q_n(a_1, \dots, a_n) z^n$ if $f = \sum a_n z^n$; the polynomial function Q_n is given by the following *inversion formula*:

$$\frac{1}{a_1^n} \sum_{k_1, k_2, \dots} (-1)^{k_1 + k_2 + \dots} \cdot \frac{(n+1) \cdots (n-1+k_1+k_2+\dots)}{k_1! k_2! \cdots} \cdot \left(\frac{a_2}{a_1}\right)^{k_1} \left(\frac{a_3}{a_1}\right)^{k_2} \cdots$$

where $a_i = A_i(f)$ and the sum is over all sequences of integers k_i such that

$$k_1 + 2k_2 + 3k_3 + \cdots = n - 1.$$

We refer to [Jen] where this is proved for f and g tangent to the identity; the general case easily follows.

To encapsulate this kind of properties, we introduce the following definition. Let m be a positive integer. By definition, a function $Q\colon \widehat{\mathsf{Diff}}\,(\mathbf{k},0)^m \to \mathbf{k}$ is a *polynomial function* with integer coefficients, if there is an integer n, and a polynomial $q \in \mathbf{Z}[A_{1,1}, A_{2,1}, \dots, A_{m-1,n}, A_{m,n}][A_{1,1}^{-1}, \dots, A_{m,1}^{-1}]$ such that

$$Q(f_1, \dots, f_m) = q(A_1(f_1), \dots, A_n(f_m)) \tag{2.3}$$

for all m-tuples $(f_1, \dots, f_m) \in \widehat{\mathsf{Diff}}\,(\mathbf{k},0)^m$; we denote by $\mathbf{Z}[\widehat{\mathsf{Diff}}\,(\mathbf{k},0)^m]$ this ring of polynomial functions.

Let $\mathbf{F}_m = \langle e_1, \dots, e_m \rangle$ be the free group of rank m. To every word $w = e_{i_1}^{n_1} \dots e_{i_k}^{n_k}$ in $\mathbf{F}_m$, with exponents $n_i \in \mathbf{Z}$, we associate the *word map* $w : \widehat{\mathsf{Diff}}\,(\mathbf{k},0)^m \to \widehat{\mathsf{Diff}}\,(\mathbf{k},0)$,

$$(g_1, \dots, g_m) \mapsto w(g_1, \dots, g_m) \overset{\text{def}}{=} g_{i_1}^{n_1} \circ \dots \circ g_{i_k}^{n_k}. \tag{2.4}$$

Since composition and inversion are polynomial functions on $\widehat{\mathsf{Diff}}\,(\mathbf{k},0)$, we get:

Lemma 2.1. *Let $w : \widehat{\mathsf{Diff}}\,(\mathbf{k},0)^m \to \widehat{\mathsf{Diff}}\,(\mathbf{k},0)$ be the word map given by some element of the free group $\mathbf{F}_m$. For each $n \geq 1$, there is a polynomial function $Q_{w,n} \in \mathbf{Z}[\widehat{\mathsf{Diff}}\,(\mathbf{k},0)^m]$ such that*

$$A_n\big(w(g_1, \dots, g_m)\big) = Q_{w,n}(g_1, \dots, g_m)$$

for all $g_1, \dots, g_m \in \widehat{\mathsf{Diff}}\,(\mathbf{k},0)$.

2.2. Diffeomorphisms and Koenigs linearization Theorem. Suppose now that $\mathbf{k}$ is endowed with an absolute value $|\cdot|\colon \mathbf{k} \to \mathbf{R}_+$. Then $\mathbf{k}$ becomes a metric space with the distance induced by $|\cdot|$. We shall almost always assume that

- $\mathbf{k}$ is not discrete, equivalently there is an element $x \in \mathbf{k}$ with $|x| \neq 0, 1$;
- $\mathbf{k}$ is complete.

Let $\mathbf{k}\{z\}$ be the subring of $\mathbf{k}[\![z]\!]$ consisting of power series $f(z) = \sum a_n z^n$ whose radius of convergence $\operatorname{rad}(f)$ is positive (see Equation (1.1)). When $\mathbf{k}$ is complete, the series $\sum a_n z^n$ converges uniformly in the closed disk $\mathbb{D}_r = \{z \in \mathbf{k} \mid |z| \leq r\}$ for every $r < \operatorname{rad}(f)$. The group of germs of analytic diffeomorphisms is the intersection $\mathsf{Diff}(\mathbf{k}, 0) := \widehat{\mathsf{Diff}}(\mathbf{k}, 0) \cap \mathbf{k}\{z\}$; it is a subgroup of $\widehat{\mathsf{Diff}}(\mathbf{k}, 0)$.

A germ $f \in \mathsf{Diff}(\mathbf{k}, 0)$ is *hyperbolic* if $|A_1(f)| \neq 1$. The following result is proved in [Mil, Chapter 8] and [HY, Theorem 1, p. 423] (see also [Sie, Theorem 1] or [Koe]).

Theorem 2.2 (Koenigs linearization theorem). *Let $(\mathbf{k}, |\cdot|)$ be a complete, non-discrete valued field. Let $f \in \mathsf{Diff}(\mathbf{k}, 0)$ be a hyperbolic germ of diffeomorphism. There is a unique germ of diffeomorphism $h \in \mathsf{Diff}(\mathbf{k}, 0)$ such that $h(f(z)) = A_1(f) \cdot h(z)$ and $A_1(h) = 1$.*

3. Embedding orientable surface groups

3.1. Abstract setting. Our strategy to construct embeddings of surface groups relies on the following simple remark. Let Γ be a countable group, and G be any group. Consider a non-empty topological space $\mathcal{R}$, with a map $\Phi : s \in \mathcal{R} \mapsto \Phi_s \in \operatorname{Hom}(\Gamma, G)$. Given $g \in \Gamma$, set $\mathcal{R}_g = \{s \in \mathcal{R} \mid \Phi_s(g) = 1\}$.

Lemma 3.1. *Assume that $\mathcal{R}$ has the following three properties:*

(1) ***Baire:*** *$\mathcal{R}$ is a Baire space;*

(2) ***Separation:*** *for every $g \neq 1$ in Γ, $\Phi_s(g) \neq 1$ for some $s \in \mathcal{R}$;*

(3) ***Irreducibility:*** *for every $g \in \Gamma$, either $\mathcal{R}_g = \mathcal{R}$ or $\mathcal{R}_g$ is closed with empty interior.*

Then the set of $s \in \mathcal{R}$ such that Φ_s is an injective homomorphism is a dense G_δ in $\mathcal{R}$; in particular, it is non-empty.

Proof. For any $g \in \Gamma \setminus \{1\}$, one has $\mathcal{R}_g \neq \mathcal{R}$ by (2), so $\mathcal{R}_g$ is closed with empty interior by (3). By the Baire property, $\mathcal{R} \setminus (\cup_{g \in \Gamma \setminus \{1\}} \mathcal{R}_g)$ is a dense G_δ. But $\mathcal{R} \setminus (\cup_{g \in \Gamma \setminus \{1\}} \mathcal{R}_g)$ is precisely the set of $s \in \mathcal{R}$ such that Φ_s is injective. $\square$

3.2. Baumslag Lemma. As explained in the introduction, it is proved in [Bau] that the fundamental group of an orientable surface is fully-residually free. We need a precise version of this result; to obtain it, the main technical input is the Baumslag's Lemma (see [Ols, Lemma 2.4]):

Lemma 3.2 (Baumslag's Lemma). *Let $n \geq 1$ be a positive integer. Let $g_0, \ldots, g_n$ be elements of $\mathbf{F}_k$, and let $c_1, \ldots, c_n$ be elements of $\mathbf{F}_k \setminus \{1\}$. Assume that for all $1 \leq i \leq n-1$, $g_i^{-1} c_i g_i$ does not commute with c_{i+1}. Then for N large enough,*

$$g_0 c_1^N g_1 c_2^N \ldots c_{n-1}^N g_{n-1} c_n^N g_n \neq 1.$$

Sketch of proof (*I*). The group $\mathsf{PSL}_2(\mathbf{R})$ acts on the hyperbolic plane $\mathbb{H}$ by isometries, and contains a subgroup Γ such that (0) Γ is isomorphic to $\mathbf{F}_k$, (1) every element $g \neq \mathrm{Id}$ in Γ is a loxodromic isometry of $\mathbb{H}$, and (2) two elements g and h in $\Gamma \setminus \{\mathrm{Id}\}$ commute if and only if they have the same axis, which happens if and only if they share a common fixed point on $\partial\mathbb{H}$. One can find such a group in any lattice of $\mathsf{PSL}_2(\mathbf{R})$. To prove the lemma, we prove it in Γ.

Fix a base point $x \in \mathbb{H}$, denote by α_i and ω_i the repulsive and attracting fixed points of c_i in $\partial\mathbb{H}$, and consider the word

$$g_0 c_1^N g_1 c_2^N g_2.$$

For m large enough, $c_2^m g_2$ maps x to a point which is near ω_2. If $g_1(\omega_2)$ were equal to α_1, then c_1 and $g_1 c_2 g_1^{-1}$ would share the common fixed point α_1, and they would commute. Thus, $g_1(\omega_2) \neq \alpha_1$ and then $g_0 c_1^{m'} g_1 c_2^m g_2$ maps x to a point which is near $g_0(\omega_1)$ if m' is large enough. Thus, $g_0 c_1^N g_1 c_2^N g_2(x) \neq x$ for large N. The proof is similar if n is larger than 2. □

Sketch of proof (*II*). We rephrase this proof, using the action of $\mathbf{F}_k$ on its boundary, because this boundary will also be used in the proof of Proposition 3.3.

Fix a basis $a_1, \ldots, a_k$ of $\mathbf{F}_k$, and denote by $\partial\mathbf{F}_k$ the boundary of $\mathbf{F}_k$. The elements of $\partial\mathbf{F}_k$ are represented by infinite reduced words in the generators a_i and their inverses. If g is an element of $\mathbf{F}_k$ and α is an element of $\partial\mathbf{F}_k$ the concatenation $g \cdot \alpha$ is an element of $\partial\mathbf{F}_k$: this defines an action of $\mathbf{F}_k$ by homeomorphisms on the Cantor set $\partial\mathbf{F}_k$. If g is a non-trivial, its action on $\partial\mathbf{F}_k$ has exactly two fixed points, given by the infinite words $\omega(g) = g \cdots g \cdots$ and $\alpha(g) = g^{-1} \cdots g^{-1} \cdots$ (there are no simplifications if g is given by a reduced and cyclically reduced word). Then we get: (1) every element $g \neq \mathrm{Id}$ in $\mathbf{F}_k$ has a north-south dynamics on $\partial\mathbf{F}_k$, every orbit $g^n \cdot \beta$ converging to $\omega(g)$, except when $\beta = \alpha(g)$, and (2) two elements g and h in $\mathbf{F}_k \setminus \{\mathrm{Id}\}$ commute if and only if they have the same fixed points, which happens if and only if they share

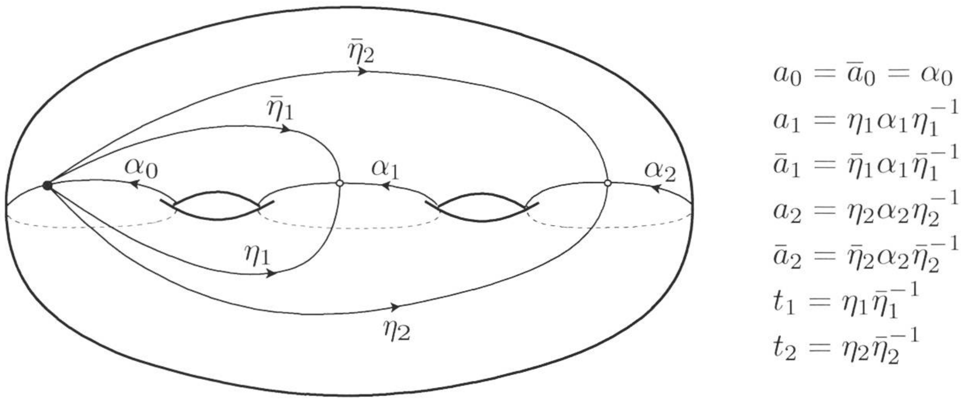

FIGURE 1

The fundamental group Γ_2. – The α_i are three loops, while the η_j and $\overline{\eta_j}$ are four paths. The figure is symmetric with respect to the plane cutting the surface along the loops α_i.

a common fixed point on $\partial\mathbf{F}_k$. One can then repeat the previous proof with the action of $\mathbf{F}_k$ on its boundary. $\square$

Write the surface of genus 2 as the union of two pairs of pants as in Figure 1, with respective fundamental groups

$$\langle a_0, a_1, a_2 \mid a_0a_1a_2 = 1\rangle \text{ and } \langle \overline{a}_0, \overline{a}_1, \overline{a}_2 \mid \overline{a}_0\overline{a}_1\overline{a}_2 = 1\rangle. \tag{3.1}$$

This gives the presentation

$$\Gamma_2 = \left\langle \begin{matrix} a_0, a_1, a_2, \\ \overline{a}_0, \overline{a}_1, \overline{a}_2, \\ t_1, t_2 \end{matrix} \;\middle|\; \begin{matrix} a_0a_1a_2 = 1, \\ \overline{a}_0\overline{a}_1\overline{a}_2 = 1, \\ \overline{a}_0 = a_0,\ \overline{a}_1 = t_1^{-1}a_1t_1,\ \overline{a}_2 = t_2^{-1}a_2t_2 \end{matrix} \right\rangle \tag{3.2}$$

which can be rewritten as

$$\Gamma_2 = \langle a_0, a_1, a_2, t_1, t_2 \mid a_0a_1a_2 = 1,\ a_0t_1^{-1}a_1t_1t_2^{-1}a_2t_2 = 1\ \rangle. \tag{3.3}$$

Denote by $p : \Gamma_2 \to \langle a_0, a_1, a_2\rangle \simeq \mathbf{F}_2$ the morphism defined by $p(a_i) = a_i$, $p(\overline{a}_i) = a_i$, and $p(t_1) = p(t_2) = 1$. Let $\tau : \Gamma_2 \to \Gamma_2$ be the (left) Dehn twist along the three curves a_0, a_1, and a_2, i.e. the automorphism fixing a_i and sending t_i to $a_it_ia_0^{-1}$ for $i = 1, 2$. Note the following facts:

- τ sends $\overline{a}_i$ to $a_0\overline{a}_ia_0^{-1}$; in particular, if g is a word in the $\overline{a}_i$, then $\tau^N(g) = a_0^N g a_0^{-N}$;
- $p \circ \tau^N$ fixes a_i for every $i = 0,\ 1,\ 2$, and

  $$p \circ \tau^N(t_j) = a_j^N a_0^{-N} \tag{3.4}$$

  for $j = 1,\ 2$.

Proposition 3.3. *For every $g \in \Gamma_2 \setminus \{1\}$, there exists a positive integer n_0 such that $p \circ \tau^N(g) \neq 1$ for all $N \geq n_0$.*

Proof. To uniformize notations, we define $t_0 = 1$ so that for all $i \in \{0, 1, 2\}$ the relation $t_i \overline{a}_i t_i^{-1} = a_i$ holds, and τ maps t_i to $a_i t_i a_0^{-1}$. Let $A = \langle a_0, a_1, a_2 \rangle$ and $\overline{A} = \langle \overline{a}_0, \overline{a}_1, \overline{a}_2 \rangle$. Write g as a shortest possible word of the following form:

$$g = g_0 t_{i_1} g_1 t_{i_2}^{-1} g_2 t_{i_3} \dots g_{n-1} t_{i_n}^{-1} g_n \tag{3.5}$$

where n is even, $i_k \in \{0, 1, 2\}$ for all $k \leq n$, $g_k \in A$ for k even, $g_k \in \overline{A}$ for k odd, and the exponent of t_{i_k} is $(-1)^{k+1}$ (we allow $g_k = 1$). One easily checks that g can be written in this form because all generators can (for instance $t_1 = 1 \cdot t_1 \cdot 1 \cdot t_0^{-1} \cdot 1$).

If k is such that $i_k = i_{k+1}$, then $g_k \notin \langle a_{i_k} \rangle$ if k is even (resp $g_k \notin \langle \overline{a}_{i_k} \rangle$ if k is odd) as otherwise, one could shorten the word using the relation $t_{i_k} a_{i_k} t_{i_k}^{-1} = \overline{a}_{i_k}$.

First claim. *If $k \in \{2, \dots, n-2\}$ is even, $g_k^{-1} a_{i_k} g_k$ does not commute to $a_{i_{k+1}}$.*

If $i_k \neq i_{k+1}$, this is because $g_k \in A \simeq \mathbf{F}_2$ and no pair of A-conjugates of a_{i_k} and $a_{i_{k+1}}$ commute. If $i_k = i_{k+1}$, then $g_k \notin \langle a_{i_k} \rangle$ as we have just seen; since a_{i_k} is not a proper power in A, this shows that $g_k.(a_{i_k}^{+\infty}) \neq a_{i_k}^{+\infty}$ in the boundary at infinity of the free group A, so $g_k^{-1} a_{i_k} g_k$ does not commute with a_{i_k}, and the claim follows.

Similarly, using the fact that $g_k \in \overline{A}$ for odd indices, we obtain:

Second claim. *If $k \leq n-1$ is odd, $g_k^{-1} \overline{a}_{i_k} g_k$ does not commute to $\overline{a}_{i_{k+1}}$.*

We have $\tau^N(g_k) = g_k$ if k is even, and $\tau^N(g_k) = a_0^N g_k a_0^{-N}$ if k is odd. After simplifications, one has

$$\tau^N(g) = g_0 a_{i_1}^N t_{i_1} g_1 t_{i_2}^{-1} a_{i_2}^{-N} g_2 a_{i_3}^N t_{i_3} \dots g_{n-1} t_{i_n}^{-1} a_{i_n}^{-N} g_n. \tag{3.6}$$

For k odd, denote by $g'_k \in \mathbf{F}_r$ the image of g_k under p. Applying p, we thus get

$$p \circ \tau^N(g) = g_0 a_{i_1}^N g'_1 a_{i_2}^{-N} g_2 a_{i_3}^N g'_3 \dots g'_{n-1} a_{i_n}^{-N} g_n, \tag{3.7}$$

with $g'_i := p(g_i)$. Let us check that the hypotheses of the Baumslag Lemma 3.2 apply. For k even, the first claim shows that $g_k^{-1} a_{i_k} g_k$ does not commute to $a_{i_{k+1}}$, as required. For k odd, we use that p is injective on $\overline{A}$ and that $\overline{A}$ contains $g_k^{-1} \overline{a}_{i_k} g_k$ and $\overline{a}_{i_{k+1}}$, and we apply the second claim to deduce that $g_k'^{-1} a_{i_k} g'_k$ does not commute to $a_{i_{k+1}}$. Applying Baumslag's Lemma, we conclude that $p \circ \tau^N(g) \neq 1$ for N large enough. □

3.3. Embeddings of free groups. The group $\mathsf{Diff}(\mathbf{R},0)$ contains non-abelian free groups. This has been proved by arithmetic means [Whi, Gla], by looking at the monodromy of generic polynomial planar vector fields [IP], and by a dynamical argument [MRR]. We shall need the following precise version of that result.

Theorem 3.4. *Let $(\mathbf{k},|\cdot|)$ be a complete non-discrete valued field. For every pair (λ_1,λ_2) in $\mathbf{k}^*$, there exists a pair f_1, $f_2 \in \mathsf{Diff}(\mathbf{k},0)$ that generates a free group and satisfies $f_1'(0)=\lambda_1$ and $f_2'(0)=\lambda_2$.*

This result is proved in [BCLN, Proposition 4.3] for generic pairs of derivatives (λ_1,λ_2). We provide a proof of Theorem 3.4 in the Appendix, extending the argument of [MRR]. We refer to Section 7.1 below for other approaches.

3.4. Embedding orientable surface groups. We can now prove Theorem A for orientable surfaces:

Theorem 3.5. *Let Γ_g be the fundamental group of a closed, orientable surface of genus g. Then, there exists an injective morphism $\Gamma_g \to \mathsf{Diff}(\mathbf{R},0)$.*

The group Γ_0 is trivial. The group Γ_1 is isomorphic to $\mathbf{Z}^2$, hence it embeds in the group of homotheties $z\mapsto \lambda z$, $\lambda\in\mathbf{R}_+^*$. If $g\geq 2$, then Γ_g embeds in Γ_2. To see this, fix a surjective morphism $\Gamma_2\to\mathbf{Z}$, and take the preimage $\Lambda\subset\Gamma_2$ of the subgroup $(g-1)\mathbf{Z}\subset\mathbf{Z}$. Then, Λ is a normal subgroup of index $g-1$ in Γ_2, and it is the fundamental group of a closed surface Σ, given by a Galois cover of degree $g-1$ of the surface of genus 2. Since the Euler characteristic is multiplicative, the genus of Σ satisfies $-2(g-1)=2-2g(\Sigma)$. Thus, $g(\Sigma)=g$ and Λ is isomorphic to Γ_g. Thus, we now restrict to the case $g=2$.

By Theorem 3.4, we can fix an injective morphism

$$\rho_0 : \mathbf{F}_2 = \langle a_0, a_1, a_2 \mid a_0a_1a_2 = 1\rangle \to \mathsf{Diff}(\mathbf{R},0) \tag{3.8}$$

such that the images $f_1=\rho_0(a_1)$, $f_2=\rho_0(a_2)$, and $f_0=\rho_0(a_0)=f_2^{-1}f_1^{-1}$ satisfy

$$f_1'(0)=\lambda_1>1,\quad f_2'(0)=\lambda_2>1,\quad f_0'(0)=\lambda_0<1 \tag{3.9}$$

for some real numbers λ_1 and $\lambda_2>1$ and $\lambda_0=(\lambda_1\lambda_2)^{-1}$. In particular, f_0, f_1, and f_2 are hyperbolic. For $\lambda\in\mathbf{R}^*$, denote by $m_\lambda(z)=\lambda z$ the corresponding homothety. For $i\in\{0,1,2\}$, the Koenigs linearization theorem shows that f_i is conjugate to the homothety m_{λ_i} : there is a germ of diffeomorphism $h_i\in\mathsf{Diff}(\mathbf{R},0)$ such that $f_i=h_i\circ m_{\lambda_i}\circ h_i^{-1}$. Thus f_i extends to the multiplicative flow $\varphi_i:\mathbf{R}_+^*\to\mathsf{Diff}(\mathbf{R},0)$ defined by $\varphi_i^s=h_i\circ m_s\circ h_i^{-1}$ for $s\in\mathbf{R}_+^*$; by contruction,

$\varphi_i^{\lambda_i} = f_i$ and φ_i^s commutes with f_i for all $s > 0$. We note that $s \mapsto \varphi_i^s$ is polynomial in the sense that for all $k \in \mathbf{N}$, $s \mapsto A_k(\varphi_i^s)$ is a polynomial function with real coefficients in the variables s and s^{-1}.

Set $\mathcal{R} = (\mathbf{R}_+^*)^3$. As in Section 3.2, consider the presentation

$$\Gamma_2 = \langle a_0, a_1, a_2, t_1, t_2 \mid a_0 a_1 a_2 = 1,\ a_0 t_1^{-1} a_1 t_1 t_2^{-1} a_2 t_2 = 1 \rangle. \tag{3.10}$$

Given $s = (s_0, s_1, s_2) \in (\mathbf{R}_+^*)^3$, we define a morphism $\Phi_s : \Gamma_2 \to \mathsf{Diff}(\mathbf{R}, 0)$ by

$$\Phi_s(a_i) = f_i \quad \text{for } i \in \{0, 1, 2\} \tag{3.11}$$

$$\Phi_s(t_i) = \varphi_i^{s_i} \varphi_0^{s_0} \quad \text{for } i \in \{1, 2\} \tag{3.12}$$

This provides a well defined homomorphism because φ_i commutes with f_i. As we shall see below, this morphism Φ_s is constructed to coincide with $\rho_0 \circ p \circ \tau^N$ for $s = (\lambda_0^N, \lambda_1^N, \lambda_2^N)$ (see Equation (3.4)).

Remark 3.6. For every $s \in \mathcal{R}$, the image of Φ_s contains f_1 and f_2, hence the free group $\rho_0(\mathbf{F}_2)$. This will be used in Section 4.3.

To conclude, we check that the three assumptions of Lemma 3.1 hold for this family of morphisms $(\Phi_s)_{s \in \mathcal{R}}$.

Clearly, $\mathcal{R}$ is a Baire space.

To check the irreducibility property, consider $g \in \Gamma_2$ and assume that $\mathcal{R}_g \neq \mathcal{R}$: this means that there exists a parameter $s \in \mathcal{R}$ and an index $k \geq 1$ such that $A_k(\Phi_s(g)) \neq A_k(\mathrm{Id})$. The map $s = (s_0, s_1, s_2) \mapsto A_k(\Phi_s(g)) - A_k(\mathrm{Id})$ is a polynomial function in the variables $s_0^{\pm 1}$, $s_1^{\pm 1}$, and $s_2^{\pm 1}$ that does not vanish identically on $\mathcal{R}$, so its zero set is a closed subset with empty interior.

We now check that $\mathcal{R}$ has the separation property. As in Section 3.2, denote by $p : \Gamma_2 \to \mathbf{F}_2 = \langle a_0, a_1, a_2 \mid a_0 a_1 a_2 = 1 \rangle$ the morphism obtained by killing t_1 and t_2. For the parameter $s = (1, 1, 1)$, Φ_s is equal to $\rho_0 \circ p$. More generally, setting $s_N = (\lambda_0^N, \lambda_1^N, \lambda_2^N)$ for $N \in \mathbf{N}$, the morphism $\Phi_{s_N} : \Gamma_2 \to \mathsf{Diff}(\mathbf{R}, 0)$ satisfies

$$\Phi_{s_N}(a_i) = f_i \quad \text{for } i \in \{0, 1, 2\} \tag{3.13}$$

$$\Phi_{s_N}(t_i) = \varphi_i^{u_i^N} \varphi_0^{u_0^N} = f_i^N f_0^N \quad \text{for } i \in \{1, 2\}. \tag{3.14}$$

This means that $\Phi_{s_N} = \rho_0 \circ p \circ \tau^N$ where, as in Section 3.2, $\tau : \Gamma_2 \to \Gamma_2$ is the Dehn twist along the three curves a_i. By Proposition 3.3, for all $g \in \Gamma_2 \setminus \{1\}$ there exists $N \in \mathbf{N}$ such that $p \circ \tau^N(g) \neq 1$. Since ρ_0 is injective, this implies that $\Phi_{s_N}(g) \neq 1$ which shows that $\mathcal{R}$ has the separation property.

4. Non-orientable surface groups

Theorem 4.1. *Let N_g be the fundamental group of a closed non-orientable surface of genus $g \geq 4$. There exists an injective morphism $N_g \to \mathsf{Diff}(\mathbf{R}, 0)$.*

Remark 4.2. The fundamental group N_3 of the non-orientable surface of genus 3 is not fully residually free, and our methods do not apply to this group. (See [Lyn, Proposition 9].)

4.1. Even genus. We first treat the case of an even genus $g \geq 4$. In this case, the group N_g embeds in N_4. Indeed, the non-orientable surface of genus 4 is the connected sum of a torus $\mathbf{R}^2/\mathbf{Z}^2$ with two projective planes $\mathbb{P}^2(\mathbf{R})$. Taking a cyclic cover of the torus of degree k, we get a surface homeomorphic to the connected sum of $\mathbf{R}^2/\mathbf{Z}^2$ with $2k$ copies of $\mathbb{P}^2(\mathbf{R})$, hence a non-orientable surface of genus $2(k+1)$. Thus, it suffices to prove that N_4 embeds in $\mathsf{Diff}(\mathbf{R}, 0)$.

The non-orientable surface of genus 4 is homeomorphic to the connected sum of 4 copies of $\mathbb{P}^2(\mathbf{R})$, and this gives the presentation (see Figure 2)

$$N_4 = \langle a_1, a_2, b_1, b_2 \mid a_1^2 a_2^2 b_2^2 b_1^2 = 1 \rangle. \tag{4.1}$$

Let $p : N_4 \to \langle a_1, a_2 \rangle$ be the morphism fixing a_1, a_2 and sending b_1 and b_2 to a_1^{-1} and a_2^{-1} respectively. Let $\tau : N_4 \to N_4$ be the Dehn twist around the curve $\gamma = (a_1^2 a_2^2)^{-1}$, i.e., the automorphism that fixes a_1 and a_2 and sends b_1 and b_2 to $\gamma b_1 \gamma^{-1}$ and $\gamma b_2 \gamma^{-1}$ respectively.

Lemma 4.3. *Given any $g \in N_4 \setminus \{1\}$, there exists $n_0 \in \mathbf{N}$ such that for all $N \geq n_0$, $p \circ \tau^N(g) \neq 1$.*

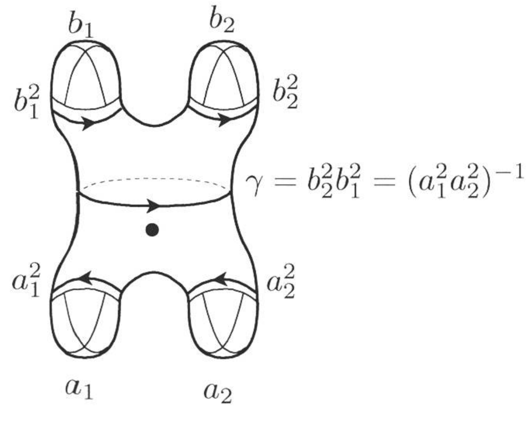

FIGURE 2
The fundamental group N_4. – The base point is represented by $\bullet$, the 4 generators are a_1, a_2, b_1, b_2, and the curve γ is used to construct the Dehn twist τ.

For the proof. The proof of this statement is completely analogous to the proof of [BGSS, Corollary 2.2], using Baumslag Lemma, we leave it as an exercise to the reader. See also [CG, Proposition 4.13]. □

Using Theorem 3.4, we fix two germs of diffeomorphisms f_1 and $f_2 \in \mathsf{Diff}(\mathbf{R},0)$ generating a free group and satisfying $f_1'(0) > 1$ and $f_2'(0) > 1$. We denote by

$$\rho_0 : \mathbf{F}_2 = \langle a_1, a_2 \rangle \to \mathsf{Diff}(\mathbf{R},0) \tag{4.2}$$

the injective morphism sending a_i to f_i for $i \in \{1,2\}$. In particular,

$$\rho_0(\gamma) = (f_1^2 \circ f_2^2)^{-1} \tag{4.3}$$

is a hyperbolic germ: its derivative $\lambda = ((f_1^2 \circ f_2^2)'(0))^{-1}$ is < 1. The Koenigs linearization theorem gives an element $h \in \mathsf{Diff}(\mathbf{R},0)$ such that $\rho_0(\gamma) = h \circ m_\lambda \circ h^{-1}$. Consider the multiplicative flow $\varphi : \mathbf{R}_+^* \to \mathsf{Diff}(\mathbf{R},0)$ defined by $\varphi^s = g \circ m_s \circ g^{-1}$. As above, $\varphi^\lambda = \rho_0(\gamma)$, φ^s commutes with $\rho_0(\gamma)$ for all $s > 0$, and $s \mapsto \varphi^s$ is a polynomial map: for all $k \in \mathbf{N}$, $s \mapsto A_k(\varphi^s)$ is a polynomial in the variables s and s^{-1}.

Set $\mathcal{R} = \mathbf{R}_+^*$. Given $s \in \mathbf{R}_+^*$, consider the morphism $\rho_s : N_4 \to \mathsf{Diff}(\mathbf{R},0)$ defined by

$$\begin{aligned} a_1 &\mapsto f_1 & \qquad a_2 &\mapsto f_2 \\ b_1 &\mapsto \varphi^s f_1^{-1} \varphi^{-s} & \qquad b_2 &\mapsto \varphi^s f_2^{-1} \varphi^{-s}. \end{aligned}$$

This gives a well defined homomorphism because φ^s commutes with $f_1^2 f_2^2$.

We now check the three assumptions of Lemma 3.1. Clearly, $\mathcal{R}$ is a Baire space. The irreducibility is a consequence of the fact that for any $g \in N_4$, and any $k \in \mathbf{N}$ the map $s \mapsto A_k(\varphi_s(g))$ is a polynomial function in the variables $s^{\pm 1}$. The separation property follows from Lemma 4.3 together with the fact that $\rho_{\lambda^N} = \rho_0 \circ p \circ \tau^N$ and that ρ_0 is injective.

4.2. Odd genus. We now treat the case of a non-orientable surface of odd genus $g = 2k+1$, $k \geq 2$. One can write N_{2k+1} as (see Figure 3 below)

$$N_{2k+1} = \langle a_1, \ldots, a_k, c, b_1, \ldots, b_k \mid a_1^2 \ldots a_k^2 c^2 b_k^2 \ldots b_1^2 = 1 \rangle. \tag{4.4}$$

This group splits as a double amalgam of free groups

$$N_{2k+1} = \langle a_1, \ldots a_k \rangle \underset{a_1^2 \ldots a_k^2 = \gamma^{-1}}{*} \langle \gamma, c \rangle \underset{c^{-2}\gamma = b_k^2 \ldots b_1^2}{*} \langle b_1, \ldots, b_k \rangle. \tag{4.5}$$

We shall use the following notation to refer to this amalgam structure:

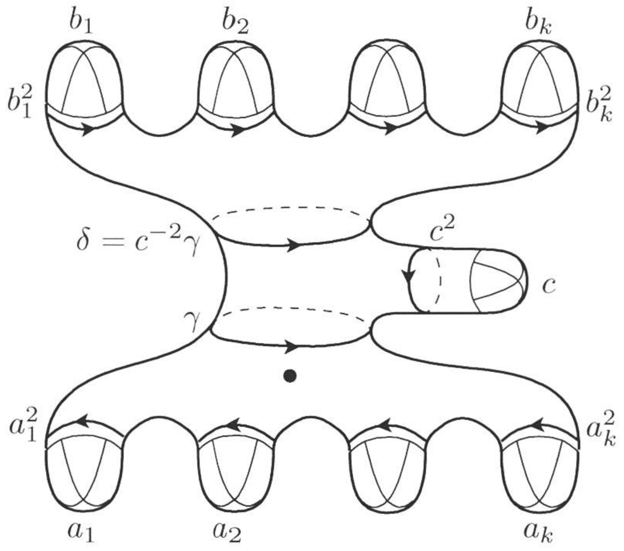

FIGURE 3
The fundamental group N_{2k+1}

- $A_1 = \langle a_1, \dots a_k \rangle$ and $e_{1,2} = (a_1^2 \dots a_k^2)^{-1}$;
- $A_2 = \langle \gamma, c \rangle$ and $e_{2,1} = \gamma$ and $e_{2,3} = c^{-2}\gamma = \delta$;
- $A_3 = \langle b_1, \dots, b_k \rangle$ and $e_{3,2} = b_k^2 \dots b_1^2$.

So, each of the A_i is a free group and the amalgamation is given by $e_{1,2} = e_{2,1}$ and $e_{2,3} = e_{3,2}$.

Define a morphism $p\colon N_{2k+1} \to \langle a_1, \dots, a_k \rangle \simeq \mathbf{F}_k$ by

$$a_i \mapsto a_i \quad \text{for } i \le k \qquad\qquad c \mapsto a_k^{-2}$$
$$b_i \mapsto a_i^{-1} \quad \text{for } i \le k-1 \qquad\qquad b_k \mapsto a_k$$

(the structure of almagam shows that p is well defined).

Lemma 4.4. *The morphism $p\colon N_{2k+1} \to \mathbf{F}_k$ is injective in restriction to each of the three subgroups of the amalgam* (4.5).

Proof. By construction, it is injective in restriction to $\langle a_1, \dots, a_k \rangle$ and in restriction to $\langle b_1, \dots, b_k \rangle$. Then, note that $p(\langle \gamma, c \rangle) = \langle a_1^2 \dots a_k^2, a_k^{-2} \rangle$ is isomorphic to $\mathbf{F}_2$ because it is a non-abelian subgroup of a free group. Since $\mathbf{F}_2$ is Hopfian, p is necessarily injective in retriction to $\langle \gamma, c \rangle$. □

Consider $\delta = b_k^2 \cdots b_1^2 = c^{-2}\gamma$ and note that $p(\delta) = a_k^2 a_{k-1}^{-2} \cdots a_1^{-2}$. Let τ be the Dehn twist corresponding to the decomposition above, i.e. the automorphism fixing a_i, sending c to $\gamma c \gamma^{-1}$ and sending b_i to $(\gamma\delta) b_i (\gamma\delta)^{-1}$. Since τ is the composition of the twists given by γ and δ and these two twists commute we get

$$\tau^N(b) = \left(\gamma^N\delta^N\right)b\left(\gamma^N\delta^N\right)^{-1}, \quad \forall b \in A_3.$$

In this situation, one can prove the following lemma in a similar way to Proposition 3.3.

Lemma 4.5. *Given any $g \in N_{2g+1} \setminus \{1\}$, there exists $n_0 \in \mathbf{N}$ such that for all $N \geq n_0$, $p \circ \tau^N(g) \neq 1$.*

Proof. Write g as a word in the graph of groups, i.e., $g = s_0 \dots s_n$ with $s_k \in A_{r_k}$ (we allow $s_k = 1$) for some $r_k \in \{1,2,3\}$, with $r_{k+1} = r_k \pm 1$, and $r_0 = r_n = 1$. We take this word of minimal possible length among words satisfying these contraints. If k is such that $r_{k-1} = r_{k+1}$, then $s_k \notin \langle e_{r_k,r_{k+1}}\rangle$ since otherwise, one could shorten the word using the structure of amalgam (in particular $s_k \neq 1$ in this case). Now one easily checks that

$$\tau^N(g) = s_0 d_1^{\varepsilon_1 N} s_1 d_2^{\varepsilon_2 N} s_2 \cdots d_n^{\varepsilon_n N} s_n \tag{4.6}$$

where $d_k = e_{r_{k-1},r_k} \in \{\gamma, \delta\}$, and $\varepsilon_k = r_k - r_{k-1} \in \{\pm 1\}$.

We claim that $s_k^{-1} d_k s_k$ does not commute with d_{k+1}. If $d_k \neq d_{k+1}$, this follows from the fact that γ commutes with no conjugate of δ in $A_2 = \langle c, \delta\rangle$. If $d_k = d_{k+1}$, then $r_{k-1} = r_{k+1}$, so $s_k \notin \langle e_{r_k,r_{k+1}}\rangle = \langle d_k\rangle$. If $[s_k^{-1} d_k s_k, d_k] = 1$, then s_k preserves the axis of d_k in the Cayley graph of the free group A_{r_k}, so s_k is a power of d_k, because $d_k \in \{\gamma, \delta\}$ is not a proper power; this contradicts that $s_k \notin \langle d_k\rangle$.

Denote by $\overline{s}_k$, $\overline{d}_k \in \mathbf{F}_r$ the images of s_k, d_k under p. Since p is injective on each A_{r_k}, $\overline{s}_k^{-1}\overline{d}_k\overline{s}_k$ does not commute with $\overline{s}_{k+1}$, so the hypotheses of Baumslag Lemma apply to the word

$$p \circ \tau^N(g) = \overline{d}_0^{\varepsilon_0 N}\overline{s}_1\overline{d}_1^{\varepsilon_1 N}\overline{s}_2 \dots \overline{d}_{n-1}^{\varepsilon_{n-1} N}\overline{s}_n\overline{d}_n^{\varepsilon_n N} \tag{4.7}$$

so $p \circ \tau^N(g) \neq 1$ for N large enough. □

Now consider k elements $f_1, \dots, f_k$ of $\mathsf{Diff}(\mathbf{R}, 0)$ generating a free group of rank k with $f_i'(0) > 1$ for all $i \in \{1, \dots, k\}$, and $f_k'(0) < f_1'(0)$. Such a set can be obtained from two generators g_1 and g_2 of a free group of rank 2 with $g_i'(0) > 1$, as in Theorem 3.4, by taking $f_i = g_1^i \circ g_2^2 \circ g_1^{-i}$ for $i < k$ and $f_k = g_1^k \circ g_2 \circ g_1^{-k}$. Let $\rho_0 : \mathbf{F}_k = \langle a_1, \dots, a_k\rangle \to \mathsf{Diff}(\mathbf{R}, 0)$ be the injective morphism sending a_i to f_i for $i \leq k$. In particular, $\rho_0(\gamma) = (f_1^2 \circ \dots \circ f_k^2)^{-1}$ and $\rho_0(p(\delta)) = f_k^2 \circ f_{k-1}^{-2} \circ \dots \circ f_1^{-2}$ are hyperbolic. Using Koenigs linearization theorem as above, there exists two multiplicative flows φ and $\psi : \mathbf{R}_+^* \to \mathsf{Diff}(\mathbf{R}, 0)$ and a pair of positive real numbers λ and μ such that (1) $\varphi^\lambda = \rho_0(\gamma)$ and $\psi^\mu = \rho_0(p(\delta))$, and (2) $s \mapsto \varphi^s$ and $s \mapsto \psi^s$ are polynomial mappings.

Set $\mathcal{R} = (\mathbf{R}_+^*)^2$ and, for every $(s, s') \in \mathcal{R}$, define a morphism $\rho_{s,s'} : \mathbf{N}_{2k+1} \to \mathsf{Diff}(\mathbf{R}, 0)$ by

$$a_i \mapsto f_i \quad \text{for } i \leq k \qquad\qquad c \mapsto \varphi^s f_k^{-2} \varphi^{-s}$$
$$b_i \mapsto \varphi^s \psi^{s'} f_i^{-1} (\varphi^s \psi^{s'})^{-1} \quad \text{for } i \leq k-1 \qquad b_k \mapsto \varphi^s \psi^{s'} f_k (\varphi^s \psi^{s'})^{-1}$$

(this is well defined because φ^s and $\psi^{s'}$ commute with $\rho_0(\gamma) = (f_1^2 \circ \cdots \circ f_k^2)^{-1}$ and $\rho_0(p(\delta)) = f_k^2 \circ f_{k-1}^{-2} \circ \cdots \circ f_1^{-2}$ respectively).

The assumptions of Lemma 3.1 hold: $\mathcal{R}$ is a Baire space, and the irreducibility follows from the fact that the maps $s \mapsto \varphi_s$ and $s' \mapsto \varphi_{s'}$ are polynomials in the variables $s^{\pm 1}, s'^{\pm 1}$. The separation property follows from Lemma 4.5 together with the fact that $\rho_{\lambda^N, \mu^N} = \rho_0 \circ p \circ \tau^N$, and that ρ_0 is injective.

4.3. Embeddings in $\mathsf{Diff}(\mathbf{k}, 0)$. The proofs just given provide the following statement.

Theorem B. *Let $(\mathbf{k}, |\cdot|)$ be a non-discrete and complete valued field.*

(1) *Let Γ be the fundamental group of a closed orientable surface, or a closed non-orientable surface of genus ≥ 4. Then, there is an embedding of Γ into $\mathsf{Diff}(\mathbf{k}, 0)$.*

(2) *Let $F \subset \mathsf{Diff}(\mathbf{k}, 0)$ be a free group of rank 2, generated by two germs f and g with $|f'(0)| > 1$ and $|g'(0)| > 1$. Then, there is an embedding of Γ_2, the fundamental group of a closed, orientable surface of genus 2, into $\mathsf{Diff}(\mathbf{k}, 0)$ whose image contains F.*

Proof. For the first assertion, we just have to replace $\mathbf{R}$ by $\mathbf{k}$ in the proofs of Theorem 3.5 and 4.1. The parameter space is $\mathcal{R} = (\mathbf{k}^*)^3$ or $\mathbf{k}^*$ or $(\mathbf{k}^*)^2$, and it is a Baire space because $(\mathbf{k}, |\cdot|)$ is complete.

For the second assertion, we start with a representation ρ_0 in Equation (3.8) whose image is equal to F. Remark 3.6 shows that all the injective morphisms Φ_s that we get satisfy also $\Phi_s(\Gamma_2) \supset F$. □

Part II

5. The final topology on germs of diffeomorphisms

Let $(\mathbf{k}, |\cdot|)$ be a complete field. This section introduces a new topology on $\mathbf{k}\{z\}$ and $\mathsf{Diff}(\mathbf{k}, 0)$, which will be used in our second proof of Theorem A. The reader may very well skip this section on a first reading.

5.1. The final topology over the complex numbers. Until Section 5.4, we focus on the case $\mathbf{k} = \mathbf{C}$. Let r be a positive real number. Consider the subalgebra $\mathcal{A}_r$ of $\mathbf{C}\{z\}$ consisting of those power series $f(z) = \sum_n a_n z^n$ which converge on the open unit disk $\mathbb{D}_r$ (i.e., $\mathrm{rad}(f) \geq r$) and extend continuously to the closed unit disk $\overline{\mathbb{D}}_r$. When endowed with the norm

$$\|f\|_{\mathcal{A}_r} = \max_{z \in \overline{\mathbb{D}}_r} |f(z)|, \tag{5.1}$$

$\mathcal{A}_r$ is a Banach algebra. If $s < r$, the restriction of functions $f \in \mathcal{A}_r$ to the smaller disk $\overline{\mathbb{D}}_s$ determines a 1-Lipschitz embedding $\mathcal{A}_r \to \mathcal{A}_s$.

The space $\mathbf{C}\{z\}$ is the union of the algebras $\mathcal{A}_r$ and can be thus endowed with the final topology associated to the colimit

$$\mathbf{C}\{z\} = \varinjlim \mathcal{A}_r. \tag{5.2}$$

This means that a subset $\mathcal{U} \subset \mathbf{C}\{z\}$ is open if its intersection with $\mathcal{A}_r$ is open for every $r > 0$. Equivalently, a map $\varphi \colon \mathbf{C}\{z\} \to X$ to a topological space is continuous if and only if its composition with the embedding $\mathcal{A}_r \to \mathbf{C}\{z\}$ is continuous for all r. Unless we say it explicitly, open sets, neighborhoods, and continuous maps refer, from now on, to this topology. A word of warning: for $r > s$, the inclusion $\mathcal{A}_r \to \mathcal{A}_s$ is not a homeomorphism to its image, and neither is the inclusion $\mathcal{A}_r \to \mathbf{C}\{z\}$.

The goal of this section is to obtain several basic properties of this topology. For instance, we are going to prove that there is a filtration of $\mathbf{C}\{z\}$ by compact subsets $\mathbf{C}_c\{z\}$ so that the continuity can be checked in restriction to each $\mathbf{C}_c\{z\}$.

Remark 5.1. If $s < r$, the homomorphism $\mathcal{A}_r \to \mathcal{A}_s$ is compact: by Montel theorem, the ball of radius 1 in $\mathcal{A}_r$ is mapped into a compact subset K_1 of $\mathcal{A}_s$.

Let $K \subset \mathcal{A}_r$ be a bounded subset. Then, the closure $cl_s(K)$ of (the image of) K in $\mathcal{A}_s$ is compact. If $t \leq s$, the image of $cl_s(K)$ in $\mathcal{A}_t$ is compact, hence closed; this implies that $cl_s(K) = cl_t(K)$ in $\mathbf{C}\{z\}$. Thus, the closure $\overline{K}$ of K in $\mathbf{C}\{z\}$ coincides with the closure $cl_s(K)$ of K in $\mathcal{A}_s$ for any $s < r$. As a consequence, $\overline{K}$ is compact.

We denote by $B_{\mathcal{A}_r}(\varepsilon)$ the open ball centred at 0 and of radius ϵ in $\mathcal{A}_r$, which we also view as a subset of $\mathbf{C}\{z\}$. If $s \leq r$ and $\varepsilon \leq \varepsilon'$, then $B_{\mathcal{A}_r}(\varepsilon) \subset B_{\mathcal{A}_s}(\varepsilon') \subset \mathbf{C}\{z\}$. Given any finite set of such balls $B_{\mathcal{A}_{r_j}}(\epsilon_j)$, the sum $\sum_{j\geq 1}^n B_{\mathcal{A}_{r_j}}(\epsilon_j)$ is the subset of $\mathbf{C}\{z\}$ whose elements are sums $f_1 + \cdots + f_n$ with $f_j \in B_{\mathcal{A}_{r_j}}(\epsilon_j)$ for all j.

Lemma 5.2. *A subset $\mathcal{U}$ of $\mathbf{C}\{z\}$ is a neighborhood of* 0 *if and only if there are decreasing sequences (r_n) and (ϵ_n) tending to* 0 *such that $\mathcal{U}$ contains the set*

$$\mathcal{B} = \bigcup_n \sum_{j\geq 1}^n B_{\mathcal{A}_{r_j}}(\epsilon_j).$$

Lemma 5.2 shows that the topology defined in this section is the same as the topology introduced by Leslie in [Les], except that we consider germs of analytic functions at the origin in $\mathbf{C}$ instead of real analytic functions on a compact analytic manifold.

Proof. First we argue that any set $\mathcal{B}$ as in the statement of Lemma 5.2 is a neighborhood of 0 in $\mathbf{C}\{z\}$. To do so we need to check that $\mathcal{B} \cap \mathcal{A}_r$ contains a neighborhood of 0 for all r. The sum $\sum_{j=1}^n B_{\mathcal{A}_{r_j}}(\epsilon_j)$ is a subset of $\mathbf{C}\{z\}$ which is contained in $\mathcal{A}_{r_n}$. It is open in $\mathcal{A}_{r_n}$ because one of the summands, namely $B_{\mathcal{A}_{r_n}}(\epsilon_n)$, is itself open. Now, the continuity of the inclusion $\mathcal{A}_r \to \mathcal{A}_{r_n}$ for $r_n < r$ implies that $\sum_{j=1}^n B_{\mathcal{A}_{r_j}}(\epsilon_j) \cap \mathcal{A}_r$ is also open in $\mathcal{A}_r$. Since $\sum_{j=1}^n B_{\mathcal{A}_{r_j}}(\epsilon_j) \cap \mathcal{A}_r$ is contained in $\mathcal{B} \cap \mathcal{A}_r$, the latter is a neighborhood of 0, as we needed to prove.

Suppose now that $\mathcal{U}$ is a neighborhood of the origin in $\mathbf{C}\{z\}$, and fix a decreasing sequence (r_n) tending to 0. For each $n \geq 1$, set $\mathcal{U}_n = \mathcal{U} \cap \mathcal{A}_{r_n}$.

We first claim that there is a ball B_1 in $\mathcal{A}_{r_1}$ such that $\overline{B}_1 \subset \mathcal{U}$. Since $\mathcal{U}_2$ is open in $\mathcal{A}_{r_2}$, consider $\varepsilon > 0$ such that $B_{\mathcal{A}_{r_2}}(\varepsilon) \subset \mathcal{U}_2$. Now let $B_1 = B_{\mathcal{A}_{r_1}}(\varepsilon/2)$. Then for all $\eta > 0$, $\overline{B}_1 \subset B_1 + B_{\mathcal{A}_{r_2}}(\eta)$ so taking $\eta = \varepsilon/2$, we get $\overline{B}_1 \subset B_{\mathcal{A}_{r_2}}(\varepsilon/2) + B_{\mathcal{A}_{r_2}}(\varepsilon/2) \subset B_{\mathcal{A}_{r_2}}(\varepsilon) \subset \mathcal{U}$, which proves our claim.

We now construct by induction open balls $B_n \subset \mathcal{A}_{r_n}$ such that for all n, $\overline{B}_1 + \cdots + \overline{B}_n \subset \mathcal{U}$. Given such a set of balls $B_1, \ldots, B_n$, the set $K = \overline{B}_1 + \cdots + \overline{B}_n$ provides a compact subset of $\mathcal{A}_{r_{n+2}}$ contained in $\mathcal{U}_{n+2}$. Let ϵ be the distance from K to the complement of $\mathcal{U}_{n+2}$ in $\mathcal{A}_{r_{n+2}}$; by compactness, $\varepsilon > 0$, and $K + B_{\mathcal{A}_{r_{n+2}}}(\varepsilon/2) \subset U$. We then define $B_{n+1} = B_{\mathcal{A}_{r_{n+1}}}(\varepsilon/4)$. Then $K + \overline{B}_{n+1} \subset K + B_{\mathcal{A}_{r_{n+2}}}(\varepsilon/2) \subset \mathcal{U}$. This concludes the induction step and the proof. □

5.2. Coefficient functions. Recall that the coefficients of $f \in \mathcal{A}_r$ can be computed via the Cauchy integral formula:

$$A_n(f) = \frac{1}{2\pi i} \int_{\{|z|=r\}} \frac{f(z)}{z^{n+1}} dz. \tag{5.3}$$

This implies that the linear form A_n is continuous on each algebra $\mathcal{A}_r$ with operator norm $\|A_n\|_{\mathcal{A}_r^*} \leq \frac{1}{2\pi} r^{-(n+1)}$, i.e. $|A_n(f)| \leq \frac{1}{2\pi} r^{-(n+1)} \|f\|_{\mathcal{A}_r}$ for all $f \in \mathcal{A}_r$. Since the maps A_n separate points in $\mathbf{C}\{z\}$, we obtain:

Lemma 5.3. *For each $n \geq 0$, the map $A_n : \mathbf{C}\{z\} \to \mathbf{C}$ is continuous. The topological space $\mathbf{C}\{z\}$ is Hausdorff.*

More generally, we have:

Lemma 5.4. *If $\sum_n \theta_n z^n$ is a power series with infinite convergence radius, then the quantity*

$$\Theta(f) = \sum_n \theta_n |A_n(f)| \tag{5.4}$$

is well defined for every $f \in \mathbf{C}\{z\}$ and the function $\Theta\colon \mathbf{C}\{z\} \to \mathbf{R}_+$ is continuous.

Proof. The estimate $\|A_n\|_{\mathcal{A}_r^*} \leq \frac{1}{2\pi} r^{-(n+1)}$ implies that the map

$$\mathcal{A}_r \to \mathbf{C}, \quad f \mapsto \sum_n \theta_n |A_n(f)| \tag{5.5}$$

is continuous for any power series $\sum_n \theta_n z^n$ with convergence radius greater than $\frac{1}{r}$. By definition of the topology on $\mathbf{C}\{z\}$ we get that this map is continuous on the whole space if the power series in question has infinite convergence radius. □

5.3. Another filtration. We now introduce another filtration of $\mathbf{C}\{z\}$. If c is any positive real number, we define

$$\mathbf{C}_c\{z\} = \left\{ f \in \mathbf{C}\{z\} \text{ with } |A_n(f)| \leq c^{n+1} \text{ for all } n \right\}. \tag{5.6}$$

Then $\mathbf{C}_c\{z\} \subset \mathbf{C}_{c'}\{z\}$ for $c \leq c'$, and $\mathbf{C}\{z\}$ is the increasing union of all $\mathbf{C}_c\{z\}$.

Lemma 5.5. *$\mathbf{C}_c\{z\}$ is compact, and contained in $\mathcal{A}_r$ for all $r < c^{-1}$. Every compact subset $\Lambda \subset \mathbf{C}\{z\}$ is contained in some $\mathbf{C}_c\{z\}$.*

By compactness, the topology on $\mathbf{C}_c\{z\}$ induced by $\mathcal{A}_r$ and by $\mathbf{C}\{z\}$ agree.

Proof. From Lemma 5.3, we deduce that $\mathbf{C}_c\{z\}$ is closed in $\mathbf{C}\{z\}$. If $f \in \mathbf{C}_c\{z\}$ and $r < c^{-1}$, then

$$\|f\|_{\mathcal{A}_r} \leq \sum_n c^{n+1} r^n \leq \frac{c}{1 - cr} \tag{5.7}$$

This means that $\mathbf{C}_c\{z\}$ is a bounded subset in $\mathcal{A}_r$. Since the inclusion $\mathcal{A}_r \to \mathcal{A}_s$ is compact for $r > s$, $\mathbf{C}_c\{z\}$ has compact closure in $\mathcal{A}_s$, hence in $\mathbf{C}\{z\}$. Since $\mathbf{C}_c\{z\}$ is closed, it is compact.

To prove the second assertion, assume by contradiction that there is a compact subset $\Lambda \subset \mathbf{C}\{z\}$ such that for every integer $m > 0$ there exists $f_m \in \Lambda \setminus \mathbf{C}_m\{z\}$.

By definition, there is an index $n_m \geq 0$ with $|A_{n_m}(f_m)| > m^{n_m+1}$. By Lemma 5.4, each individual coefficient is continuous and thus bounded on the compact Λ. It follows that n_m goes to $+\infty$ as m does. We can thus assume, passing to a subsequence if necessary, that the n_m's are pairwise distinct.

Set $\theta_{n_m} = (\frac{1}{m})^{n_m}$, and $\theta_n = 0$ if n is not one of the indices n_m. Then $\theta_n^{1/n}$ converges towards 0 as n goes to $+\infty$, meaning that the power series $\sum_n \theta_n z^n$ has infinite convergence radius. By Lemma 5.4, the map $f \mapsto \Theta(f) = \sum_n \theta_n |A_n(f)|$ is continuous on $\mathbf{C}\{z\}$ and thus bounded on our compact set Λ. On the other hand we have

$$\Theta(f_m) \geq \theta_{n_m} |A_{n_m}(f_m)| \geq m.$$

This yields the desired contradiction. □

Remark 5.6. Given $r > 0$, introduce

$$\mathcal{B}_r = \left\{ f \in \mathbf{C}\{z\} \mid \operatorname{rad}(f) \geq r,\ \sup_{\mathbb{D}_r} |f| \leq \frac{1}{r} \right\}. \tag{5.8}$$

This is the closure in $\mathbf{C}\{z\}$ of a ball in $\mathcal{A}_r$ and is therefore compact (Remark 5.1). There are functions c_1, c_2, r_1, and $r_2 : \mathbf{R}_+^* \to \mathbf{R}_+^*$ such that

$$\mathbf{C}_{c_1(r)}\{z\} \subset \mathcal{B}_r \subset \mathbf{C}_{c_2(r)}\{z\} \text{ and } \mathcal{B}_{r_1(c)} \subset \mathbf{C}_c\{z\} \subset \mathcal{B}_{r_2(c)}.$$

It follows that one could equivalently state the results of this section in terms of the filtration $(\mathcal{B}_r)_{r>0}$ instead of $(\mathbf{C}_c\{z\})_{c>0}$.

The following corollary allows us to view the final topology on $\mathbf{C}\{z\}$ as the weak topology associated to the filtration by the compact sets $\mathbf{C}_c\{z\}$.

Corollary 5.7. *A subset $F \subset \mathbf{C}\{z\}$ is closed if and only if for all $c > 0$, $F \cap \mathbf{C}_c\{z\}$ is closed. A map $F : \mathbf{C}\{z\} \to X$ to a topological space is continuous if and only if its restriction to $\mathbf{C}_c\{z\}$ is continuous for all $c > 0$.*

Proof. Clearly, it suffices to prove the first assertion. If F is closed, so is $F \cap \mathbf{C}_c\{z\}$. Assume conversely that $F \cap \mathbf{C}_c\{z\}$ is closed for all $c > 0$, and let us prove that F is closed. By definition of the final topology, we need to prove that given $r > 0$, its preimage $j_r^{-1}(F)$ under the inclusion $j_r : \mathcal{A}_r \to \mathbf{C}\{z\}$ is closed. It suffices to prove that for any $R > 0$, its intersection with the ball $B_{\mathcal{A}_r}(R)$ is closed in $\mathcal{A}_r$. Since $B_{\mathcal{A}_r}(R)$ has compact closure, there exists $c > 0$ such that $B_{\mathcal{A}_r}(R) \subset \mathbf{C}_c\{z\}$. Since $F \cap \mathbf{C}_c\{z\}$ is closed, $B_{\mathcal{A}_r}(R) \cap j_r^{-1}(F) = B_{\mathcal{A}_r}(R) \cap j_r^{-1}(F \cap \mathbf{C}_c\{z\})$ is a closed subset of $\mathcal{A}_r$ which concludes the proof. □

Although one can show that the topology on $\mathbf{C}\{z\}$ is not metrizable, each space $\mathbf{C}_c\{z\}$ is a metric space. Being compact, the topology on $\mathbf{C}_c\{z\}$ can be described in many equivalent ways:

Proposition 5.8. *Let c be a positive real number. Let (f_m) be a sequence in $\mathbf{C}_c\{z\}$ and let f_∞ be an element of $\mathbf{C}_c\{z\}$. The following are equivalent:*

(1) *(f_m) converges to f_∞ in $\mathbf{C}_c\{z\}$;*

(2) *for some (any) $r < c^{-1}$, (f_m) converges uniformly toward f_∞ on $\overline{\mathbb{D}}_r$;*

(3) *(f_m) converges toward f_∞ uniformly on every compact subset of $\mathbb{D}_{c^{-1}}$;*

(4) *for every index n, $A_n(f_m)$ converges toward $A_n(f_\infty)$.*

Proof. As seen before, $\mathbf{C}_c\{z\}$ is contained in $\mathcal{A}_r$ for all $r < c^{-1}$ and the topology induced by $\|\cdot\|_{\mathcal{A}_r}$ agrees with the topology induced by $\mathbf{C}_c\{z\}$. This proves the equivalence of the first three assertions.

To prove the equivalence with the last assertion, consider the map $\Phi : \mathbf{C}_c\{z\} \to [0,1]^{\mathbf{N}}$ defined by $\Phi(f) = (\frac{A_n(f)}{c^{n+1}})_{n\in\mathbf{N}}$, where $[0,1]^{\mathbf{N}}$ is endowed with the product topology. This map being continuous and injective, it is a homeomorphism to its image, and the result follows. □

5.4. The final topology on a field with an absolute value. In this section, we explain that the final topology induced by the filtration $\mathbf{C}_c\{z\}$ makes sense for every field $\mathbf{k}$ with an absolute value $|\cdot|$; but the results based on Montel theorem (Remark 5.1) may fail for fields $\mathbf{k} \neq \mathbf{C}$.

Let $\mathbf{k}$ be a complete field $\mathbf{k}$ for some absolute value $|\cdot| : \mathbf{k} \to \mathbf{R}_+$. By Ostrowski's Theorem, $\mathbf{k}$ is either $\mathbf{R}$ or $\mathbf{C}$, or the absolute value is non-archimedean: $|x+y| \leq \max(|x|,|y|)$ for all $x, y \in \mathbf{k}$. The algebra $\mathbf{k}\{z\}$ of convergent power series is filtrated by the family of subsets

$$\mathbf{k}_c\{z\} = \left\{ f \in \mathbf{k}\{z\} \text{ with } |A_n(f)| \leq c^{n+1} \text{ for all } n \right\} \tag{5.9}$$

for $c > 0$. We endow $\mathbf{k}_c\{z\}$ with the product topology, via the embedding $f \in \mathbf{k}_c\{z\} \to (A_n(f))_n \in \mathbf{k}^{\mathbf{N}}$: a sequence $(f_k)_{k\in\mathbf{N}}$ of elements of $\mathbf{k}_c\{z\}$ converges to $f_\infty \in \mathbf{k}_c\{z\}$ if and only if $A_n(f_k) \to A_n(f_\infty)$ for all n. For $c \leq c'$, $\mathbf{k}_c\{z\}$ is closed in $\mathbf{k}_{c'}\{z\}$ and the inclusion is a homeomorphism to its image. We then endow $\mathbf{k}\{z\}$ with the topology associated to this filtration: a subset $F \subset \mathbf{k}\{z\}$ is closed if and only if $F \cap \mathbf{k}_c\{z\}$ is closed in $\mathbf{k}_c\{z\}$. Equivalently, a map $\varphi : \mathbf{k}\{z\} \to X$ to a topological space is continuous if and only if its restriction to $\mathbf{k}_c\{z\}$ is continuous for every $c > 0$. By construction, the maps $f \mapsto A_n(f)$ are continuous on $\mathbf{k}\{z\}$. Proposition 5.8 shows that, when $\mathbf{k} = \mathbf{C}$, this topology agrees with the final topology defined in Section 5.1.

If $\mathbf{k}$ is locally compact, each $\mathbf{k}_c\{z\}$ is compact. In general, since $\mathbf{k}_c\{z\}$ is a countable product of complete metric spaces, we get:

Proposition 5.9. *If $\mathbf{k}$ is a complete field, then $\mathbf{k}_c\{z\}$ is a metrizable complete space. In particular, it is a Baire space.*

On the other hand, $\mathbf{k}\{z\}$ is not a Baire space since it is a countable union of $\mathbf{k}_c\{z\}$, each of which is closed and has an empty interior.

5.5. The topological group of germs of diffeomorphisms. Any $f \in \mathsf{Diff}(\mathbf{k}, 0)$ can be written as $f = \lambda(z + z^2\tilde{f})$ for some $\tilde{f} \in \mathbf{k}\{z\}$ or equivalently as

$$f = \lambda(z + \tilde{a}_2 z^2 + \cdots + \tilde{a}_k z^k + \dots) \tag{5.10}$$

for some $\lambda \in \mathbf{k}^*$ and $\tilde{a}_n \in \mathbf{k}$. Thus, we define the maps $\tilde{A}_n : \mathsf{Diff}(\mathbf{k}, 0) \to \mathbf{k}$ by

$$\tilde{A}_n(f) = A_n(f)/A_1(f) = \tilde{a}_n. \tag{5.11}$$

Given two real numbers $c > 0$ and $\lambda_0 > 1$, we define the two subsets

$$\mathsf{Diff}_c(\mathbf{k}, 0) = \left\{f \in \mathsf{Diff}(\mathbf{k}, 0)\ ;\ |\tilde{A}_n(f)| \le c^{n-1} \text{ for all } n\right\} \tag{5.12}$$

and

$$\mathsf{Diff}_{\lambda_0,c}(\mathbf{k}, 0) = \left\{f \in \mathsf{Diff}_c(\mathbf{k}, 0)\ ;\ \frac{1}{\lambda_0} \le |A_1(f)| \le \lambda_0\right\} \tag{5.13}$$

Observe that if we denote by $m_\alpha : z \mapsto \alpha z$ the multiplication by some scalar $\alpha \in \mathbf{k}^*$ then we have

$$m_\alpha \mathsf{Diff}_c(\mathbf{k}, 0) m_\alpha^{-1} = \mathsf{Diff}_{c\alpha}(\mathbf{k}, 0) \tag{5.14}$$

and

$$m_\alpha \mathsf{Diff}_{\lambda_0,c}(\mathbf{k}, 0) m_\alpha^{-1} = \mathsf{Diff}_{\lambda_0,c\alpha}(\mathbf{k}, 0) \tag{5.15}$$

Lemma 5.10. *A map $\varphi : \mathsf{Diff}(\mathbf{k}, 0) \to X$ to a topological space is continuous if and only if it is continuous in restriction to $\mathsf{Diff}_c(\mathbf{k}, 0)$ (or equivalently to $\mathsf{Diff}_{\lambda_0,c}(\mathbf{k}, 0)$) for every $c > 0$ and $\lambda_0 > 1$.*

Proof. It suffices to check the continuity of φ on the open set $U_{\lambda_0} = \{f\ ;\ \frac{1}{\lambda_0} < |A_1(f)| < \lambda_0\}$ for all $\lambda_0 > 1$. By definition of the final topology, it suffices to check its continuity on $U_{\lambda_0} \cap \mathbf{k}_c\{z\}$ for every $c > 1$. But $U_{\lambda_0} \cap \mathbf{k}_c\{z\}$ is a subset of $\mathsf{Diff}_{\lambda_0,c'}(\mathbf{k}, 0)$ as soon as $c' \ge \max(\lambda_0, c^3)$; since we know that φ is continuous on $\mathsf{Diff}_{\lambda_0,c'}(\mathbf{k}, 0)$, this proves the lemma. □

Proposition 5.11. *If* $\mathbf{k}$ *is a complete field, then* $\mathsf{Diff}_{\lambda_0,c}(\mathbf{k},0)$ *and* $\mathsf{Diff}_c(\mathbf{k},0)$ *are complete metric spaces. In particular, they are Baire spaces.*

Proof. By definition, $\mathsf{Diff}_{c,\lambda_0}(\mathbf{k},0)$ is homeomorphic to a countable product of closed subsets of $\mathbf{k}$; so, $\mathbf{k}$ being complete, its topology is induced by a complete metric. Since $\mathbf{k}^*$ is homeomorphic to the closed subset $\{(x,y)|xy-1\}\subset\mathbf{k}^2$, the same argument applies to $\mathsf{Diff}_c(\mathbf{k},0)$. □

Theorem 5.12. *Let* $(\mathbf{k},|\cdot|)$ *be a field with a complete absolute value. With the final topology,* $\mathsf{Diff}(\mathbf{k},0)$ *is a topological group.*

Lemma 5.13. *For every real number* $c>1$*, there exists a real number* $c'>1$ *such that the following holds: if* f *and* g *are in* $\mathsf{Diff}_{c,c}(\mathbf{k},0)$*, then* $f\circ g$ *and* f^{-1} *lie in* $\mathsf{Diff}_{c',c'}(\mathbf{k},0)$.

Proof. Let $f=\lambda(z+\sum_{n\geq 2}\tilde{a}_nz^n)$, $g=\mu(z+\sum_{n\geq 2}\tilde{b}_nz^n)$ with $|\tilde{a}_n|,|\tilde{b}_n|\leq c^{n-1}$ and $|\lambda|$, $|\mu|$, $|\lambda|^{-1}$, $|\nu|^{-1}\leq c$. Let $F=c(z+\sum_{n\geq 2}c^{n-1}z^n)=\frac{cz}{1-cz}\in\mathbf{R}\{z\}$ so that the absolute value of the coefficients of f and g are bounded by the coefficients of F. Then the absolute value of the coefficients of $f\circ g=\sum_{n\geq 1}a_n(\sum_{m\geq 1}b_mz^m)^n$ are bounded by the coefficients of $F\circ F=\frac{cz(1-cz)}{1-cz-c^2z}$. Since $F\circ F$ has positive convergence radius, there exists $c'\geq c^2$ such that $\tilde{A}_n(F\circ F)\leq c'^{n-1}$ for all $n\geq 2$. The first assertion follows.

We now prove the second assertion. Let $f=\lambda(z+\sum_{n\geq 2}\tilde{a}_nz^n)$, and let $f^{-1}=\lambda^{-1}(z+\sum_{n\geq 2}\tilde{b}_nz^n)$. The inversion formula from Section 2.1 gives

$$
\begin{aligned}
|\tilde{b}_n| &\leq \frac{|\lambda|}{|\lambda|^n}\sum_{k_1,k_2,\dots}\frac{(n+1)\cdots(n-1+k_1+k_2+\dots)}{k_1!k_2!\cdots}\cdot|\tilde{a}_2|^{k_1}|\tilde{a}_3|^{k_2}\cdots\\
&\leq c^{n-1}\sum_{k_1,k_2,\dots}\frac{(n+1)\cdots(n-1+k_1+k_2+\dots)}{k_1!k_2!\cdots}(c)^{k_1}(c)^{2k_2}\cdots\\
&= c^{2n-2}\sum_{k_1,k_2,\dots}\frac{(n+1)\cdots(n-1+k_1+k_2+\dots)}{k_1!k_2!\cdots}.
\end{aligned}
$$

Thus, we have to bound the quantity

$$K_n:=\sum_{k_1,k_2,\dots}\frac{(n+1)\cdots(n-1+k_1+k_2+\dots)}{k_1!k_2!\cdots}.$$

But the numbers K_n are the coefficients of the power series expansion of the reciprocal diffeomorphism g^{-1} of

$$g(z)=z-z^2-z^3-z^4\cdots=z\left(2-\frac{1}{1-z}\right)=\frac{z-2z^2}{1-z}.$$

In close form, we obtain

$$g^{-1}(y) = \frac{(1+y)}{4} - \frac{1}{4}\sqrt{1-6y+y^2}.$$

Since g^{-1} has positive convergence radius, there exists c_0 such that for all $n \geq 2$, $K_n \leq c_0^{n-1}$ hence $|\tilde{b}_n| \leq (c_0 c^2)^{n-1}$ and the result follows. □

Proof of Theorem 5.12. By definition of the topology, given $c > 0$, one only needs to check the continuity of the group laws in restriction to $\mathsf{Diff}_{c,c}(\mathbf{k}, 0)$.

Since $A_n(f \circ g)$ and $A_n(f^{-1})$ are given by polynomials in the coefficients $\tilde{A}_i(f)$, $\tilde{A}_i(g)$, $A_1(f)^{\pm 1}$, $A_1(g)^{\pm 1}$, the maps $(f, g) \mapsto A_n(f \circ g)$ and $f \mapsto A_n(f^{-1})$ are continuous on $\mathsf{Diff}_{c,c}(\mathbf{k}, 0)$. By Lemma 5.13 there exists c' such that for all $f, g \in \mathsf{Diff}_{c,c}(\mathbf{k}, 0)$, $f \circ g$ and f^{-1} lie in $\mathsf{Diff}_{c',c'}(\mathbf{k}, 0)$. Since the topology on $\mathsf{Diff}_{c',c'}(\mathbf{k}, 0)$ is the product topology, the continuity of the coefficients implies the continuity of the group laws. □

5.6. Other topologies. First, we would like to point out that there are other reasonable and useful topologies on $\mathbf{C}\{z\}$, but for which the group laws are not continuous. This the case for the so-called *Takens topology* [MRR, BT]; this is the topology induced by the distance

$$\mathsf{dist}(f, g) = \sup_n |A_n(f) - A_n(g)|^{1/n}. \tag{5.16}$$

Note that in particular the convergence radius of $f - g$ is large if f and g are close to each other in the Takens topology, and this implies that the right translation $R_f : g \mapsto g \circ f$ is not continuous if the radius of convergence of f is finite. Indeed, a small perturbation $g(z) + \epsilon z$ is mapped to $R_f(g + \epsilon z) = g \circ f + \epsilon f$, and the difference ϵf is not small in the Takens topology because its radius of convergence does not depend on ϵ.

We comment now on another important topology on $\mathsf{Diff}(\mathbf{C}, 0)$, but for which the Baire property fails. Let $\mathsf{Jets}_\ell(\mathbf{C}, 0)$ be the group of ℓ-jets of diffeomorphisms $a_1 z + \cdots + a_\ell z^\ell \mod (z^{\ell+1})$, with $a_1 \neq 0$; it can be considered as a solvable algebraic group and thus as a solvable complex Lie group. Let

$$\mathsf{j}_\ell : \mathsf{Diff}(\mathbf{C}, 0) \to \mathsf{Jets}_\ell(\mathbf{C}, 0) \tag{5.17}$$

denote the homomorphism that maps a power series $f = \sum_n a_n z^n$ to $\sum_{n=1}^{\ell} a_n z^n$. We can then define a topology on $\mathsf{Diff}(\mathbf{C}, 0)$ (resp. on $\widehat{\mathsf{Diff}}\,(\mathbf{C}, 0)$): the weakest topology for which all projections j_ℓ are continuous. With this topology, $\mathsf{Diff}(\mathbf{C}, 0)$ is a topological group, because the projections j_ℓ are homomorphisms. Moreover, a sequence (f_m) converges toward a germ of diffeomorphism g if and only if the coefficients $A_n(f_m)$ converge to $A_n(g)$ for all n. In other words, this is the topology of simple convergence on the coefficients. In particular, $\mathsf{Diff}(\mathbf{C},0)$ is not a closed subset of $\widehat{\mathsf{Diff}}(\mathbf{C},0)$ for this topology. With this topology, $\widehat{\mathsf{Diff}}(\mathbf{C},0)$ is a Baire space, but $\mathsf{Diff}(\mathbf{C},0)$ is not (Proposition 5.11 fails if $\mathsf{Diff}(\mathbf{C},0)$ is endowed with this topology).

5.7. Continuity in the Koenigs linearization Theorem. A contraction $f \in \mathsf{Diff}(\mathbf{k},0)$ is an element with $|A_1(f)| < 1$. In this case, Koenigs theorem says that the unique formal diffeomorphism h_f tangent to the identity that conjugates f to the homothety $z \mapsto A_1(f)z$ has positive convergence radius. The following result shows that $f \mapsto h_f$ is continuous for the final topology on the set of contractions

$$\mathsf{Cont}(\mathbf{k},0) = \{f \in \mathsf{Diff}(\mathbf{k},0) \mid |A_1(f)| < 1\}. \tag{5.18}$$

Theorem 5.14. *Let* $\mathbf{k}$ *be a field with a complete non-trivial absolute value. For every germ* $f \in \mathsf{Cont}(\mathbf{k},0)$*, the unique formal diffeomorphism* h_f *such that*

$$h_f(f(z)) = A_1(f)\cdot h_f(z) \quad \textit{and} \quad A_1(h_f) = 1$$

has positive convergence radius, and the map

$$h : f \in \mathsf{Cont}(\mathbf{k},0) \mapsto h_f \in \mathsf{Diff}(\mathbf{k},0)$$

is continuous for the final topology. The coefficients of h_f *are polynomial functions with integer coefficients in the variables* $A_i(f)$ *and* $(A_1(f)^j - 1)^{-1}$*, for* $i, j \geq 1$.

When $\mathbf{k} = \mathbf{C}$, it is shown in [Mil, Chapter 8] that h_f is convergent and its coefficients depend holomorphically on f. Theorem 5.14 is just a variation on this classical result.

Proof. We refer to [Sie] for the real and complex cases, and to [HY] for the non-archimedian ones.

The coefficients of h_f can be computed inductively and turn out to be polynomials with integer coefficients in the variables $A_i(f)$ and $(A_1(f)^j - 1)^{-1}$, for $i, j \geq 1$ (see for instance [Sie, Eq. 4]). If $|A_1(f)| \leq \alpha$ for some α in the interval $[0,1[$, then

$$|(A_1(f))^n - 1| \geq 1 - \alpha \tag{5.19}$$

for all $n > 0$. By [Sie, Theorem 1] and [HY, Theorem 1] in the archimedean and non-archimedean cases respectively, h_f is convergent and for all c, $\lambda > 1$, there exists c' such that $h_f \in \mathsf{Diff}_{c'}(\mathbf{k},0)$ if $f \in \mathsf{Diff}_{\lambda,c}(\mathbf{k},0)$.

The topology on $\mathsf{Diff}_{c'}(\mathbf{C},0)$ is the product topology on the coefficients. Since the coefficients of h_f are continuous functions of f, it follows that the restriction of $f \mapsto h_f$ to $\mathsf{Diff}_{\lambda,c}(f)$ is continuous. By definition of the final topology, this proves the continuity of h. □

6. A large irreducible component of the representation variety

This section describes our second proof strategy for Theorem A. For simplicity, we consider only the fundamental group of a closed orientable surface of genus 2, but we work over any field $\mathbf{k}$ with a complete absolute value $|\cdot|$.

6.1. An irreducible set of representations. Using the presentation

$$\Gamma_2 = \langle a, b, \overline{a}, \overline{b} \mid [a,b] = [\overline{a}, \overline{b}] \rangle, \tag{6.1}$$

we get an idenfication

$$\mathrm{Hom}\big(\Gamma_2; \mathsf{Diff}(\mathbf{k},0)\big) = \big\{(f, g, \overline{f}, \overline{g}) \in \widehat{\mathsf{Diff}}\,(\mathbf{k},0)^4 \mid [f,g] = [\overline{f}, \overline{g}]\big\}. \tag{6.2}$$

Let $\mathbb{X} \subset \mathrm{Hom}(\Gamma_2, \mathsf{Diff}(\mathbf{k},0))$ be the set of representations $\rho : \Gamma_2 \to \mathsf{Diff}(\mathbf{k},0))$ such that $\rho(a)$ is tangent to Id and $\rho(b)$ is a contraction. As in Equation (5.18), we denote by $\mathsf{Cont}(\mathbf{k},0)$ the set of contractions. For $c > 0$, we let $\mathbb{X}_c = \mathbb{X} \cap \mathsf{Diff}_c(\mathbf{C},0)$. Set

$$\mathcal{R} = \mathsf{Cont}(\mathbf{k},0) \times \mathsf{Diff}(\mathbf{k},0) \times \mathsf{Diff}(\mathbf{k},0), \tag{6.3}$$

$$\mathcal{R}(c) = \mathsf{Cont}_c(\mathbf{k},0) \times \mathsf{Diff}_c(\mathbf{k},0) \times \mathsf{Diff}_c(\mathbf{k},0), \tag{6.4}$$

and denote by $\pi : \mathbb{X} \to \mathcal{R}$ the projection

$$\pi(\rho) = \big(\rho(b), \rho(\overline{a}), \rho(\overline{b})\big) \tag{6.5}$$

Proposition 6.1. *The map π is a homeomorphism for the final topology, and its inverse*

$$\pi^{-1} : \big(g, \overline{f}, \overline{g}\big) \mapsto \big(f, g, \overline{f}, \overline{g}\big)$$

is a polynomial map, in the following sense: for each $n \in \mathbf{N}^*$, the map $(g, \overline{f}, \overline{g}) \mapsto A_n(f)$ is polynomial in (finitely many of) the variables $A_k(g)$, $A_k(\overline{f})$, $A_k(\overline{g})$, $A_1(g)^{-1}$, $A_1(\overline{f})^{-1}$, $A_1(\overline{g})^{-1}$ and $(A_1(g)^k - 1)^{-1}$ $(k \geq 1)$.

Proof. The projection π is continuous because both $\mathbb{X}$ and $\mathcal{R}$ come with the topology induced by the same topology on $\mathsf{Diff}(\mathbf{k},0)$.

Consider a triple $(g, \overline{f}, \overline{g}) \in \mathsf{Cont}(\mathbf{k},0) \times \mathsf{Diff}(\mathbf{k},0) \times \mathsf{Diff}(\mathbf{k},0)$. Since $[\overline{f}, \overline{g}]$ is tangent to the identity, the germs g and $[\overline{f}, \overline{g}] \circ g$ have the same derivative $\lambda = A_1(g)$ at 0. Since $|\lambda| < 1$, we can apply Koenigs Theorem 5.14: we get two germs h_1 and $h_2 \in \mathsf{Diff}(\mathbf{k},0)$ tangent to the identity such that

$$h_1 \circ g \circ h_1^{-1} = m_\lambda \quad \text{and} \quad h_2 \circ ([\overline{f}, \overline{g}] \circ g) \circ h_2^{-1} = m_\lambda \tag{6.6}$$

where $m_\lambda(z) = \lambda z$ is the multiplication by λ. Then, the map $f := h_2^{-1} \circ h_1$ conjugates g to $[\overline{f}, \overline{g}] \circ g$ so

$$f \circ g \circ f^{-1} = [\overline{f}, \overline{g}] \circ g \quad \text{and} \quad [f, g] = [\overline{f}, \overline{g}].$$

This means that one can define the preimage $\pi^{-1}(g, \overline{f}, \overline{g}) \in \mathrm{Hom}(\Gamma_2, \mathsf{Diff}(\mathbf{k}, 0))$ by the 4-tuple $(f, g, \overline{f}, \overline{g})$: the fact that $\pi^{-1} \circ \pi = \mathrm{Id}_{\mathcal{R}}$ follows from uniqueness in Koenigs Theorem.

The continuity of π^{-1} is a consequence of the continuity of the conjugacy in Koenigs Theorem 5.14 and of the continuity of the map $(g, \overline{f}, \overline{g}) \mapsto [\overline{f}, \overline{g}] \circ g$. The fact that $A_n(f)$ is polynomial in the given variables is a direct consequence of the corresponding fact in Koenigs Theorem, and the fact that group operations are polynomial mappings. □

We denote the inverse map π^{-1} by Φ:

$$\forall s \in \mathcal{R}, \quad \Phi_s = \pi^{-1}(s). \tag{6.7}$$

Thus, if $s = (g, \overline{f}, \overline{g})$, then Φ_s is the morphism $\Gamma_2 \to \mathsf{Diff}(\mathbf{k}, 0)$ such that $\Phi_s(b) = g$, $\Phi_s(\overline{a}) = \overline{f}$, $\Phi_s(\overline{b}) = \overline{g}$, and $\Phi_s(a)$ is the unique germ of diffeomorphism f which is tangent to the identity and satisfies the relation $[f, g] = [\overline{f}, \overline{g}]$. To conclude the proof, our goal now is to prove that for every $c > 0$, the family of morphisms Φ_s, for $s \in \mathcal{R}(c)$, satisfies the assumptions of Lemma 3.1. Proposition 5.11 shows that $\mathcal{R}(c)$ is a Baire space. The following corollary proves the irreducibility of $\mathcal{R}(c)$.

Corollary 6.2. *For any $w \in \Gamma_2$, denote by $\mathcal{R}(c)_w \subset \mathcal{R}(c)$ the set of homomorphisms in $\mathcal{R}(c)$ that kill w. Then either $\mathcal{R}(c)_w = \mathcal{R}(c)$ or $\mathcal{R}(c)_w$ is a closed subset of $\mathcal{R}(c)$ with empty interior.*

Proof. Since the functions $s \in \mathcal{R} \mapsto A_k(\Phi_s(g)) - A_k(\mathrm{Id})$ are continuous, $\mathcal{R}(c)_w$ is closed. Now, assume that $\mathcal{R}(c)_w \neq \mathcal{R}(c)$: there exists $k \geq 1$ and a point $s = (g_0, \overline{f}_0, \overline{g}_0)$ in $\mathcal{R}(c)$ such that $A_k(\Phi_s(w)) \neq A_k(\mathrm{Id})$. According to Proposition 6.1 the map $s \mapsto A_k(\varphi_s(w)) - A_k(\mathrm{Id})$ is a polynomial function in finitely many of

- the coefficients $A_n(g_0)$, $A_n(\overline{f}_0)$ and $A_n(\overline{g}_0)$ $(n \geq 1)$,
- the inverses $A_1(g_0)^{-1}$, $A_1(\overline{f}_0)^{-1}$, $A_1(\overline{g}_0)^{-1}$ and $(A_1(g_0)^k - 1)^{-1}$ $(k \geq 1)$

(note that $A_1(g_0)$, $A_1(\overline{f}_0)$, $A_1(\overline{g}_0)$ and $(A_1(g_0)^k - 1)$ do not vanish on $\mathcal{R}$). Our assumption says that this function does not vanish identically on $\mathcal{R}(c)$. Assume that $\mathcal{R}(c)_w$ contains a non-empty open subset $\mathcal{U}$, and choose a point $s = (g_1, \overline{f}_1, \overline{g}_1)$ in $\mathcal{U}$. If $\mathbf{k} = \mathbf{R}$ or $\mathbf{C}$, we denote by $B_{\mathbf{k}}$ the interval $[0, 1] \subset \mathbf{R}$; in the non-archimedean case we set $B_{\mathbf{k}} = \{t \in \mathbf{k}, |t| \leq 1\}$. Then, we consider the convex combination

$$(6.8)\quad s_t = \Big(g_t = tf_1 + (1-t)f_0,\ \overline{f}_t = t\overline{f}_1 + (1-t)\overline{f}_0, \overline{g}_t = t\overline{g}_1 + (1-t)\overline{g}_0\Big)$$

with t in $B_{\mathbf{k}}$. According to Lemma 6.3 below, g_t, $\overline{f}_t$ and $\overline{g}_t$ are in $\mathcal{R}(c')$ for some $c' \geq c$, and $t \mapsto s_t$ is continuous; thus $\{t\ ;\ s_t \in \mathcal{U}\}$ is an open neighborhood of 1.

The function $t \mapsto \mathcal{A}_n(\varphi_{s_t}(w)) - A_n(\mathrm{Id})$ does not vanish for $t = 0$, it is the restriction of a rational function of the variable t to the interval $[0,1]$, and it vanishes identically on the open set $\{t\ ;\ s_t \in \mathcal{U}\}$. This is a contradiction, which shows that the interior of $\mathcal{R}(c)_w$ is empty. □

Let $B_{\mathbf{k}}$ be the interval $[0,1] \subset \mathbf{R}$ if $\mathbf{k} = \mathbf{R}$ or $\mathbf{C}$, or the ball $\{t \in \mathbf{k}, |t| \leq 1\}$ in the non-archimedean case.

Lemma 6.3. *Let $f_0 \in \mathsf{Diff}_{c_0}(\mathbf{k}, 0)$ and $f_1 \in \mathsf{Diff}_{c_1}(\mathbf{k}, 0)$, and for $t \in \mathbf{k}$, let $f_t = (1-t)f_0 + tf_1$. Let $p \in \mathbf{k}$ be the value of t (if any) such that $f_p'(0) = 0$.*

If $c_0 \leq c_1$ and $c_0|f_0'(0)| \leq c_1|f_1'(0)|$, then for all $t \in B_{\mathbf{k}} \backslash \{p\}$, $f_t \in \mathsf{Diff}_{c_1}(\mathbf{k}, 0)$.

Proof. Denote $\lambda_0 = |f_0'(0)|$ and $\lambda_1 = |f_1'(0)|$. By assumption, for all $n \geq 2$, $|A_n(f_0)| \leq \lambda_0 c_0^{n-1}$ and $|A_n(f_1)| \leq \lambda_1 c_1^{n-1}$.

Consider first the case $\mathbf{k} = \mathbf{R}$ or $\mathbf{C}$. Since $\lambda_0 c_0 \leq \lambda_1 c_1$, we get for all $t \in [0,1]$,

$$\begin{aligned} &(6.9) && |A_n(f_t)| \leq (1-t)\lambda_0 c_0^{n-1} + t\lambda_1 c_1^{n-1} \\ &(6.10) && \leq (1-t)\lambda_1 c_1 c_0^{n-2} + t\lambda_1 c_1^{n-1} \leq \lambda_1 c_1^{n-1}. \end{aligned}$$

This shows that $f_t \in \mathsf{Diff}_{c_1}(\mathbf{k})$ as soon as $f_t'(0) \neq 0$.

In the non-archimedean case, one has $|t-1| \leq 1$ for $t \in B_{\mathbf{k}}$. Similarly, we get

$$\begin{aligned} &(6.11) && |A_n(f_t)| \leq \max\{|1-t|\lambda_0 c_0^{n-1},\ |t|\lambda_1 c_1^{n-1}\} \\ &(6.12) && \leq \max\{\lambda_1 c_1 c_0^{n-2},\ \lambda_1 c_1^{n-1}\} \leq \lambda_1 c_1^{n-1}. \end{aligned}$$

This shows that $f_t \in \mathsf{Diff}_{c_1}(\mathbf{k})$ as soon as $f_t'(0) \neq 0$. Then, the continuity follows from the continuity of the coefficients $t \mapsto A_n(f_t)$. □

6.2. Separation. To conclude the proof, we fix $c > 0$ and prove that $\mathcal{R}(c)$ satisfies the separation condition of Lemma 3.1. We thus fix $g \in \Gamma_2 \setminus \{1\}$ and show that $\mathcal{R}(c)$ contains a representation that does not kill g. Write the orientable surface group of genus 2 as $\Gamma_2 = \langle a, b, \overline{a}, \overline{b} \mid [a,b] = [\overline{a}, \overline{b}]\rangle$, and let $p : \Gamma_2 \to \langle a, b\rangle$ be the morphism fixing a and b and sending $\overline{a}$ and $\overline{b}$ to a and b respectively. Let $\tau : \Gamma_2 \to \Gamma_2$ be the Dehn twist around the curve $c = [a,b]$, i.e., the automorphism that fixes a, b and sends $\overline{a}$ and $\overline{b}$ to $c\overline{a}c^{-1}$ and $c\overline{b}b^{-1}$

respectively. According to Proposition 1.2, there exists a positive integer n_0 such that $p \circ \tau^N(g) \neq 1$ for all $N \geq n_0$.

Apply Theorem 3.4 to get a pair f_1, f_2 of germs of diffeomorphisms generating a free group $\langle f_1, f_2\rangle$ of rank 2 and satisfying $f_1'(0) > 1$ and $f_2'(0) > 1$. Define a morphism $\rho : \langle a, b\rangle \to \mathsf{Diff}(\mathbf{k}, 0)$ by $\rho(a) = [f_1, f_2]$ and $\rho(b) = f_2^{-1}$. Then ρ is injective, $\rho(a)$ is tangent to the identity, and $\rho(b)$ is a contraction. Set $\rho_N := \rho \circ p \circ \tau^N$. For $N \geq n_0$ $\rho_N(g) \neq 1$. Thus, $\pi(\rho_N)$ lies in $\mathcal{R} \setminus \mathcal{R}_g$; but it might not lie in $\mathcal{R}(c)$.

Let $c_N > 0$ be such that $\pi(\rho_N) \in \mathcal{R}(c_N)$. Given $\alpha \in \mathbf{k}^*$, let ad_α be the inner automorphism of $\mathsf{Diff}(\mathbf{k}, 0)$ given by $f \mapsto m_\alpha \circ f \circ m_\alpha^{-1}$. As noticed in Equation (5.12), we have $\mathrm{ad}_\alpha(\mathsf{Diff}_{c_N}(\mathbf{k}, 0)) = \mathsf{Diff}_{\alpha c_N}(\mathbf{k}, 0)$. Thus, the representation $\rho'_N = \mathrm{ad}_\alpha \circ \rho_N$ satisfies $\pi(\rho'_N) \in \mathcal{R}(c)$ if α is sufficiently small. Since $\rho'_N(g) \neq 1$ this concludes that $\mathcal{R}(c)$ satisfies the separation condition of Lemma 3.1.

6.3. Conclusion. The family of representations Φ_s, with $s \in \mathcal{R}(c)$ satisfies the Baire property, the irreducibility property, and the separation property of Lemma 3.1. This lemma implies that a generic element of $s \in \mathcal{R}(c)$ gives an embedding $\Phi_s \colon \Gamma_2 \to \mathsf{Diff}(\mathbf{k}, 0)$, proving Theorem A for the group Γ_2.

Part III

7. A p-adic proof

7.1. Free groups with integer coefficients. A theorem of White [Whi] shows that the homeomorphisms of $\mathbf{R}$ defined by $f : z \mapsto z+1$ and $g : z \mapsto z^3$ generate a free group. Conjugating the maps f and gfg^{-1} by $z \mapsto \frac{1}{3z}$, as in [Gla][1], one gets two formal diffeomorphisms

$$\begin{aligned} f_0(z) &= \frac{z}{1+3z} = \sum_{n=1}^{\infty} (-3)^{n-1} z^n \\ g_0(z) &= \frac{z}{(1+(3z)^3)^{1/3}} = \sum_{n=0}^{\infty} \begin{pmatrix} \frac{-1}{3} \\ n \end{pmatrix} 3^{3n} z^{3n+1} \end{aligned} \tag{7.1}$$

that generate a non-abelian free group $\langle f_0, g_0\rangle \subset \widehat{\mathsf{Diff}}(\mathbf{Q}, 0) \subset \widehat{\mathsf{Diff}}(\mathbf{k}, 0)$. It is remarkable that f_0 and g_0 are tangent to the identity at the origin and have integer coefficients:

[1] This conjugacy is called the "Wilson trick" in [Gla].

Theorem 7.1. *The group* $\widehat{\mathsf{Diff}}(\mathbf{Q},0)$ *contains a non-abelian free group, all of whose elements are tangent to the identity and have integer coefficients.*

Thus, one can produce an explicit free group in $\widehat{\mathsf{Diff}}(\mathbf{k},0)$ for every field $\mathbf{k}$ of characteristic 0. In characteristic $p>0$, Szegedy proved that almost every pair of elements in the Nottingham group $\widehat{\mathsf{Diff}}(\mathbf{Z}/p\mathbf{Z},0)$ generates a free group [Sze].

7.2. Subgroups of $\mathsf{Diff}(\mathbf{Q}_p,0)$**.** In this section, p is a prime number, and $\mathbf{Q}_p$ is the field of p-adic numbers, with its absolute value $|\cdot|$ normalized by $|p|=1/p$.

Let G_p denote the set of elements $f=\sum_{n\geq 1}a_nz^n$ in $\mathsf{Diff}(\mathbf{Q}_p,0)$ such that

$$a_n\in\mathbf{Z}_p\ \forall n \quad\text{and}\quad |a_1|=1. \tag{7.2}$$

Every element $f\in G_p$ satisfies $\mathrm{rad}(f)\geq 1$. The ultrametric inequality and the Inversion formula show that G_p is a subgroup of $\mathsf{Diff}(\mathbf{Q}_p,0)$. With the product topology on coefficients (as in Section 5.6), it is a compact topological group, and the morphism $\mathrm{j}_\ell\colon G_p\to\mathsf{Jets}_\ell(\mathbf{Q}_p,0)$ is continuous for every integer $\ell\geq 1$. The kernel of j_ℓ will be denoted $G_{p,\ell}$.

From Theorem 7.1, we know that G_p contains a free group of rank two generated by two germs f_0 and g_0 whose coefficients are in $\mathbf{Z}$.

Corollary 7.2. *Let* p *be a prime number and* ℓ *be a positive integer. The group* $G_{p,\ell}$ *contains a non-abelian free group. The group* G_p *contains a free group* $\langle f,g\rangle$ *of rank* 2 *such that* $A_1(f)$ *is a transcendental number while* g *is tangent to the identity up to order* ℓ.

Proof. Start with a non-abelian free group F in G_p. Since the group of jets $\mathsf{Jets}_\ell(\mathbf{Q}_p,0)$ is solvable, the restriction of j_ℓ to F is not injective. Its kernel is a free group (as any subgroup of F), and if ℓ is large it is not cyclic. Thus, the kernel is a non-abelian free group. This proves the first statement.

Set $\mathcal{R}=\{t\in\mathbf{Z}_p\ ;\ |t|=1\}$. Now, take a pair of generators f_0 and g_0 of a free group of rank 2 in $G_{p,\ell}$, and for $t\in\mathcal{R}$ consider the family of representations $\rho_t\colon\mathbf{F}_2=\langle a,b\rangle\to G_p$ defined by $\rho_t(a)=m_t\circ f_0$ and $\rho_t(b)=g_0$ (here, as usual, $m_t(z)=tz$). If w is an element of $\mathbf{F}_2$, and n is a positive integer, then $A_n(\rho_t(w))$ is a polynomial function in t and $1/t$ (see Section 2.1). If $w\neq 1$, there is an integer $n\geq 1$ such that $A_n(\rho_1(w))\neq A_n(\mathrm{Id})$. Thus, the set $\mathcal{R}_w\subset\mathcal{R}$ of parameters s such that $\rho_s(w)=\mathrm{Id}$ is finite, the union $\cup_{w\neq 1}\mathcal{R}_w$ is at most countable, and there are transcendental numbers in its complement. For such a parameter t, ρ_t is injective and $A_1(\rho_t(a))=t$ is transcendental. □

Now, we apply the result of [BGSS] described in Section 1.2 to get:

Theorem 7.3. *Let p be a prime number. Let $\Gamma_2 = \langle a, b, \overline{a}, \overline{b} \mid [a,b] = [\overline{a}, \overline{b}] \rangle$ be the fundamental group of a closed orientable surface of genus 2. Then*

(1) *For every integer $\ell \geq 1$, the group Γ_2 embeds in the compact group $G_{p,\ell}$.*

(2) *There is an embedding $\rho\colon \Gamma_2 \to G_p$ such that $\rho(a)'(0) = \rho(\overline{a})'(0)$ is a transcendental number while $\rho(b)$ and $\rho(\overline{b})$ are tangent to the identity up to order ℓ.*

7.3. Back to complex coefficients. The field $\mathbf{Q}_p$, and thus the ring $\mathbf{Z}_p$, embeds (although not continuously) into $\mathbf{C}$; such an embedding induces an embedding, coefficient by coefficient, of $\mathbf{Z}_p[[z]]$ into $\mathbf{C}[[z]]$. Thus, the surface groups constructed in Theorem 7.3 provide surface groups in $\widehat{\mathsf{Diff}}\,(\mathbf{C}, 0)$. This construction does not preserve the convergence of power series, but it preserves the order of tangency to Id. Since there are transcendental complex numbers with modulus < 1, we obtain:

Corollary 7.4. *Let ℓ be a positive integer. There is an embedding $\rho\colon \Gamma_2 \to \widehat{\mathsf{Diff}}\,(\mathbf{C}, 0)$ such that $|\rho(a)'(0)| = |\rho(\overline{a})'(0)| < 1$ while $\rho(b)$ and $\rho(\overline{b})$ are tangent to the identity up to order ℓ.*

We can now prove the following version of Theorem A. This will be our third and last proof of it.

Theorem 7.5. *There is an embedding of Γ_2 in $\mathsf{Diff}(\mathbf{C}, 0)$ such that $|\rho(a)'(0)| = |\rho(\overline{a})'(0)| < 1$ while $\rho(b)$ and $\rho(\overline{b})$ are tangent to the identity up to order ℓ.*

Proof. The first step is to choose a sequence $\mathcal{C} = (a_1, a_2, a_3, \ldots)$ of complex numbers such that

(a) the set $\{a_1, a_2, \ldots\}$ is algebraically free: if $m \geq 1$ and $P \in \mathbf{Z}[x_1, \ldots, x_m]$, and if $P(a_1, \ldots, a_m) = 0$, then $P = 0$;

(b) $|a_n| \leq 2^{-n}$ for all $n \geq 1$.

Such a sequence exists because $\mathbf{C}$ is uncountable. Concrete examples can be obtained from the Lindemann-Weierstrass theorem (see also [Wal] for the constructions of von Neumann, Perron, Kneser, and Durand of uncountably many, algebraically free complex numbers). We shall consider the a_i as indeterminates for the field of rational functions $\mathbf{Q}(a_1, a_2, \ldots)$. Armed with such a set we consider the following three formal diffeomorphisms

(7.3)

$$g = a_1 z + \sum_{i=1}^{\infty} a_{3i+1} z^{i+1}, \quad \bar{f} = z + \sum_{i=\ell}^{\infty} a_{3i+2} z^{i+1}, \quad \bar{g} = z + \sum_{i=\ell}^{\infty} a_{3i+3} z^{i+1}.$$

From the decay relation (b), these three power series have a positive radius of convergence. Since $|a_1| \leq 1/2$, the Koenigs linearization theorem gives a unique element $\bar{f} \in \mathsf{Diff}(\mathbf{C}, 0)$ with $\bar{f}'(0) = 1$ such that

$$fgf^{-1} = [\bar{f}, \bar{g}]g. \tag{7.4}$$

The four elements $(f, g, \bar{f}, \bar{g})$ determine a representation φ of Γ_2 into $\mathsf{Diff}(\mathbf{C}, 0)$.

Let us prove that this representation is faithfull. Fix a non-trivial element w of Γ_2, and write it as a word in a, b, $\overline{a}$, $\overline{b}$ and their inverses. For every integer n, the coefficient $A_n(\varphi(w))$ is a polynomial function $Q_{w,n}$ in the variables a_n (for $n \geq 1$), a_1^{-1}, and the $(a_1^k - 1)^{-1}$ (for $k \geq 1$) with integer coefficients.

Now, take a faithful representation $\rho \colon \Gamma_2 \to \widehat{\mathsf{Diff}}(\mathbf{C}, 0)$ that satisfies the conclusion of Corollary 7.4. There is an integer $n \geq 1$ such that $A_n(\rho(w)) \neq A_n(\mathrm{Id})$. This implies that $Q_{w,n} \neq A_n(\mathrm{Id})$ when we specialize the indeterminates a_i to the coefficients of the generators $\rho(\overline{a})$, $\rho(b)$, and $\rho(\overline{b})$. Since $Q_{w,n} \neq A_n(\mathrm{Id})$, $\varphi(w) \neq \mathrm{Id}$ and φ is the identity. □

Part IV

8. Complements and open questions

8.1. Takens' theorem and smooth diffeomorphisms. To conclude this chapter, we mention the following result which allows to realize *any* faithful representation of a surface group in the group of formal germs as a group of C^∞ germs. Note that the p-adic method provides many embeddings of surface groups in $\widehat{\mathsf{Diff}}(\mathbf{R}, 0)$ (see Corollary 7.4).

Recall that Γ_g denotes the fundamental group of the closed orientable surface of genus g.

Theorem C. *Let $\hat{\rho} \colon \Gamma_g \to \widehat{\mathsf{Diff}}(\mathbf{R}, 0)$ be a faithful representation of the surface group Γ_g in the group of formal diffeomorphisms in one real variable. Then, there exists a faithful representation $\rho \colon \Gamma_g \to \mathsf{Diff}^\infty(\mathbf{R}, 0)$ into the group of germs of C^∞-diffeomorphisms such that the Taylor expansion of $\rho(w)$ coincides with $\hat{\rho}(w)$ for every $w \in \Gamma_g$.*

The proof will be a consequence of the following result (this theorem is easily derived from the Sternberg linearization theorem and Theorem 2 of [Tak]):

Theorem 8.1 (Sternberg [Ste], Takens, [Tak]). *Let $f, g\colon (\mathbf{R},0) \to (\mathbf{R},0)$ be two germs of C^∞-diffeomorphisms, and let $\hat{f}$ and $\hat{g}$ denote their Taylor expansions. Suppose that f is not flat to the identity, that is $\hat{f} \neq \mathrm{Id}$. Then, if $\hat{f}$ and $\hat{g}$ are conjugate by a formal diffeomorphism $\hat{h}$, there exists a germ of C^∞-diffeomorphism $h\colon (\mathbf{R},0) \to (\mathbf{R},0)$ such that*

- *the Taylor expansion of h coincides with $\hat{h}$;*
- *h conjugates f to g.*

Proof of Theorem C. Denote by $\hat{a}_i$, $\hat{b}_i$, $1 \le i \le g$ the images of the standard generators of Γ_g by the representation ρ; they satisfy the relation

$$\hat{a}_1 \circ \hat{b}_1 \circ \hat{a}_1^{-1} = \Big(\prod_{j=2}^{g} [\hat{a}_j, \hat{b}_j]\Big) \circ \hat{b}_1. \tag{8.1}$$

By the theorem of Borel and Peano, one can find germs of diffeomorphisms b_1 and a_j, b_j, $j \ge 2$, whose respective Taylor expansions coincide with $\hat{b}_1$, $\hat{a}_j$, and $\hat{b}_j$ respectively. Then, Theorem 8.1 provides a germ of diffeomorphism a_1 such that $a_1 \circ b_1 \circ a_1^{-1} = (\prod_{j=2}^{g}[a_j, a_j]) \circ b_1$. Thus, one gets a representation ρ of Γ_2 into $\mathsf{Diff}^\infty(\mathbf{R},0)$ with Taylor expansion equal to $\hat{\rho}$. Since the initial representation $\hat{\rho}$ is injective, so is ρ. □

8.2. Conjugacy classes. Two subgroups Γ and Γ' of $\mathsf{Diff}(\mathbf{C},0)$ are *topologically conjugate* if there is a germ of homeomorphism $\varphi\colon (\mathbf{C},0) \to (\mathbf{C},0)$ such that $\varphi \circ \Gamma \circ \varphi^{-1} = \Gamma'$, and are *formally conjugate* if there is a formal diffeomorphism $\hat{\varphi}$ such that $\hat{\varphi} \circ \Gamma \circ \hat{\varphi}^{-1} = \Gamma'$. A germ of homeomorphism φ is *anti-holomorphic* if its complex conjugate $z \mapsto \overline{\varphi(z)}$ is holomorphic.

Theorem 8.2 (Nakai, Cerveau-Moussu). *Let Γ and Γ' be two subgroups of* $\mathsf{Diff}(\mathbf{C},0)$ *which are not solvable.*

(1) *If φ is a local homeomorphism that conjugates Γ to Γ', then φ is holomorphic, or anti-holomorphic.*

(2) *If $\hat{\varphi}$ is a formal conjugacy between Γ and Γ', then $\hat{\varphi}$ converges and is therefore a holomorphic conjugacy.*

Thus, (the images of) two embeddings of Γ_g in $\mathsf{Diff}(\mathbf{C},0)$ are topologically or formally conjugate if and only if they are analytically conjugate.

8.3. Two questions.

8.3.1. It would be interesting to exhibit an embedding α of the group Γ_g, $g \geq 2$, into the group of analytic diffeomorphisms of the circle $\mathbf{R}/Z$ fixing the origin $o \in \mathbf{R}/\mathbf{Z}$. If such an embedding exists, the suspension of this representation α gives a compact manifold M_α of dimension 3 that fibers over Σ_g, together with a foliation $\mathcal{F}_\alpha$ of co-dimension 1 which is transverse to the fibration $\pi : M_\alpha \to \Sigma_g$ and whose monodromy is given by τ. The fixed point gives a compact leaf of $\mathcal{F}_\alpha$ with holonomy given by the same representation τ.

Question. Does there exist an embedding of Γ_2 into the group of analytic diffeomorphisms of the circle fixing the origin ?

This question was the original motivation of Cerveau and Ghys when they asked for a proof of Theorem A (see [Cer]).

Remark 8.3. According to Theorem 7.3, there is an embedding ρ of Γ_2 in $\mathsf{Diff}(\mathbf{Q}_p, 0)$ such that $\rho(a)'(0)$ and $\rho(\overline{a})'(0)$ have modulus 1 while $\rho(b)$ and $\rho(\overline{b})$ are tangent to the identity. Conjugate ρ by the homothety $a \mapsto p^N z$ for some positive integer N. If N is large enough, the coefficients a_n, $n \geq 2$, of all elements of $\rho(\Gamma_2)$ have norm < 1, and the ultrametric inequality shows that $\rho(\Gamma_2)$ preserves the open disks $\{z \in \mathbf{C}_p \; ; \; |z| < 1 - \epsilon\}$ for every $\epsilon > 0$. Thus, it preserves arbitrary thin annuli $\{z \in \mathbf{C}_p \; ; \; 1{-}\epsilon|z| \leq 1\}$. (Here $\mathbf{C}_p$ is the completion of the algebraic closure of $\mathbf{Q}_p$.)

A related, but a priori simpler question is: does there exist an embedding of Γ_2 into the group of increasing, real analytic diffeomorphisms of $[0, 1]$ fixing 0 and 1 ? Here, we demand that the diffeomorphisms extend to germs of real analytic diffeomorphisms on neighbourhoods of 0 and 1. If we replace real analytic diffeomorphisms by $\mathcal{C}^\infty$ diffeomorphisms, interesting examples have been constructed in [MS]. We refer to the introduction of [MS] for a description of the difficulties in trying to apply the strategy of [BGSS]: this is related to the question of deciding when a diffeomorphism f of $[0, 1]$ is contained in the flow of a smooth vector field, hence to Mather's invariant (see [EBN, Yoc]).

8.3.2. The derived subgroup of $\mathsf{Diff}(\mathbf{C}, 0)$ is the kernel of the morphism $\mathsf{j}_1 : f \mapsto f'(0)$:

Theorem 8.4. *Let* $\mathbf{k}$ *be a complete, non-discrete valued field. An element* f *of* $\mathsf{Diff}(\mathbf{k}, 0)$ *is a commutator if and only if* $f'(0) = 1$. *All higher terms of the lower central series coincide with the kernel of* $\mathsf{j}_1 : \mathsf{Diff}(\mathbf{k}, 0) \to \mathsf{Jets}_1(\mathbf{k}, 0)$.

Proof. If f is a commutator $[g, h]$ then $f'(0) = 1$. If $f'(0) = 1$, compose f with the homothety $m_\lambda(z) = \lambda z$ for some $\lambda \in \mathbf{k}^*$ of norm $|\lambda| \neq 1$, and apply Koenigs linearization theorem to find an element $h \in \mathsf{Diff}(\mathbf{k}, 0)$ such that $m_\lambda \circ f = h \circ m_\lambda h^{-1}$ and $h'(0) = 1$. Then $f = [h, m_\lambda]$. This proves that the derived subgroup of $\mathsf{Diff}(\mathbf{k}, 0)$ is the kernel of j_1; since h is in the kernel of j_1, all subsequent terms of the lower central series coincide with the derived subgroup. □

Now, consider the upper central series. The first terms are $\mathsf{Diff}(\mathbf{C}, 0)$ and its derived subgroup $\mathsf{Diff}(\mathbf{k}, 0)^{(1)}$. Then comes

$$\mathsf{Diff}(\mathbf{k}, 0)^{(2)} := [\mathsf{Diff}(\mathbf{k}, 0)^{(1)}, \mathsf{Diff}(\mathbf{k}, 0)^{(1)}]. \tag{8.2}$$

The group of jets of order 3 which are tangent to identity, i.e., jets of the form $j(z) = z + a_2 z^2 + a_3 z^3$ modulo z^4, is an abelian group; at the level of formal germs, it is known that the kernel of j_3 in $\widehat{\mathsf{Diff}}\,(\mathbf{k}, 0)^{(1)}$ coincides with the derived subgroup $\widehat{\mathsf{Diff}}\,(\mathbf{k}, 0)^{(2)}$ (see [Cam], §3, for the description of the upper central series of $\widehat{\mathsf{Diff}}\,(\mathbf{k}, 0)$). We don't know if a similar statement holds for germs of diffeomorphisms:

Question. Does the kernel of j_3 coincide with the second derived subgroup of $\mathsf{Diff}(\mathbf{C}, 0)^{(2)}$? More generally, what is the upper central series of $\mathsf{Diff}(\mathbf{C}, 0)$?

9. Appendix: Free groups

The following theorem, and its proof, are strongly inspired by [MRR]. The proof given in [MRR] is somewhat difficult because it makes use of a topology on $\mathsf{Diff}(\mathbf{C}, 0)$ which is not compatible with the group law. We adapt the same proof, without reference to such a topology.

Theorem 9.1. *Let $(\mathbf{k}, |\cdot|)$ be a complete, non-discrete valued field. Let f and g be elements of $\mathsf{Diff}(\mathbf{k}, 0)$ of infinite order. Let w be a non-trivial element of the free group $\mathbf{F}_2$. Then, there is a polynomial germ of diffeomorphism h such that $w(hfh^{-1}, g) \neq \mathrm{Id}$.*

If $w = a^{n_\ell} b^{n_{\ell-1}} \cdots a^{n_2} b^{n_1}$, one can choose h of the form $z + \epsilon z^2 P(z)$ with an arbitrarily small ϵ and a polynomial function $P \in \mathbf{k}[z]$ such that $\deg(P) \leq (2\ell)!$ and $|P(x)| \leq 1$ for all $x \in \mathbb{D}_1$.

Before proving this result, let us introduce some vocabulary and notation. Write w as a reduced word in the generators a and b of the free group:

$$w = a^{n_\ell} b^{n_{\ell-1}} \cdots a^{n_2} b^{n_1} \tag{9.1}$$

where the n_i are in $\mathbf{Z}\setminus\{0\}$, except maybe if n_1 or n_ℓ is zero, but conjugating w by a power of a, we only need to consider the case $n_1 n_\ell \neq 0$. Set $N = \max|n_i|$.

Let h be an element of $\mathsf{Diff}(\mathbf{k}, 0)$, and set $f_h = h^{-1} \circ f \circ h$. Let $r > 0$ be smaller than the convergence radius of h, f, g and their inverses. Choose $R > 0$ such that all these germs, and all their compositions of length $\leq 3N\ell$ map $\mathbb{D}_R$ inside $\mathbb{D}_r$. If z is a point in $\mathbb{D}_R$, then its orbit under the action of f_h and g stays in $\mathbb{D}_r$ for all compositions of these germs given by words of length $\leq N\ell$ in $\mathbf{F}_2$; in this situation, we say that the orbit of z is *well defined* up to length $N\ell$. In particular, if we look at the composition $w(f_h, g)$, and pick a point z in $\mathbb{D}_R$, we get a sequence of points

$$z_0 = z,\ z_1 = g^{n_1}(z_0),\ z_2 = f_h^{n_2}(z_1),\ \ldots,\ z_\ell = w(f_h, g)(z_0). \tag{9.2}$$

To prove the theorem, we construct a triple (h, R, z) such that the orbit of z is well defined and the z_i are pairwise distinct; in particular, $z_\ell \neq z_0$ and $w(f_h, g) \neq \mathrm{Id}$.

Proof. We do a recursion on the length ℓ, proving the existence of a triple (h, R, z) such that the z_i are pairwise distinct for $0 \leq i \leq \ell$. Since f and g have infinite order, the union of all fixed points of f^m and g^m in $\mathbb{D}_r$ for $-N \leq m \leq N$ is a finite set F. For $j = 1$, we just pick a point z_0 sufficiently near the origin with $z_1 := g^{n_1}(z_0) \neq z_0$; the only constraint is to take z_0 in the complement of F. The points z_0 and z_1 will be kept fixed in the recursion.

Assume that a polynomial germ of diffeomorphism h_k has been constructed, in such a way that (a) the points z_0, z_1, z_2, $\ldots$, z_{2k}, and z_{2k+1} are pairwise distinct (we just initialized the recursion for $k = 0$), and (b) $h_k(z) = z + \epsilon_k R_k(z)$ for some small $\epsilon_k \in \mathbf{k}$ and some element $R_k \in \mathbf{k}[z]$ of degree $\leq (2k)!$ which is divisible by z^2. Consider a polynomial germ

$$P_k(z) = z + \eta_k z^2 \prod_{j=0}^{2k} (z - z_j) \tag{9.3}$$

with a small $\eta_k \in \mathbf{k}$; then

- P_k fixes z_j for all $j \leq 2k$,
- $P_k(z_{2k+1}) = a_k \eta_k + b_k$ for some pair $(a_k, b_k) \in \mathbf{k}^2$ with $a_k \neq 0$,
- as η_k goes to 0, the radius of convergence of P_k and its inverse P_k^{-1} go to infinity.

If we compose h_k with P_k then $H = h_k \circ P_k$ is a new polynomial germ such that the orbit of z_0 under f_H and g gives the same sequence z_0, z_1, $\ldots$, up to z_{2k+1}. The next point is

$$z_{2k+2} = f_H^{n_{2k+2}}(z_{2k+1}) \tag{9.4}$$

and we want to exclude the possibility $z_{2k+2} \in \{z_0, \ldots, z_{2k+1}\}$; since

$$z_{2k+2} = (P_k^{-1} \circ f_{h_k}^{n_{2k+2}} \circ P_k)(z_{2k+1}) \tag{9.5}$$

we want to avoid the inclusion

$$f_{h_k}^{n_{2k+2}}(P_k(z_{2k+1})) \subset P_k\{z_0, \ldots, z_{2k+1}\}, \tag{9.6}$$

and for that we just need to choose the parameter η_k in the definition of P_k in such a way that $f_{h_k}^{n_{2k+2}}(P_k(z_{2k+1}))$ is not in $\{z_0, \ldots, z_{2k}\}$ and $P_k(z_{2k+1})$ is not a fixed point of $f_{h_k}^{n_{2k+2}}$. These constraints are satisfied for all small non-zero values of η_k because $f_{h_k}^{n_{2k+2}}$ is not the identity and the coefficient a_k in $P_k(z_{2k+1}) = a_k\eta_k + b_k$ is not zero.

The next point is $z_{2k+3} = g^{n_{2k+3}}(z_{2k+2})$ and we want it to be disjoint from $\{z_0, \ldots, z_{2k+1}, z_{2k+2}\}$. For this, we do a second perturbation of the conjugacy. Let

$$Q_k(z) = z + \beta_k z^2 \prod_{j=0}^{2k+1} (z - z_j) \tag{9.7}$$

with a small $\beta_k \in \mathbf{k}$; then

- Q_k fixes z_j for all $j \leq 2k+1$,
- $Q_k(z_{2k+2}) = c_k\beta_k + d_k$ for some pair $(c_k, d_k) \in \mathbf{k}^2$ with $c_k \neq 0$,
- as β_k goes to 0, the radius of convergence of Q_k and its inverse Q_k^{-1} go to infinity.

Now, we set $h_{k+1} = Q_k \circ H$. This does not change the sequence z_i for $0 \leq i \leq 2k+1$, but the last point z_{2k+2} is replaced by $c_k\beta_k + d_k$. Since $g^{n_{2k+3}} \neq \mathrm{Id}$ and $c_k \neq 0$ any non-zero, small enough value of β_k assures that $z_{2k+3} \notin \{z_0, \ldots, z_{2k+1}, z_{2k+2}\}$.

To sum up, if we set $h_{k+1} = Q_k \circ P_k \circ h_k$ then the sequence z_0, ..., z_{2k+3} is now made of pairwise distinct points. Moreover, when the parameters η_k and β_k go to zero, the germ $Q_k \circ P_k$ and its inverse converge uniformly to the identity on the disk $\mathbb{D}_{2R}$, so we can assume that the orbit of z_0 is well defined for all composition of h_{k+1}, f, g, and their inverses of length $\leq 3N\ell$. The germ $Q_k \circ P_k$ is equal to $z + S_k(z)$ where S_k is divisible by z^2 and $\deg(S_k) \leq (2k+1) \times (2k+2)$. Thus,

$$h_{k+1}(z) = z + P_{k+1}(z) \tag{9.8}$$

where z^2 divides P_{k+1} and

$$\deg(P_{k+1}) \leq \deg(P_k) \times (2k+1) \times (2k+2) \leq (2k+2)! \tag{9.9}$$

This proves the recursion and finishes the proof of the theorem. □

Theorem 9.2. *Let* $(\mathbf{k}, |\cdot|)$ *be a complete, non-discrete valued field. If* f *and* g *are elements of* $\mathsf{Diff}(\mathbf{k}; 0)$ *of infinite order, there exists an element* h *of* $\mathsf{Diff}(\mathbf{k}; 0)$ *such that* $f_h := h \circ f \circ h^{-1}$ *and* g *generate a free group of rank* 2. *One can choose* h *such that* $h'(0) = 1$.

Note that Theorem 3.4 is a direct corollary of that result; one just need to start with $f = \lambda_1 z$ or $\lambda_1 z + z^2$ if λ_1 is a root of unity, and similarly for g.

Proof. Denote by a_n and b_n the coefficients of f and g respectively. Let $L \subset \mathbf{k}$ be the field generated by the a_n and b_n. Since $\mathbf{k}$ is not discrete, it has no isolated point; being complete with no isolated point, it is uncountable (a simple consequence of Baire's theorem [Oxt]), and it follows that its transcendental degree over L is infinite: it contains an infinite sequence (c_i) of algebraically independent numbers (over the prime field of $\mathbf{k}$, see [Lan, Chapter VIII]). We can moreover assume that all c_i are in the unit disk. Set $h_0(z) = \sum_{n \geq 1} c_n z^n$.

Consider a non-trivial element w of $\mathbf{F}_2$. The N-th coefficient function

$$h \mapsto A_N(w(h \circ f \circ h^{-1}, g)) \tag{9.10}$$

is a polynomial function on $\widehat{\mathsf{Diff}}(\mathbf{k}, 0)$ in the sense of Section 2.1; this means that it is a polynomial function in the coefficients of h and $A_1(h)^{-1}$ (here, f and g are fixed). If $A_N(w(h_0 \circ f \circ h_0^{-1}, g))$ vanishes (resp. is equal to 1), then $A_N(w(h \circ f \circ h^{-1}, g)) = 0$ (resp. 1) for all formal diffeomorphisms h, because the c_i are algebraically independent over $\mathbf{k}$. Thus, Theorem 9.1 implies that $w(h_0 \circ f \circ h_0^{-1}, g) \neq \mathrm{Id}$, and this shows that $f_{h_0} := h_0 \circ f \circ h_0^{-1}$ and g generate a free group of rank 2.

In this argument, we could start with $h_0 = z + \sum_{n \geq 2} c_n z^n$, because we can choose the germ h in Theorem 9.1 with the additional constraint $h'(0) = 1$. □

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