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**Autor:** Cantat, Serge / Cerveau, Dominique / Guirardel, Vincent  
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# Surface groups in the group of germs of analytic diffeomorphisms in one variable

Serge CANTAT, Dominique CERVEAU, Vincent GUIRARDEL  
and Juan SOUTO

**Abstract.** We construct embeddings of surface groups into the group of germs of analytic diffeomorphisms in one variable.

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**Keywords.** Germs of diffeomorphisms, fundamental groups of surfaces, codimension one foliations.

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## 1. Introduction

**1.1. The main result.** Let  $\mathbf{C}$  be the field of complex numbers and  $\text{Diff}(\mathbf{C}, 0)$  the group of germs of analytic diffeomorphisms at the origin  $0 \in \mathbf{C}$ . Choosing a local coordinate  $z$  near the origin, every element  $f \in \text{Diff}(\mathbf{C}, 0)$  is determined by a unique power series

$$f(z) = a_1 z + a_2 z^2 + a_3 z^3 + \dots + a_n z^n + \dots$$

with  $f'(0) = a_1 \neq 0$  and with a positive radius of convergence

$$(1.1) \quad \text{rad}(f) = \left( \limsup_{n \rightarrow +\infty} |a_n|^{1/n} \right)^{-1}.$$

We denote by  $\text{Diff}(\mathbf{R}, 0) \subset \text{Diff}(\mathbf{C}, 0)$  the subgroup of real germs in this chart, i.e., with  $a_i \in \mathbf{R}$  for all  $i \in \mathbf{N}$  (this inclusion depends on the choice of the coordinate  $z$ ). The main goal of this note is the following result, that answers a question raised by E. Ghys (see [Cer], §3.3, or also [Bru1], Problem 4.15).

**Theorem A.** *Let  $\Gamma$  be the fundamental group of a closed orientable surface, or of a closed non-orientable surface of genus  $\geq 4$ . Then  $\Gamma$  embeds in the group  $\text{Diff}(\mathbf{R}, 0)$  and in particular in  $\text{Diff}(\mathbf{C}, 0)$ .*

We shall present three proofs of Theorem A. For simplicity, in this introduction, we restrict to the case where  $\Gamma$  is the fundamental group of an orientable surface of genus 2, and we consider the presentation

$$(1.2) \quad \Gamma_2 = \langle a_1, b_1, a_2, b_2 \mid [a_1, b_1] = [a_2, b_2] \rangle.$$

Our proofs of theorem A are inspired by [BGSS], where it is proved that a compact topological group or a connected Lie group which contains a dense free group of rank 2 contains a dense subgroup isomorphic to  $\Gamma_2$ .

The surface groups considered in Theorem A are examples of limit groups. Recently, and independently, A. Brudnyi proved a related embedding theorem: limit groups embed into the group of (non converging) formal germs of diffeomorphisms (see [Bru2])

**1.2. Compact groups.** Let us describe the argument used in [BGSS] to prove the following result.

**Theorem 1.1** ([BGSS]). *If a compact group  $G$  contains a free group  $F$  of rank 2, then there is an embedding  $\rho: \Gamma_2 \rightarrow G$  such that  $F \subset \rho(\Gamma_2)$ .*

*Proof.* Denote by  $\mathbf{F}_m$  the free group on  $m$  generators. The first ingredient is a result by Baumslag [Bau] saying that  $\Gamma_2$  is *fully residually free*; this means that there exists a sequence of morphisms  $p_N: \Gamma_2 \rightarrow \mathbf{F}_2$  which is asymptotically injective: for every  $g \in \Gamma_2 \setminus \{1\}$ ,  $p_N(g) \neq 1$  if  $N$  is large enough.

To be more explicit, we use the presentation (1.2) of  $\Gamma_2$ , and we note that the subgroup  $\langle a_1, b_1 \rangle$  of  $\Gamma_2$  is a free group  $\mathbf{F}_2 = \langle a_1, b_1 \rangle$ . Let  $p: \Gamma_2 \rightarrow \langle a_1, b_1 \rangle$  be the morphism fixing  $a_1$  and  $b_1$  and sending  $a_2$  and  $b_2$  to  $a_1$  and  $b_1$

respectively. Let  $\tau : \Gamma_2 \rightarrow \Gamma_2$  be the Dehn twist around the curve  $c = [a_1, b_1]$ , i.e. the automorphism that fixes  $a_1$  and  $b_1$  and sends  $a_2$  and  $b_2$  to  $ca_2c^{-1}$  and  $cb_2c^{-1}$  respectively.

**Proposition 1.2** (see [BGSS, Corollary 2.2]). *Given any  $g \in \Gamma_2 \setminus \{1\}$ , there exists a positive integer  $n_0$  such that  $p \circ \tau^N(g) \neq 1$  for all  $N \geq n_0$ .*

Now, fix an embedding  $\iota : \langle a_1, b_1 \rangle \rightarrow G$  such that  $\iota(\langle a_1, b_1 \rangle) = F$ . Composing  $p \circ \tau^N$  with  $\iota$ , we get a sequence of points  $p_N := \iota \circ p \circ \tau^N$  in  $\text{Hom}(\Gamma_2, G)$ . Now, consider the element  $h = \iota(p(c))$  of  $G$ , and let  $T$  be the closure of the cyclic group  $\langle h \rangle$  in the compact group  $G$ . For  $t \in T$ , define a morphism  $\rho_t : \Gamma_2 \rightarrow G$  by

$$(1.3) \quad \rho_t(a_1) = \iota(p(a_1)), \quad \rho_t(a_2) = t \circ (\iota(p(a_1))) \circ t^{-1}$$

$$(1.4) \quad \rho_t(b_1) = \iota(p(b_1)), \quad \rho_t(b_2) = t \circ (\iota(p(b_1))) \circ t^{-1}$$

these representations are well defined and satisfy  $\rho_t = \iota \circ p \circ \tau^N$  when  $t = h^N$ . Moreover, on the subgroup  $\langle a_1, b_1 \rangle$ ,  $\rho_t$  coincides with  $\iota \circ p$ , so  $F \subset \rho_t(\Gamma_2)$ . Thus,  $(\rho_t)_{t \in T}$  is a compact subset  $\mathcal{R}(T) \subset \text{Hom}(\Gamma_2, G)$  that contains the sequence of points  $p_N$ . For every  $g$  in  $\Gamma_2 \setminus \{1\}$ , the subset  $\mathcal{R}(T)_g = \{\rho_t \mid \rho_t(g) \neq 1\}$  is open, and Proposition 1.2 shows that it is dense because  $\{h^n \mid n \geq n_0\}$  is dense in  $T$  for every integer  $n_0$ . By the Baire theorem, the subset of injective representations  $\rho_t$  is a dense  $G_\delta$  in  $\mathcal{R}(T)$ , and this proves Theorem 1.1.  $\square$

The group  $\text{Diff}(\mathbf{R}, 0)$  contains non-abelian free groups (this is well known, see Section 3.3), and one may want to copy the above argument for  $G = \text{Diff}(\mathbf{R}, 0)$  instead of a compact group. The Koenigs linearization theorem says that if  $f \in \text{Diff}(\mathbf{R}, 0)$  satisfies  $f'(0) > 1$ , then  $f$  is conjugate to the homothety  $z \mapsto f'(0)z$ ; in particular, there is a flow of diffeomorphisms  $(\varphi^t)_{t \in \mathbf{R}}$  for which  $\varphi^1 = f$ . In our argument, the compact group  $T$  introduced to prove Theorem 1.1 will be replaced by such a flow, hence by a group isomorphic to  $(\mathbf{R}, +)$ . Also, in that proof,  $h = \iota(p(c))$  was a commutator, and the derivative of any commutator in  $\text{Diff}(\mathbf{R}, 0)$  is equal to 1 at the origin, so that Koenigs theorem can not be applied to a commutator. Thus, we need to change  $p_N$  into a different sequence of morphisms: the Dehn twist  $\tau$  will be replaced by another automorphism of  $\Gamma_2$ , twisting along three non-separating curves.

This argument will be described in details in Sections 2 and 3; the reader who wants the simplest proof of Theorem A in the case of orientable surfaces only needs to read these sections. Non orientable surfaces are dealt with in Section 4.

**1.3. Lie groups.** Now, let us look at representations in a linear algebraic subgroup  $G$  of  $\mathrm{GL}_m(\mathbf{R})$ . Assuming that there is a faithful representation  $\iota: \mathbf{F}_2 \rightarrow G$  with dense image, we shall construct a faithful representation  $\Gamma_2 \rightarrow G$ .

The representation variety  $\mathrm{Hom}(\Gamma_2, G)$  is an algebraic subset of  $G^4$ . Let  $\mathcal{R}$  be the irreducible component containing the trivial representation. Let  $p_N: \Gamma_2 \rightarrow \mathbf{F}_2$  be an asymptotically injective sequence of morphisms, as given by Baumslag's proposition. When the image of  $\rho$  is dense, one can prove that  $\iota \circ p_N$  is in  $\mathcal{R}$  for arbitrarily large values of  $N$ . For  $g \in \Gamma_2 \setminus \{1\}$ , the subset  $\mathcal{R}_g \subset \mathcal{R}$  of homomorphisms killing  $g$  is algebraic, and it is a proper subset because it does not contain  $\iota \circ p_N$  for some large  $N$ . Then, a Baire category argument in  $\mathcal{R}$  implies that a generic choice of  $\rho \in \mathcal{R}$  is faithful.

To apply this argument to  $G = \mathrm{Diff}(\mathbf{R}, 0)$ , one needs a good topology on  $\mathrm{Diff}(\mathbf{R}, 0)$ , and a good “irreducible variety”  $\mathcal{R} \subset \mathrm{Hom}(\Gamma_2, G)$  containing  $\iota \circ p_N$ , in which a Baire category argument can be used. This approach may seem difficult because  $\mathrm{Hom}(\Gamma_2, G)$  is a priori far from being an irreducible analytic variety but, again, the Koenigs linearization theorem will provide the key ingredient.

First, we shall adapt an idea introduced by Leslie in [Les] to define a useful group topology on  $\mathrm{Diff}(\mathbf{R}, 0)$  (see Section 5). With this topology,  $\mathrm{Diff}(\mathbf{R}, 0)$  is an increasing union of Baire spaces, which will be enough for our purpose. Denote by  $\mathrm{Cont}(\mathbf{R}, 0) \subset \mathrm{Diff}(\mathbf{R}, 0)$  the set of elements  $f \in \mathrm{Diff}(\mathbf{R}, 0)$  with  $|f'(0)| < 1$ ;  $\mathrm{Cont}$  stands for “contractions”. Consider the set  $\mathcal{R}$  of representations  $\rho: \Gamma_2 \rightarrow \mathrm{Diff}(\mathbf{R}, 0)$  with  $\rho(a_1)$  tangent to the identity, and  $\rho(b_1) \in \mathrm{Cont}(\mathbf{R}, 0)$ . Then, the key fact is that the map

$$\begin{aligned} \Psi: \mathcal{R} &\rightarrow \mathrm{Cont}(\mathbf{R}, 0) \times \mathrm{Diff}(\mathbf{R}, 0) \times \mathrm{Diff}(\mathbf{R}, 0) \\ \rho &\mapsto (\rho(b_1), \rho(a_2), \rho(b_2)) \end{aligned}$$

is a continuous bijection. Indeed, the defining relation of  $\Gamma$  is equivalent to  $a_1 b_1 a_1^{-1} = [a_2, b_2] b_1$ . Given  $(g_1, f_2, g_2) \in \mathrm{Cont}(\mathbf{R}, 0) \times \mathrm{Diff}(\mathbf{R}, 0) \times \mathrm{Diff}(\mathbf{R}, 0)$ , the germs  $g_1$  and  $[f_2, g_2] g_1$  have the same derivative at the origin and, from the Koenigs linearization theorem, there is a unique  $f_1 \in \mathrm{Diff}(\mathbf{R}, 0)$  tangent to the identity solving the equation  $f_1 g_1 f_1^{-1} = [f_2, g_2] g_1$ : by construction there is a unique morphism  $\rho: \Gamma_2 \rightarrow \mathrm{Diff}(\mathbf{R}, 0)$  that maps the  $a_i$  to the  $f_i$ , and the  $b_i$  to the  $g_i$ , and this representation satisfies  $\Psi(\rho) = (g_1, f_2, g_2)$ . With this bijection  $\Psi$  and the topology of Leslie, we can identify  $\mathcal{R}$  with a union of Baire spaces, in which the Baire category argument applies.

**1.4. Other fields.** Let  $\mathbf{k}$  be a finite field with  $p$  elements. The group  $\mathrm{Diff}^1(\mathbf{k}, 0)$ , also known as the Nottingham group, is the group of power series tangent to the identity and with coefficients in the finite field  $\mathbf{k}$ . It is a compact group containing a free group (see [Sze]). Thus, by [BGSS], it contains a surface group.

Now, let  $p$  be a prime number, and let  $\mathbf{Q}_p$  be the field of  $p$ -adic numbers.

Consider the subgroup  $\text{Diff}^1(\mathbf{Z}_p, 0) \subset \text{Diff}(\mathbf{Q}_p, 0)$  of formal power series tangent to the identity and with coefficients in  $\mathbf{Z}_p$ . First, note that all elements  $f$  of  $\text{Diff}^1(\mathbf{Z}_p, 0)$  satisfy  $\text{rad}(f) \geq 1$ , so that  $\text{Diff}^1(\mathbf{Z}_p, 0)$  acts faithfully as a group of ( $p$ -adic analytic) homeomorphisms on  $\{z \in \mathbf{Z}_p ; |z| < 1\}$ . So, in that respect,  $\text{Diff}^1(\mathbf{Z}_p, 0)$  is much better than the group of germs of diffeomorphisms  $\text{Diff}(\mathbf{C}, 0)$ . Moreover, with the topology given by the product topology on the coefficients  $a_n \in \mathbf{Z}_p$  of the power series, the group  $\text{Diff}^1(\mathbf{Z}_p, 0)$  becomes a compact group. And this compact group contains a free group. By the result of [BGSS] described in Section 1.2, it contains a copy of the surface group  $\Gamma_2$ . So, we get a surface group acting faithfully as a group of  $p$ -adic analytic homeomorphisms on  $\{z \in \mathbf{Z}_p ; |z| < 1\}$ . In Section 7 we give a third proof of Theorem A that starts with the case of  $p$ -adic coefficients.

**1.5. Organisation.** The article is split in four parts.

- I. Sections 2 to 4 give a first proof of Theorem A; Section 4 is the only place where we deal with non-orientable surfaces. We refer to Theorem B in Section 4.3 for a stronger result, in which the field  $\mathbf{R}$  is replaced by any non-discrete, complete valued field  $\mathbf{k}$ .
- II. Section 5 and 6 present our second proof, based on the construction of a group topology on  $\text{Diff}(\mathbf{C}, 0)$ .
- III. Then, our  $p$ -adic proof is presented in Section 7.
- IV. Section 8 draws some consequences and list a few open problems, while the appendix shows how to construct free groups in  $\text{Diff}(\mathbf{C}, 0)$ , or  $\text{Diff}(\mathbf{k}, 0)$  for any non-discrete and complete valued field.

## Part I

### 2. Germs of diffeomorphisms and the Koenigs Linearization Theorem

**2.1. Formal diffeomorphisms.** Let  $\mathbf{k}$  be a field (of arbitrary characteristic). Denote by  $\mathbf{k}[[z]]$  the ring of formal power series in one variable with coefficients in  $\mathbf{k}$ . For every integer  $n \geq 0$ , let  $A_n: \mathbf{k}[[z]] \rightarrow \mathbf{k}$  denote the  $n$ -th coefficient function:

$$(2.1) \quad A_n: f = \sum a_n z^n \mapsto A_n(f) = a_n.$$

A *formal diffeomorphism* is a formal power series  $f \in \mathbf{k}[[z]]$  such that  $A_0(f) = 0$  and  $A_1(f) \neq 0$ . The composition  $f \circ g$  determines a group law on the set

$$(2.2) \quad \widehat{\text{Diff}}(\mathbf{k}, 0) = \{f \in \mathbf{k}[[z]] \mid A_0(f) = 0 \text{ and } A_1(f) \neq 0\}$$

of all formal diffeomorphisms.

For each  $n \geq 1$ , there is a polynomial  $P_n \in \mathbf{Z}[A_1, B_1, \dots, A_n, B_n]$  such that if  $f = \sum a_n z^n$  and  $g = \sum b_n z^n$  then  $f \circ g = \sum_{n \geq 1} P_n(a_1, b_1, \dots, a_n, b_n) z^n$ . Similarly, there are polynomials  $Q_n \in \mathbf{Z}[A_1, \dots, A_n][A_1^{-1}]$  such that  $f^{-1} = \sum_{n \geq 1} Q_n(a_1, \dots, a_n) z^n$  if  $f = \sum a_n z^n$ ; the polynomial function  $Q_n$  is given by the following *inversion formula*:

$$\frac{1}{a_1^n} \sum_{k_1, k_2, \dots} (-1)^{k_1+k_2+\dots} \cdot \frac{(n+1) \cdots (n-1+k_1+k_2+\dots)}{k_1! k_2! \cdots} \cdot \left(\frac{a_2}{a_1}\right)^{k_1} \left(\frac{a_3}{a_1}\right)^{k_2} \cdots$$

where  $a_i = A_i(f)$  and the sum is over all sequences of integers  $k_i$  such that

$$k_1 + 2k_2 + 3k_3 + \cdots = n-1.$$

We refer to [Jen] where this is proved for  $f$  and  $g$  tangent to the identity; the general case easily follows.

To encapsulate this kind of properties, we introduce the following definition. Let  $m$  be a positive integer. By definition, a function  $Q: \widehat{\text{Diff}}(\mathbf{k}, 0)^m \rightarrow \mathbf{k}$  is a *polynomial function* with integer coefficients, if there is an integer  $n$ , and a polynomial  $q \in \mathbf{Z}[A_{1,1}, A_{2,1}, \dots, A_{m-1,n}, A_{m,n}][A_{1,1}^{-1}, \dots, A_{m,1}^{-1}]$  such that

$$(2.3) \quad Q(f_1, \dots, f_m) = q(A_1(f_1), \dots, A_n(f_m))$$

for all  $m$ -tuples  $(f_1, \dots, f_m) \in \widehat{\text{Diff}}(\mathbf{k}, 0)^m$ ; we denote by  $\mathbf{Z}[\widehat{\text{Diff}}(\mathbf{k}, 0)^m]$  this ring of polynomial functions.

Let  $\mathbf{F}_m = \langle e_1, \dots, e_m \rangle$  be the free group of rank  $m$ . To every word  $w = e_{i_1}^{n_1} \cdots e_{i_k}^{n_k}$  in  $\mathbf{F}_m$ , with exponents  $n_i \in \mathbf{Z}$ , we associate the *word map*  $w: \widehat{\text{Diff}}(\mathbf{k}, 0)^m \rightarrow \widehat{\text{Diff}}(\mathbf{k}, 0)$ ,

$$(2.4) \quad (g_1, \dots, g_m) \mapsto w(g_1, \dots, g_m) \stackrel{\text{def}}{=} g_{i_1}^{n_1} \circ \cdots \circ g_{i_k}^{n_k}.$$

Since composition and inversion are polynomial functions on  $\widehat{\text{Diff}}(\mathbf{k}, 0)$ , we get:

**Lemma 2.1.** *Let  $w: \widehat{\text{Diff}}(\mathbf{k}, 0)^m \rightarrow \widehat{\text{Diff}}(\mathbf{k}, 0)$  be the word map given by some element of the free group  $\mathbf{F}_m$ . For each  $n \geq 1$ , there is a polynomial function  $Q_{w,n} \in \mathbf{Z}[\widehat{\text{Diff}}(\mathbf{k}, 0)^m]$  such that*

$$A_n(w(g_1, \dots, g_m)) = Q_{w,n}(g_1, \dots, g_m)$$

for all  $g_1, \dots, g_m \in \widehat{\text{Diff}}(\mathbf{k}, 0)$ .

**2.2. Diffeomorphisms and Koenigs linearization Theorem.** Suppose now that  $\mathbf{k}$  is endowed with an absolute value  $|\cdot|: \mathbf{k} \rightarrow \mathbf{R}_+$ . Then  $\mathbf{k}$  becomes a metric space with the distance induced by  $|\cdot|$ . We shall almost always assume that

- $\mathbf{k}$  is not discrete, equivalently there is an element  $x \in \mathbf{k}$  with  $|x| \neq 0, 1$ ;
- $\mathbf{k}$  is complete.

Let  $\mathbf{k}\{z\}$  be the subring of  $\mathbf{k}[[z]]$  consisting of power series  $f(z) = \sum a_n z^n$  whose radius of convergence  $\text{rad}(f)$  is positive (see Equation (1.1)). When  $\mathbf{k}$  is complete, the series  $\sum a_n z^n$  converges uniformly in the closed disk  $\mathbb{D}_r = \{z \in \mathbf{k} \mid |z| \leq r\}$  for every  $r < \text{rad}(f)$ . The group of germs of analytic diffeomorphisms is the intersection  $\text{Diff}(\mathbf{k}, 0) := \widehat{\text{Diff}}(\mathbf{k}, 0) \cap \mathbf{k}\{z\}$ ; it is a subgroup of  $\widehat{\text{Diff}}(\mathbf{k}, 0)$ .

A germ  $f \in \text{Diff}(\mathbf{k}, 0)$  is *hyperbolic* if  $|A_1(f)| \neq 1$ . The following result is proved in [Mil, Chapter 8] and [HY, Theorem 1, p. 423] (see also [Sie, Theorem 1] or [Koe]).

**Theorem 2.2** (Koenigs linearization theorem). *Let  $(\mathbf{k}, |\cdot|)$  be a complete, non-discrete valued field. Let  $f \in \text{Diff}(\mathbf{k}, 0)$  be a hyperbolic germ of diffeomorphism. There is a unique germ of diffeomorphism  $h \in \text{Diff}(\mathbf{k}, 0)$  such that  $h(f(z)) = A_1(f) \cdot h(z)$  and  $A_1(h) = 1$ .*

### 3. Embedding orientable surface groups

**3.1. Abstract setting.** Our strategy to construct embeddings of surface groups relies on the following simple remark. Let  $\Gamma$  be a countable group, and  $G$  be any group. Consider a non-empty topological space  $\mathcal{R}$ , with a map  $\Phi: s \in \mathcal{R} \mapsto \Phi_s \in \text{Hom}(\Gamma, G)$ . Given  $g \in \Gamma$ , set  $\mathcal{R}_g = \{s \in \mathcal{R} \mid \Phi_s(g) = 1\}$ .

**Lemma 3.1.** *Assume that  $\mathcal{R}$  has the following three properties:*

- (1) **Baire:**  $\mathcal{R}$  is a Baire space;
- (2) **Separation:** for every  $g \neq 1$  in  $\Gamma$ ,  $\Phi_s(g) \neq 1$  for some  $s \in \mathcal{R}$ ;
- (3) **Irreducibility:** for every  $g \in \Gamma$ , either  $\mathcal{R}_g = \mathcal{R}$  or  $\mathcal{R}_g$  is closed with empty interior.

*Then the set of  $s \in \mathcal{R}$  such that  $\Phi_s$  is an injective homomorphism is a dense  $G_\delta$  in  $\mathcal{R}$ ; in particular, it is non-empty.*

*Proof.* For any  $g \in \Gamma \setminus \{1\}$ , one has  $\mathcal{R}_g \neq \mathcal{R}$  by (2), so  $\mathcal{R}_g$  is closed with empty interior by (3). By the Baire property,  $\mathcal{R} \setminus (\bigcup_{g \in \Gamma \setminus \{1\}} \mathcal{R}_g)$  is a dense  $G_\delta$ . But  $\mathcal{R} \setminus (\bigcup_{g \in \Gamma \setminus \{1\}} \mathcal{R}_g)$  is precisely the set of  $s \in \mathcal{R}$  such that  $\Phi_s$  is injective.  $\square$

**3.2. Baumslag Lemma.** As explained in the introduction, it is proved in [Bau] that the fundamental group of an orientable surface is fully-residually free. We need a precise version of this result; to obtain it, the main technical input is the Baumslag's Lemma (see [Ols, Lemma 2.4]):

**Lemma 3.2** (Baumslag's Lemma). *Let  $n \geq 1$  be a positive integer. Let  $g_0, \dots, g_n$  be elements of  $\mathbf{F}_k$ , and let  $c_1, \dots, c_n$  be elements of  $\mathbf{F}_k \setminus \{1\}$ . Assume that for all  $1 \leq i \leq n-1$ ,  $g_i^{-1}c_i g_i$  does not commute with  $c_{i+1}$ . Then for  $N$  large enough,*

$$g_0 c_1^N g_1 c_2^N \dots c_{n-1}^N g_{n-1} c_n^N g_n \neq 1.$$

*Sketch of proof (I).* The group  $\mathrm{PSL}_2(\mathbf{R})$  acts on the hyperbolic plane  $\mathbb{H}$  by isometries, and contains a subgroup  $\Gamma$  such that (0)  $\Gamma$  is isomorphic to  $\mathbf{F}_k$ , (1) every element  $g \neq \mathrm{Id}$  in  $\Gamma$  is a loxodromic isometry of  $\mathbb{H}$ , and (2) two elements  $g$  and  $h$  in  $\Gamma \setminus \{\mathrm{Id}\}$  commute if and only if they have the same axis, which happens if and only if they share a common fixed point on  $\partial\mathbb{H}$ . One can find such a group in any lattice of  $\mathrm{PSL}_2(\mathbf{R})$ . To prove the lemma, we prove it in  $\Gamma$ .

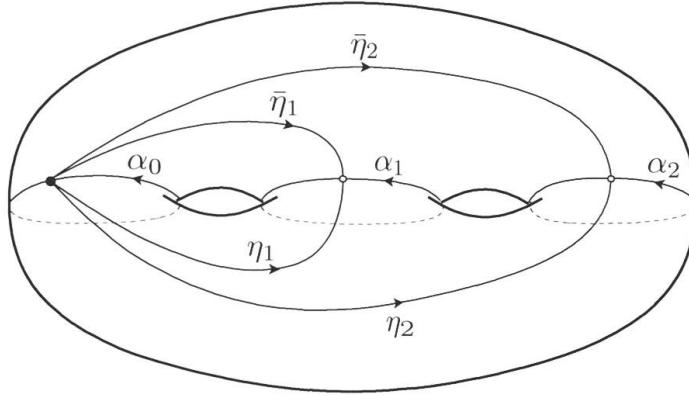
Fix a base point  $x \in \mathbb{H}$ , denote by  $\alpha_i$  and  $\omega_i$  the repulsive and attracting fixed points of  $c_i$  in  $\partial\mathbb{H}$ , and consider the word

$$g_0 c_1^N g_1 c_2^N g_2.$$

For  $m$  large enough,  $c_2^m g_2$  maps  $x$  to a point which is near  $\omega_2$ . If  $g_1(\omega_2)$  were equal to  $\alpha_1$ , then  $c_1$  and  $g_1 c_2 g_1^{-1}$  would share the common fixed point  $\alpha_1$ , and they would commute. Thus,  $g_1(\omega_2) \neq \alpha_1$  and then  $g_0 c_1^{m'} g_1 c_2^m g_2$  maps  $x$  to a point which is near  $g_0(\omega_1)$  if  $m'$  is large enough. Thus,  $g_0 c_1^N g_1 c_2^N g_2(x) \neq x$  for large  $N$ . The proof is similar if  $n$  is larger than 2.  $\square$

*Sketch of proof (II).* We rephrase this proof, using the action of  $\mathbf{F}_k$  on its boundary, because this boundary will also be used in the proof of Proposition 3.3.

Fix a basis  $a_1, \dots, a_k$  of  $\mathbf{F}_k$ , and denote by  $\partial\mathbf{F}_k$  the boundary of  $\mathbf{F}_k$ . The elements of  $\partial\mathbf{F}_k$  are represented by infinite reduced words in the generators  $a_i$  and their inverses. If  $g$  is an element of  $\mathbf{F}_k$  and  $\alpha$  is an element of  $\partial\mathbf{F}_k$  the concatenation  $g \cdot \alpha$  is an element of  $\partial\mathbf{F}_k$ : this defines an action of  $\mathbf{F}_k$  by homeomorphisms on the Cantor set  $\partial\mathbf{F}_k$ . If  $g$  is a non-trivial, its action on  $\partial\mathbf{F}_k$  has exactly two fixed points, given by the infinite words  $\omega(g) = g \dots g \dots$  and  $\alpha(g) = g^{-1} \dots g^{-1} \dots$  (there are no simplifications if  $g$  is given by a reduced and cyclically reduced word). Then we get: (1) every element  $g \neq \mathrm{Id}$  in  $\mathbf{F}_k$  has a north-south dynamics on  $\partial\mathbf{F}_k$ , every orbit  $g^n \cdot \beta$  converging to  $\omega(g)$ , except when  $\beta = \alpha(g)$ , and (2) two elements  $g$  and  $h$  in  $\mathbf{F}_k \setminus \{\mathrm{Id}\}$  commute if and only if they have the same fixed points, which happens if and only if they share



$$\begin{aligned}
 a_0 &= \bar{a}_0 = \alpha_0 \\
 a_1 &= \eta_1 \alpha_1 \eta_1^{-1} \\
 \bar{a}_1 &= \bar{\eta}_1 \alpha_1 \bar{\eta}_1^{-1} \\
 a_2 &= \eta_2 \alpha_2 \eta_2^{-1} \\
 \bar{a}_2 &= \bar{\eta}_2 \alpha_2 \bar{\eta}_2^{-1} \\
 t_1 &= \eta_1 \bar{\eta}_1^{-1} \\
 t_2 &= \eta_2 \bar{\eta}_2^{-1}
 \end{aligned}$$

FIGURE 1

The fundamental group  $\Gamma_2$ . – The  $\alpha_i$  are three loops, while the  $\eta_j$  and  $\bar{\eta}_j$  are four paths. The figure is symmetric with respect to the plane cutting the surface along the loops  $\alpha_i$ .

a common fixed point on  $\partial \mathbf{F}_k$ . One can then repeat the previous proof with the action of  $\mathbf{F}_k$  on its boundary.  $\square$

Write the surface of genus 2 as the union of two pairs of pants as in Figure 1, with respective fundamental groups

$$(3.1) \quad \langle a_0, a_1, a_2 \mid a_0 a_1 a_2 = 1 \rangle \text{ and } \langle \bar{a}_0, \bar{a}_1, \bar{a}_2 \mid \bar{a}_0 \bar{a}_1 \bar{a}_2 = 1 \rangle.$$

This gives the presentation

$$(3.2) \quad \Gamma_2 = \left\langle \begin{array}{c} a_0, a_1, a_2, \\ \bar{a}_0, \bar{a}_1, \bar{a}_2, \\ t_1, t_2 \end{array} \mid \begin{array}{c} a_0 a_1 a_2 = 1, \\ \bar{a}_0 \bar{a}_1 \bar{a}_2 = 1, \\ \bar{a}_0 = a_0, \bar{a}_1 = t_1^{-1} a_1 t_1, \bar{a}_2 = t_2^{-1} a_2 t_2 \end{array} \right\rangle$$

which can be rewritten as

$$(3.3) \quad \Gamma_2 = \langle a_0, a_1, a_2, t_1, t_2 \mid a_0 a_1 a_2 = 1, a_0 t_1^{-1} a_1 t_1 t_2^{-1} a_2 t_2 = 1 \rangle.$$

Denote by  $p : \Gamma_2 \rightarrow \langle a_0, a_1, a_2 \rangle \cong \mathbf{F}_2$  the morphism defined by  $p(a_i) = a_i$ ,  $p(\bar{a}_i) = a_i$ , and  $p(t_1) = p(t_2) = 1$ . Let  $\tau : \Gamma_2 \rightarrow \Gamma_2$  be the (left) Dehn twist along the three curves  $a_0$ ,  $a_1$ , and  $a_2$ , i.e. the automorphism fixing  $a_i$  and sending  $t_i$  to  $a_i t_i a_0^{-1}$  for  $i = 1, 2$ . Note the following facts:

- $\tau$  sends  $\bar{a}_i$  to  $a_0 \bar{a}_i a_0^{-1}$ ; in particular, if  $g$  is a word in the  $\bar{a}_i$ , then  $\tau^N(g) = a_0^N g a_0^{-N}$ ;
- $p \circ \tau^N$  fixes  $a_i$  for every  $i = 0, 1, 2$ , and

$$(3.4) \quad p \circ \tau^N(t_j) = a_j^N a_0^{-N}$$

for  $j = 1, 2$ .

**Proposition 3.3.** *For every  $g \in \Gamma_2 \setminus \{1\}$ , there exists a positive integer  $n_0$  such that  $p \circ \tau^N(g) \neq 1$  for all  $N \geq n_0$ .*

*Proof.* To uniformize notations, we define  $t_0 = 1$  so that for all  $i \in \{0, 1, 2\}$  the relation  $t_i \bar{a}_i t_i^{-1} = a_i$  holds, and  $\tau$  maps  $t_i$  to  $a_i t_i a_0^{-1}$ . Let  $A = \langle a_0, a_1, a_2 \rangle$  and  $\bar{A} = \langle \bar{a}_0, \bar{a}_1, \bar{a}_2 \rangle$ . Write  $g$  as a shortest possible word of the following form:

$$(3.5) \quad g = g_0 t_{i_1} g_1 t_{i_2}^{-1} g_2 t_{i_3} \dots g_{n-1} t_{i_n}^{-1} g_n$$

where  $n$  is even,  $i_k \in \{0, 1, 2\}$  for all  $k \leq n$ ,  $g_k \in A$  for  $k$  even,  $g_k \in \bar{A}$  for  $k$  odd, and the exponent of  $t_{i_k}$  is  $(-1)^{k+1}$  (we allow  $g_k = 1$ ). One easily checks that  $g$  can be written in this form because all generators can (for instance  $t_1 = 1 \cdot t_1 \cdot 1 \cdot t_0^{-1} \cdot 1$ ).

If  $k$  is such that  $i_k = i_{k+1}$ , then  $g_k \notin \langle a_{i_k} \rangle$  if  $k$  is even (resp  $g_k \notin \langle \bar{a}_{i_k} \rangle$  if  $k$  is odd) as otherwise, one could shorten the word using the relation  $t_{i_k} a_{i_k} t_{i_k}^{-1} = \bar{a}_{i_k}$ .

**First claim.** *If  $k \in \{2, \dots, n-2\}$  is even,  $g_k^{-1} a_{i_k} g_k$  does not commute to  $a_{i_{k+1}}$ .*

If  $i_k \neq i_{k+1}$ , this is because  $g_k \in A \simeq \mathbf{F}_2$  and no pair of  $A$ -conjugates of  $a_{i_k}$  and  $a_{i_{k+1}}$  commute. If  $i_k = i_{k+1}$ , then  $g_k \notin \langle a_{i_k} \rangle$  as we have just seen; since  $a_{i_k}$  is not a proper power in  $A$ , this shows that  $g_k \cdot (a_{i_k}^{+\infty}) \neq a_{i_k}^{+\infty}$  in the boundary at infinity of the free group  $A$ , so  $g_k^{-1} a_{i_k} g_k$  does not commute with  $a_{i_k}$ , and the claim follows.

Similarly, using the fact that  $g_k \in \bar{A}$  for odd indices, we obtain:

**Second claim.** *If  $k \leq n-1$  is odd,  $g_k^{-1} \bar{a}_{i_k} g_k$  does not commute to  $\bar{a}_{i_{k+1}}$ .*

We have  $\tau^N(g_k) = g_k$  if  $k$  is even, and  $\tau^N(g_k) = a_0^N g_k a_0^{-N}$  if  $k$  is odd. After simplifications, one has

$$(3.6) \quad \tau^N(g) = g_0 a_{i_1}^N t_{i_1} g_1 t_{i_2}^{-1} a_{i_2}^{-N} g_2 a_{i_3}^N t_{i_3} \dots g_{n-1} t_{i_n}^{-1} a_{i_n}^{-N} g_n.$$

For  $k$  odd, denote by  $g'_k \in \mathbf{F}_r$  the image of  $g_k$  under  $p$ . Applying  $p$ , we thus get

$$(3.7) \quad p \circ \tau^N(g) = g_0 a_{i_1}^N g'_1 a_{i_2}^{-N} g_2 a_{i_3}^N g'_3 \dots g_{n-1} a_{i_n}^{-N} g_n,$$

with  $g'_i := p(g_i)$ . Let us check that the hypotheses of the Baumslag Lemma 3.2 apply. For  $k$  even, the first claim shows that  $g_k^{-1} a_{i_k} g_k$  does not commute to  $a_{i_{k+1}}$ , as required. For  $k$  odd, we use that  $p$  is injective on  $\bar{A}$  and that  $\bar{A}$  contains  $g_k^{-1} \bar{a}_{i_k} g_k$  and  $\bar{a}_{i_{k+1}}$ , and we apply the second claim to deduce that  $g_k'^{-1} a_{i_k} g'_k$  does not commute to  $a_{i_{k+1}}$ . Applying Baumslag's Lemma, we conclude that  $p \circ \tau^N(g) \neq 1$  for  $N$  large enough.  $\square$

**3.3. Embeddings of free groups.** The group  $\text{Diff}(\mathbf{R}, 0)$  contains non-abelian free groups. This has been proved by arithmetic means [Whi, Gla], by looking at the monodromy of generic polynomial planar vector fields [IP], and by a dynamical argument [MRR]. We shall need the following precise version of that result.

**Theorem 3.4.** *Let  $(\mathbf{k}, |\cdot|)$  be a complete non-discrete valued field. For every pair  $(\lambda_1, \lambda_2)$  in  $\mathbf{k}^*$ , there exists a pair  $f_1, f_2 \in \text{Diff}(\mathbf{k}, 0)$  that generates a free group and satisfies  $f'_1(0) = \lambda_1$  and  $f'_2(0) = \lambda_2$ .*

This result is proved in [BCLN, Proposition 4.3] for generic pairs of derivatives  $(\lambda_1, \lambda_2)$ . We provide a proof of Theorem 3.4 in the Appendix, extending the argument of [MRR]. We refer to Section 7.1 below for other approaches.

**3.4. Embedding orientable surface groups.** We can now prove Theorem A for orientable surfaces:

**Theorem 3.5.** *Let  $\Gamma_g$  be the fundamental group of a closed, orientable surface of genus  $g$ . Then, there exists an injective morphism  $\Gamma_g \rightarrow \text{Diff}(\mathbf{R}, 0)$ .*

The group  $\Gamma_0$  is trivial. The group  $\Gamma_1$  is isomorphic to  $\mathbf{Z}^2$ , hence it embeds in the group of homotheties  $z \mapsto \lambda z$ ,  $\lambda \in \mathbf{R}_+^*$ . If  $g \geq 2$ , then  $\Gamma_g$  embeds in  $\Gamma_2$ . To see this, fix a surjective morphism  $\Gamma_2 \rightarrow \mathbf{Z}$ , and take the preimage  $\Lambda \subset \Gamma_2$  of the subgroup  $(g-1)\mathbf{Z} \subset \mathbf{Z}$ . Then,  $\Lambda$  is a normal subgroup of index  $g-1$  in  $\Gamma_2$ , and it is the fundamental group of a closed surface  $\Sigma$ , given by a Galois cover of degree  $g-1$  of the surface of genus 2. Since the Euler characteristic is multiplicative, the genus of  $\Sigma$  satisfies  $-2(g-1) = 2 - 2g(\Sigma)$ . Thus,  $g(\Sigma) = g$  and  $\Lambda$  is isomorphic to  $\Gamma_g$ . Thus, we now restrict to the case  $g = 2$ .

By Theorem 3.4, we can fix an injective morphism

$$(3.8) \quad \rho_0 : \mathbf{F}_2 = \langle a_0, a_1, a_2 \mid a_0 a_1 a_2 = 1 \rangle \rightarrow \text{Diff}(\mathbf{R}, 0)$$

such that the images  $f_1 = \rho_0(a_1)$ ,  $f_2 = \rho_0(a_2)$ , and  $f_0 = \rho_0(a_0) = f_2^{-1} f_1^{-1}$  satisfy

$$(3.9) \quad f'_1(0) = \lambda_1 > 1, \quad f'_2(0) = \lambda_2 > 1, \quad f'_0(0) = \lambda_0 < 1$$

for some real numbers  $\lambda_1$  and  $\lambda_2 > 1$  and  $\lambda_0 = (\lambda_1 \lambda_2)^{-1}$ . In particular,  $f_0$ ,  $f_1$ , and  $f_2$  are hyperbolic. For  $\lambda \in \mathbf{R}^*$ , denote by  $m_\lambda(z) = \lambda z$  the corresponding homothety. For  $i \in \{0, 1, 2\}$ , the Koenigs linearization theorem shows that  $f_i$  is conjugate to the homothety  $m_{\lambda_i}$ : there is a germ of diffeomorphism  $h_i \in \text{Diff}(\mathbf{R}, 0)$  such that  $f_i = h_i \circ m_{\lambda_i} \circ h_i^{-1}$ . Thus  $f_i$  extends to the multiplicative flow  $\varphi_i : \mathbf{R}_+^* \rightarrow \text{Diff}(\mathbf{R}, 0)$  defined by  $\varphi_i^s = h_i \circ m_s \circ h_i^{-1}$  for  $s \in \mathbf{R}_+^*$ ; by construction,

$\varphi_i^{\lambda_i} = f_i$  and  $\varphi_i^s$  commutes with  $f_i$  for all  $s > 0$ . We note that  $s \mapsto \varphi_i^s$  is polynomial in the sense that for all  $k \in \mathbb{N}$ ,  $s \mapsto A_k(\varphi_i^s)$  is a polynomial function with real coefficients in the variables  $s$  and  $s^{-1}$ .

Set  $\mathcal{R} = (\mathbb{R}_+^*)^3$ . As in Section 3.2, consider the presentation

$$(3.10) \quad \Gamma_2 = \langle a_0, a_1, a_2, t_1, t_2 \mid a_0 a_1 a_2 = 1, a_0 t_1^{-1} a_1 t_1 t_2^{-1} a_2 t_2 = 1 \rangle.$$

Given  $s = (s_0, s_1, s_2) \in (\mathbb{R}_+^*)^3$ , we define a morphism  $\Phi_s : \Gamma_2 \rightarrow \text{Diff}(\mathbb{R}, 0)$  by

$$(3.11) \quad \Phi_s(a_i) = f_i \quad \text{for } i \in \{0, 1, 2\}$$

$$(3.12) \quad \Phi_s(t_i) = \varphi_i^{s_i} \varphi_0^{s_0} \quad \text{for } i \in \{1, 2\}$$

This provides a well defined homomorphism because  $\varphi_i$  commutes with  $f_i$ . As we shall see below, this morphism  $\Phi_s$  is constructed to coincide with  $\rho_0 \circ p \circ \tau^N$  for  $s = (\lambda_0^N, \lambda_1^N, \lambda_2^N)$  (see Equation (3.4)).

**Remark 3.6.** For every  $s \in \mathcal{R}$ , the image of  $\Phi_s$  contains  $f_1$  and  $f_2$ , hence the free group  $\rho_0(\mathbf{F}_2)$ . This will be used in Section 4.3.

To conclude, we check that the three assumptions of Lemma 3.1 hold for this family of morphisms  $(\Phi_s)_{s \in \mathcal{R}}$ .

Clearly,  $\mathcal{R}$  is a Baire space.

To check the irreducibility property, consider  $g \in \Gamma_2$  and assume that  $\mathcal{R}_g \neq \mathcal{R}$ : this means that there exists a parameter  $s \in \mathcal{R}$  and an index  $k \geq 1$  such that  $A_k(\Phi_s(g)) \neq A_k(\text{Id})$ . The map  $s = (s_0, s_1, s_2) \mapsto A_k(\Phi_s(g)) - A_k(\text{Id})$  is a polynomial function in the variables  $s_0^{\pm 1}$ ,  $s_1^{\pm 1}$ , and  $s_2^{\pm 1}$  that does not vanish identically on  $\mathcal{R}$ , so its zero set is a closed subset with empty interior.

We now check that  $\mathcal{R}$  has the separation property. As in Section 3.2, denote by  $p : \Gamma_2 \rightarrow \mathbf{F}_2 = \langle a_0, a_1, a_2 \mid a_0 a_1 a_2 = 1 \rangle$  the morphism obtained by killing  $t_1$  and  $t_2$ . For the parameter  $s = (1, 1, 1)$ ,  $\Phi_s$  is equal to  $\rho_0 \circ p$ . More generally, setting  $s_N = (\lambda_0^N, \lambda_1^N, \lambda_2^N)$  for  $N \in \mathbb{N}$ , the morphism  $\Phi_{s_N} : \Gamma_2 \rightarrow \text{Diff}(\mathbb{R}, 0)$  satisfies

$$(3.13) \quad \Phi_{s_N}(a_i) = f_i \quad \text{for } i \in \{0, 1, 2\}$$

$$(3.14) \quad \Phi_{s_N}(t_i) = \varphi_i^{u_i^N} \varphi_0^{u_0^N} = f_i^N f_0^N \quad \text{for } i \in \{1, 2\}.$$

This means that  $\Phi_{s_N} = \rho_0 \circ p \circ \tau^N$  where, as in Section 3.2,  $\tau : \Gamma_2 \rightarrow \Gamma_2$  is the Dehn twist along the three curves  $a_i$ . By Proposition 3.3, for all  $g \in \Gamma_2 \setminus \{1\}$  there exists  $N \in \mathbb{N}$  such that  $p \circ \tau^N(g) \neq 1$ . Since  $\rho_0$  is injective, this implies that  $\Phi_{s_N}(g) \neq 1$  which shows that  $\mathcal{R}$  has the separation property.

#### 4. Non-orientable surface groups

**Theorem 4.1.** Let  $N_g$  be the fundamental group of a closed non-orientable surface of genus  $g \geq 4$ . There exists an injective morphism  $N_g \rightarrow \text{Diff}(\mathbf{R}, 0)$ .

**Remark 4.2.** The fundamental group  $N_3$  of the non-orientable surface of genus 3 is not fully residually free, and our methods do not apply to this group. (See [Lyn, Proposition 9].)

**4.1. Even genus.** We first treat the case of an even genus  $g \geq 4$ . In this case, the group  $N_g$  embeds in  $N_4$ . Indeed, the non-orientable surface of genus 4 is the connected sum of a torus  $\mathbf{R}^2/\mathbf{Z}^2$  with two projective planes  $\mathbb{P}^2(\mathbf{R})$ . Taking a cyclic cover of the torus of degree  $k$ , we get a surface homeomorphic to the connected sum of  $\mathbf{R}^2/\mathbf{Z}^2$  with  $2k$  copies of  $\mathbb{P}^2(\mathbf{R})$ , hence a non-orientable surface of genus  $2(k+1)$ . Thus, it suffices to prove that  $N_4$  embeds in  $\text{Diff}(\mathbf{R}, 0)$ .

The non-orientable surface of genus 4 is homeomorphic to the connected sum of 4 copies of  $\mathbb{P}^2(\mathbf{R})$ , and this gives the presentation (see Figure 2)

$$(4.1) \quad N_4 = \langle a_1, a_2, b_1, b_2 \mid a_1^2 a_2^2 b_2^2 b_1^2 = 1 \rangle.$$

Let  $p : N_4 \rightarrow \langle a_1, a_2 \rangle$  be the morphism fixing  $a_1, a_2$  and sending  $b_1$  and  $b_2$  to  $a_1^{-1}$  and  $a_2^{-1}$  respectively. Let  $\tau : N_4 \rightarrow N_4$  be the Dehn twist around the curve  $\gamma = (a_1^2 a_2^2)^{-1}$ , i.e., the automorphism that fixes  $a_1$  and  $a_2$  and sends  $b_1$  and  $b_2$  to  $\gamma b_1 \gamma^{-1}$  and  $\gamma b_2 \gamma^{-1}$  respectively.

**Lemma 4.3.** Given any  $g \in N_4 \setminus \{1\}$ , there exists  $n_0 \in \mathbf{N}$  such that for all  $N \geq n_0$ ,  $p \circ \tau^N(g) \neq 1$ .

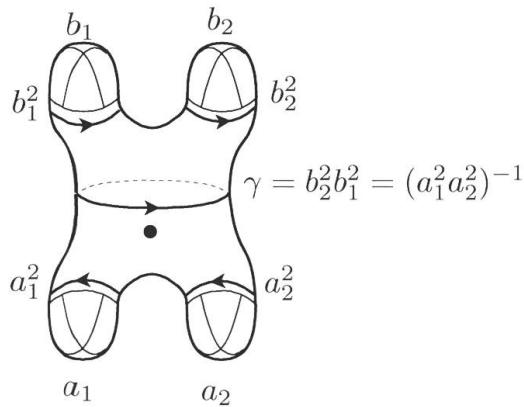


FIGURE 2

The fundamental group  $N_4$ . – The base point is represented by  $\bullet$ , the 4 generators are  $a_1, a_2, b_1, b_2$ , and the curve  $\gamma$  is used to construct the Dehn twist  $\tau$ .

*For the proof.* The proof of this statement is completely analogous to the proof of [BGSS, Corollary 2.2], using Baumslag Lemma, we leave it as an exercise to the reader. See also [CG, Proposition 4.13].  $\square$

Using Theorem 3.4, we fix two germs of diffeomorphisms  $f_1$  and  $f_2 \in \text{Diff}(\mathbf{R}, 0)$  generating a free group and satisfying  $f'_1(0) > 1$  and  $f'_2(0) > 1$ . We denote by

$$(4.2) \quad \rho_0 : \mathbf{F}_2 = \langle a_1, a_2 \rangle \rightarrow \text{Diff}(\mathbf{R}, 0)$$

the injective morphism sending  $a_i$  to  $f_i$  for  $i \in \{1, 2\}$ . In particular,

$$(4.3) \quad \rho_0(\gamma) = (f_1^2 \circ f_2^2)^{-1}$$

is a hyperbolic germ: its derivative  $\lambda = ((f_1^2 \circ f_2^2)'(0))^{-1}$  is  $< 1$ . The Koenigs linearization theorem gives an element  $h \in \text{Diff}(\mathbf{R}, 0)$  such that  $\rho_0(\gamma) = h \circ m_\lambda \circ h^{-1}$ . Consider the multiplicative flow  $\varphi : \mathbf{R}_+^* \rightarrow \text{Diff}(\mathbf{R}, 0)$  defined by  $\varphi^s = g \circ m_s \circ g^{-1}$ . As above,  $\varphi^\lambda = \rho_0(\gamma)$ ,  $\varphi^s$  commutes with  $\rho_0(\gamma)$  for all  $s > 0$ , and  $s \mapsto \varphi^s$  is a polynomial map: for all  $k \in \mathbf{N}$ ,  $s \mapsto A_k(\varphi^s)$  is a polynomial in the variables  $s$  and  $s^{-1}$ .

Set  $\mathcal{R} = \mathbf{R}_+^*$ . Given  $s \in \mathbf{R}_+^*$ , consider the morphism  $\rho_s : N_4 \rightarrow \text{Diff}(\mathbf{R}, 0)$  defined by

$$\begin{array}{ll} a_1 \mapsto f_1 & a_2 \mapsto f_2 \\ b_1 \mapsto \varphi^s f_1^{-1} \varphi^{-s} & b_2 \mapsto \varphi^s f_2^{-1} \varphi^{-s}. \end{array}$$

This gives a well defined homomorphism because  $\varphi^s$  commutes with  $f_1^2 f_2^2$ .

We now check the three assumptions of Lemma 3.1. Clearly,  $\mathcal{R}$  is a Baire space. The irreducibility is a consequence of the fact that for any  $g \in N_4$ , and any  $k \in \mathbf{N}$  the map  $s \mapsto A_k(\varphi_s(g))$  is a polynomial function in the variables  $s^{\pm 1}$ . The separation property follows from Lemma 4.3 together with the fact that  $\rho_{\lambda^N} = \rho_0 \circ p \circ \tau^N$  and that  $\rho_0$  is injective.

**4.2. Odd genus.** We now treat the case of a non-orientable surface of odd genus  $g = 2k + 1$ ,  $k \geq 2$ . One can write  $N_{2k+1}$  as (see Figure 3 below)

$$(4.4) \quad N_{2k+1} = \langle a_1, \dots, a_k, c, b_1, \dots, b_k \mid a_1^2 \dots a_k^2 c^2 b_k^2 \dots b_1^2 = 1 \rangle.$$

This group splits as a double amalgam of free groups

$$(4.5) \quad N_{2k+1} = \langle a_1, \dots, a_k \rangle \underset{a_1^2 \dots a_k^2 = \gamma^{-1}}{*} \langle \gamma, c \rangle \underset{c^{-2} \gamma = b_k^2 \dots b_1^2}{*} \langle b_1, \dots, b_k \rangle.$$

We shall use the following notation to refer to this amalgam structure:

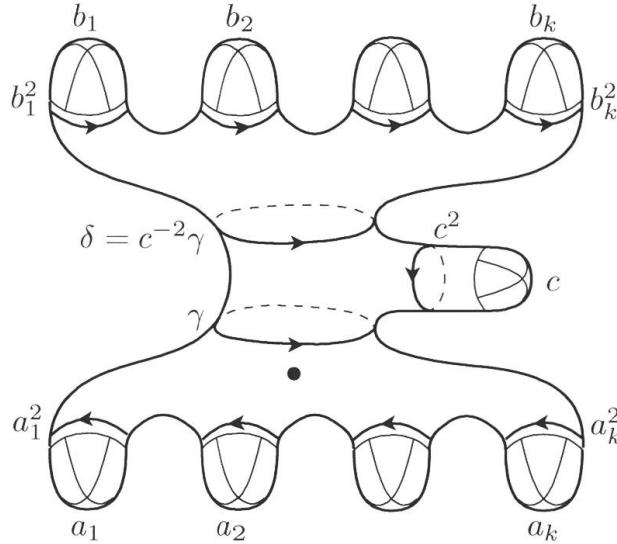


FIGURE 3  
The fundamental group  $N_{2k+1}$

- $A_1 = \langle a_1, \dots, a_k \rangle$  and  $e_{1,2} = (a_1^2 \dots a_k^2)^{-1}$ ;
- $A_2 = \langle \gamma, c \rangle$  and  $e_{2,1} = \gamma$  and  $e_{2,3} = c^{-2}\gamma = \delta$ ;
- $A_3 = \langle b_1, \dots, b_k \rangle$  and  $e_{3,2} = b_k^2 \dots b_1^2$ .

So, each of the  $A_i$  is a free group and the amalgamation is given by  $e_{1,2} = e_{2,1}$  and  $e_{2,3} = e_{3,2}$ .

Define a morphism  $p: N_{2k+1} \rightarrow \langle a_1, \dots, a_k \rangle \cong \mathbf{F}_k$  by

$$\begin{array}{ll} a_i \mapsto a_i & \text{for } i \leq k \\ b_i \mapsto a_i^{-1} & \text{for } i \leq k-1 \end{array} \quad \begin{array}{ll} c \mapsto a_k^{-2} \\ b_k \mapsto a_k \end{array}$$

(the structure of amalgam shows that  $p$  is well defined).

**Lemma 4.4.** *The morphism  $p: N_{2k+1} \rightarrow \mathbf{F}_k$  is injective in restriction to each of the three subgroups of the amalgam (4.5).*

*Proof.* By construction, it is injective in restriction to  $\langle a_1, \dots, a_k \rangle$  and in restriction to  $\langle b_1, \dots, b_k \rangle$ . Then, note that  $p(\langle \gamma, c \rangle) = \langle a_1^2 \dots a_k^2, a_k^{-2} \rangle$  is isomorphic to  $\mathbf{F}_2$  because it is a non-abelian subgroup of a free group. Since  $\mathbf{F}_2$  is Hopfian,  $p$  is necessarily injective in restriction to  $\langle \gamma, c \rangle$ .  $\square$

Consider  $\delta = b_k^2 \dots b_1^2 = c^{-2}\gamma$  and note that  $p(\delta) = a_k^2 a_{k-1}^{-2} \dots a_1^{-2}$ . Let  $\tau$  be the Dehn twist corresponding to the decomposition above, i.e. the automorphism fixing  $a_i$ , sending  $c$  to  $\gamma c \gamma^{-1}$  and sending  $b_i$  to  $(\gamma \delta) b_i (\gamma \delta)^{-1}$ . Since  $\tau$  is the composition of the twists given by  $\gamma$  and  $\delta$  and these two twists commute we get

$$\tau^N(b) = (\gamma^N \delta^N) b (\gamma^N \delta^N)^{-1}, \quad \forall b \in A_3.$$

In this situation, one can prove the following lemma in a similar way to Proposition 3.3.

**Lemma 4.5.** *Given any  $g \in N_{2g+1} \setminus \{1\}$ , there exists  $n_0 \in \mathbf{N}$  such that for all  $N \geq n_0$ ,  $p \circ \tau^N(g) \neq 1$ .*

*Proof.* Write  $g$  as a word in the graph of groups, i.e.,  $g = s_0 \dots s_n$  with  $s_k \in A_{r_k}$  (we allow  $s_k = 1$ ) for some  $r_k \in \{1, 2, 3\}$ , with  $r_{k+1} = r_k \pm 1$ , and  $r_0 = r_n = 1$ . We take this word of minimal possible length among words satisfying these constraints. If  $k$  is such that  $r_{k-1} = r_{k+1}$ , then  $s_k \notin \langle e_{r_k, r_{k+1}} \rangle$  since otherwise, one could shorten the word using the structure of amalgam (in particular  $s_k \neq 1$  in this case). Now one easily checks that

$$(4.6) \quad \tau^N(g) = s_0 d_1^{\varepsilon_1 N} s_1 d_2^{\varepsilon_2 N} s_2 \dots d_n^{\varepsilon_n N} s_n$$

where  $d_k = e_{r_{k-1}, r_k} \in \{\gamma, \delta\}$ , and  $\varepsilon_k = r_k - r_{k-1} \in \{\pm 1\}$ .

We claim that  $s_k^{-1} d_k s_k$  does not commute with  $d_{k+1}$ . If  $d_k \neq d_{k+1}$ , this follows from the fact that  $\gamma$  commutes with no conjugate of  $\delta$  in  $A_2 = \langle c, \delta \rangle$ . If  $d_k = d_{k+1}$ , then  $r_{k-1} = r_{k+1}$ , so  $s_k \notin \langle e_{r_k, r_{k+1}} \rangle = \langle d_k \rangle$ . If  $[s_k^{-1} d_k s_k, d_k] = 1$ , then  $s_k$  preserves the axis of  $d_k$  in the Cayley graph of the free group  $A_{r_k}$ , so  $s_k$  is a power of  $d_k$ , because  $d_k \in \{\gamma, \delta\}$  is not a proper power; this contradicts that  $s_k \notin \langle d_k \rangle$ .

Denote by  $\bar{s}_k, \bar{d}_k \in \mathbf{F}_r$  the images of  $s_k, d_k$  under  $p$ . Since  $p$  is injective on each  $A_{r_k}$ ,  $\bar{s}_k^{-1} \bar{d}_k \bar{s}_k$  does not commute with  $\bar{s}_{k+1}$ , so the hypotheses of Baumslag Lemma apply to the word

$$(4.7) \quad p \circ \tau^N(g) = \bar{d}_0^{\varepsilon_0 N} \bar{s}_1 \bar{d}_1^{\varepsilon_1 N} \bar{s}_2 \dots \bar{d}_{n-1}^{\varepsilon_{n-1} N} \bar{s}_n \bar{d}_n^{\varepsilon_n N}$$

so  $p \circ \tau^N(g) \neq 1$  for  $N$  large enough.  $\square$

Now consider  $k$  elements  $f_1, \dots, f_k$  of  $\text{Diff}(\mathbf{R}, 0)$  generating a free group of rank  $k$  with  $f'_i(0) > 1$  for all  $i \in \{1, \dots, k\}$ , and  $f'_k(0) < f'_1(0)$ . Such a set can be obtained from two generators  $g_1$  and  $g_2$  of a free group of rank 2 with  $g'_i(0) > 1$ , as in Theorem 3.4, by taking  $f_i = g_1^i \circ g_2^2 \circ g_1^{-i}$  for  $i < k$  and  $f_k = g_1^k \circ g_2 \circ g_1^{-k}$ . Let  $\rho_0 : \mathbf{F}_k = \langle a_1, \dots, a_k \rangle \rightarrow \text{Diff}(\mathbf{R}, 0)$  be the injective morphism sending  $a_i$  to  $f_i$  for  $i \leq k$ . In particular,  $\rho_0(\gamma) = (f_1^2 \circ \dots \circ f_k^2)^{-1}$  and  $\rho_0(p(\delta)) = f_k^2 \circ f_{k-1}^{-2} \circ \dots \circ f_1^{-2}$  are hyperbolic. Using Koenigs linearization theorem as above, there exists two multiplicative flows  $\varphi$  and  $\psi : \mathbf{R}_+^* \rightarrow \text{Diff}(\mathbf{R}, 0)$  and a pair of positive real numbers  $\lambda$  and  $\mu$  such that (1)  $\varphi^\lambda = \rho_0(\gamma)$  and  $\psi^\mu = \rho_0(p(\delta))$ , and (2)  $s \mapsto \varphi^s$  and  $s \mapsto \psi^s$  are polynomial mappings.

Set  $\mathcal{R} = (\mathbf{R}_+^*)^2$  and, for every  $(s, s') \in \mathcal{R}$ , define a morphism  $\rho_{s, s'} : \mathbf{N}_{2k+1} \rightarrow \text{Diff}(\mathbf{R}, 0)$  by

$$\begin{aligned} a_i &\mapsto f_i \quad \text{for } i \leq k & c &\mapsto \varphi^s f_k^{-2} \varphi^{-s} \\ b_i &\mapsto \varphi^s \psi^{s'} f_i^{-1} (\varphi^s \psi^{s'})^{-1} \quad \text{for } i \leq k-1 & b_k &\mapsto \varphi^s \psi^{s'} f_k (\varphi^s \psi^{s'})^{-1} \end{aligned}$$

(this is well defined because  $\varphi^s$  and  $\psi^{s'}$  commute with  $\rho_0(\gamma) = (f_1^2 \circ \dots \circ f_k^2)^{-1}$  and  $\rho_0(p(\delta)) = f_k^2 \circ f_{k-1}^{-2} \circ \dots \circ f_1^{-2}$  respectively).

The assumptions of Lemma 3.1 hold:  $\mathcal{R}$  is a Baire space, and the irreducibility follows from the fact that the maps  $s \mapsto \varphi_s$  and  $s' \mapsto \varphi_{s'}$  are polynomials in the variables  $s^{\pm 1}, s'^{\pm 1}$ . The separation property follows from Lemma 4.5 together with the fact that  $\rho_{\lambda^N, \mu^N} = \rho_0 \circ p \circ \tau^N$ , and that  $\rho_0$  is injective.

**4.3. Embeddings in  $\text{Diff}(\mathbf{k}, 0)$ .** The proofs just given provide the following statement.

**Theorem B.** *Let  $(\mathbf{k}, |\cdot|)$  be a non-discrete and complete valued field.*

- (1) *Let  $\Gamma$  be the fundamental group of a closed orientable surface, or a closed non-orientable surface of genus  $\geq 4$ . Then, there is an embedding of  $\Gamma$  into  $\text{Diff}(\mathbf{k}, 0)$ .*
- (2) *Let  $F \subset \text{Diff}(\mathbf{k}, 0)$  be a free group of rank 2, generated by two germs  $f$  and  $g$  with  $|f'(0)| > 1$  and  $|g'(0)| > 1$ . Then, there is an embedding of  $\Gamma_2$ , the fundamental group of a closed, orientable surface of genus 2, into  $\text{Diff}(\mathbf{k}, 0)$  whose image contains  $F$ .*

*Proof.* For the first assertion, we just have to replace  $\mathbf{R}$  by  $\mathbf{k}$  in the proofs of Theorem 3.5 and 4.1. The parameter space is  $\mathcal{R} = (\mathbf{k}^*)^3$  or  $\mathbf{k}^*$  or  $(\mathbf{k}^*)^2$ , and it is a Baire space because  $(\mathbf{k}, |\cdot|)$  is complete.

For the second assertion, we start with a representation  $\rho_0$  in Equation (3.8) whose image is equal to  $F$ . Remark 3.6 shows that all the injective morphisms  $\Phi_s$  that we get satisfy also  $\Phi_s(\Gamma_2) \supset F$ .  $\square$

## Part II

### 5. The final topology on germs of diffeomorphisms

Let  $(\mathbf{k}, |\cdot|)$  be a complete field. This section introduces a new topology on  $\mathbf{k}\{z\}$  and  $\text{Diff}(\mathbf{k}, 0)$ , which will be used in our second proof of Theorem A. The reader may very well skip this section on a first reading.

**5.1. The final topology over the complex numbers.** Until Section 5.4, we focus on the case  $\mathbf{k} = \mathbf{C}$ . Let  $r$  be a positive real number. Consider the subalgebra  $\mathcal{A}_r$  of  $\mathbf{C}\{z\}$  consisting of those power series  $f(z) = \sum_n a_n z^n$  which converge on the open unit disk  $\mathbb{D}_r$  (i.e.,  $\text{rad}(f) \geq r$ ) and extend continuously to the closed unit disk  $\overline{\mathbb{D}}_r$ . When endowed with the norm

$$(5.1) \quad \|f\|_{\mathcal{A}_r} = \max_{z \in \overline{\mathbb{D}}_r} |f(z)|,$$

$\mathcal{A}_r$  is a Banach algebra. If  $s < r$ , the restriction of functions  $f \in \mathcal{A}_r$  to the smaller disk  $\overline{\mathbb{D}}_s$  determines a 1-Lipschitz embedding  $\mathcal{A}_r \rightarrow \mathcal{A}_s$ .

The space  $\mathbf{C}\{z\}$  is the union of the algebras  $\mathcal{A}_r$  and can be thus endowed with the final topology associated to the colimit

$$(5.2) \quad \mathbf{C}\{z\} = \varinjlim \mathcal{A}_r.$$

This means that a subset  $\mathcal{U} \subset \mathbf{C}\{z\}$  is open if its intersection with  $\mathcal{A}_r$  is open for every  $r > 0$ . Equivalently, a map  $\varphi: \mathbf{C}\{z\} \rightarrow X$  to a topological space is continuous if and only if its composition with the embedding  $\mathcal{A}_r \rightarrow \mathbf{C}\{z\}$  is continuous for all  $r$ . Unless we say it explicitly, open sets, neighborhoods, and continuous maps refer, from now on, to this topology. A word of warning: for  $r > s$ , the inclusion  $\mathcal{A}_r \rightarrow \mathcal{A}_s$  is not a homeomorphism to its image, and neither is the inclusion  $\mathcal{A}_r \rightarrow \mathbf{C}\{z\}$ .

The goal of this section is to obtain several basic properties of this topology. For instance, we are going to prove that there is a filtration of  $\mathbf{C}\{z\}$  by compact subsets  $\mathbf{C}_c\{z\}$  so that the continuity can be checked in restriction to each  $\mathbf{C}_c\{z\}$ .

**Remark 5.1.** If  $s < r$ , the homomorphism  $\mathcal{A}_r \rightarrow \mathcal{A}_s$  is compact: by Montel theorem, the ball of radius 1 in  $\mathcal{A}_r$  is mapped into a compact subset  $K_1$  of  $\mathcal{A}_s$ .

Let  $K \subset \mathcal{A}_r$  be a bounded subset. Then, the closure  $cl_s(K)$  of (the image of)  $K$  in  $\mathcal{A}_s$  is compact. If  $t \leq s$ , the image of  $cl_s(K)$  in  $\mathcal{A}_t$  is compact, hence closed; this implies that  $cl_s(K) = cl_t(K)$  in  $\mathbf{C}\{z\}$ . Thus, the closure  $\overline{K}$  of  $K$  in  $\mathbf{C}\{z\}$  coincides with the closure  $cl_s(K)$  of  $K$  in  $\mathcal{A}_s$  for any  $s < r$ . As a consequence,  $\overline{K}$  is compact.

We denote by  $B_{\mathcal{A}_r}(\varepsilon)$  the open ball centred at 0 and of radius  $\varepsilon$  in  $\mathcal{A}_r$ , which we also view as a subset of  $\mathbf{C}\{z\}$ . If  $s \leq r$  and  $\varepsilon \leq \varepsilon'$ , then  $B_{\mathcal{A}_r}(\varepsilon) \subset B_{\mathcal{A}_s}(\varepsilon') \subset \mathbf{C}\{z\}$ . Given any finite set of such balls  $B_{\mathcal{A}_{r_j}}(\varepsilon_j)$ , the sum  $\sum_{j \geq 1}^n B_{\mathcal{A}_{r_j}}(\varepsilon_j)$  is the subset of  $\mathbf{C}\{z\}$  whose elements are sums  $f_1 + \dots + f_n$  with  $f_j \in B_{\mathcal{A}_{r_j}}(\varepsilon_j)$  for all  $j$ .

**Lemma 5.2.** *A subset  $\mathcal{U}$  of  $\mathbf{C}\{z\}$  is a neighborhood of 0 if and only if there are decreasing sequences  $(r_n)$  and  $(\epsilon_n)$  tending to 0 such that  $\mathcal{U}$  contains the set*

$$\mathcal{B} = \bigcup_n \sum_{j \geq 1}^n B_{\mathcal{A}_{r_j}}(\epsilon_j).$$

Lemma 5.2 shows that the topology defined in this section is the same as the topology introduced by Leslie in [Les], except that we consider germs of analytic functions at the origin in  $\mathbf{C}$  instead of real analytic functions on a compact analytic manifold.

*Proof.* First we argue that any set  $\mathcal{B}$  as in the statement of Lemma 5.2 is a neighborhood of 0 in  $\mathbf{C}\{z\}$ . To do so we need to check that  $\mathcal{B} \cap \mathcal{A}_r$  contains a neighborhood of 0 for all  $r$ . The sum  $\sum_{j=1}^n B_{\mathcal{A}_{r_j}}(\epsilon_j)$  is a subset of  $\mathbf{C}\{z\}$  which is contained in  $\mathcal{A}_{r_n}$ . It is open in  $\mathcal{A}_{r_n}$  because one of the summands, namely  $B_{\mathcal{A}_{r_n}}(\epsilon_n)$ , is itself open. Now, the continuity of the inclusion  $\mathcal{A}_r \rightarrow \mathcal{A}_{r_n}$  for  $r_n < r$  implies that  $\sum_{j=1}^n B_{\mathcal{A}_{r_j}}(\epsilon_j) \cap \mathcal{A}_r$  is also open in  $\mathcal{A}_r$ . Since  $\sum_{j=1}^n B_{\mathcal{A}_{r_j}}(\epsilon_j) \cap \mathcal{A}_r$  is contained in  $\mathcal{B} \cap \mathcal{A}_r$ , the latter is a neighborhood of 0, as we needed to prove.

Suppose now that  $\mathcal{U}$  is a neighborhood of the origin in  $\mathbf{C}\{z\}$ , and fix a decreasing sequence  $(r_n)$  tending to 0. For each  $n \geq 1$ , set  $\mathcal{U}_n = \mathcal{U} \cap \mathcal{A}_{r_n}$ .

We first claim that there is a ball  $B_1$  in  $\mathcal{A}_{r_1}$  such that  $\overline{B}_1 \subset \mathcal{U}$ . Since  $\mathcal{U}_2$  is open in  $\mathcal{A}_{r_2}$ , consider  $\varepsilon > 0$  such that  $B_{\mathcal{A}_{r_2}}(\varepsilon) \subset \mathcal{U}_2$ . Now let  $B_1 = B_{\mathcal{A}_{r_1}}(\varepsilon/2)$ . Then for all  $\eta > 0$ ,  $\overline{B}_1 \subset B_1 + B_{\mathcal{A}_{r_2}}(\eta)$  so taking  $\eta = \varepsilon/2$ , we get  $\overline{B}_1 \subset B_{\mathcal{A}_{r_2}}(\varepsilon/2) + B_{\mathcal{A}_{r_2}}(\varepsilon/2) \subset B_{\mathcal{A}_{r_2}}(\varepsilon) \subset \mathcal{U}$ , which proves our claim.

We now construct by induction open balls  $B_n \subset \mathcal{A}_{r_n}$  such that for all  $n$ ,  $\overline{B}_1 + \dots + \overline{B}_n \subset \mathcal{U}$ . Given such a set of balls  $B_1, \dots, B_n$ , the set  $K = \overline{B}_1 + \dots + \overline{B}_n$  provides a compact subset of  $\mathcal{A}_{r_{n+2}}$  contained in  $\mathcal{U}_{n+2}$ . Let  $\epsilon$  be the distance from  $K$  to the complement of  $\mathcal{U}_{n+2}$  in  $\mathcal{A}_{r_{n+2}}$ ; by compactness,  $\varepsilon > 0$ , and  $K + B_{\mathcal{A}_{r_{n+2}}}(\varepsilon/2) \subset \mathcal{U}$ . We then define  $B_{n+1} = B_{\mathcal{A}_{r_{n+1}}}(\varepsilon/4)$ . Then  $K + \overline{B}_{n+1} \subset K + B_{\mathcal{A}_{r_{n+2}}}(\varepsilon/2) \subset \mathcal{U}$ . This concludes the induction step and the proof.  $\square$

**5.2. Coefficient functions.** Recall that the coefficients of  $f \in \mathcal{A}_r$  can be computed via the Cauchy integral formula:

$$(5.3) \quad A_n(f) = \frac{1}{2\pi i} \int_{\{|z|=r\}} \frac{f(z)}{z^{n+1}} dz.$$

This implies that the linear form  $A_n$  is continuous on each algebra  $\mathcal{A}_r$  with operator norm  $\|A_n\|_{\mathcal{A}_r^*} \leq \frac{1}{2\pi} r^{-(n+1)}$ , i.e.  $|A_n(f)| \leq \frac{1}{2\pi} r^{-(n+1)} \|f\|_{\mathcal{A}_r}$  for all  $f \in \mathcal{A}_r$ . Since the maps  $A_n$  separate points in  $\mathbf{C}\{z\}$ , we obtain:

**Lemma 5.3.** *For each  $n \geq 0$ , the map  $A_n : \mathbf{C}\{z\} \rightarrow \mathbf{C}$  is continuous. The topological space  $\mathbf{C}\{z\}$  is Hausdorff.*

More generally, we have:

**Lemma 5.4.** *If  $\sum_n \theta_n z^n$  is a power series with infinite convergence radius, then the quantity*

$$(5.4) \quad \Theta(f) = \sum_n \theta_n |A_n(f)|$$

*is well defined for every  $f \in \mathbf{C}\{z\}$  and the function  $\Theta : \mathbf{C}\{z\} \rightarrow \mathbf{R}_+$  is continuous.*

*Proof.* The estimate  $\|A_n\|_{\mathcal{A}_r^*} \leq \frac{1}{2\pi} r^{-(n+1)}$  implies that the map

$$(5.5) \quad \mathcal{A}_r \rightarrow \mathbf{C}, \quad f \mapsto \sum_n \theta_n |A_n(f)|$$

is continuous for any power series  $\sum_n \theta_n z^n$  with convergence radius greater than  $\frac{1}{r}$ . By definition of the topology on  $\mathbf{C}\{z\}$  we get that this map is continuous on the whole space if the power series in question has infinite convergence radius.  $\square$

**5.3. Another filtration.** We now introduce another filtration of  $\mathbf{C}\{z\}$ . If  $c$  is any positive real number, we define

$$(5.6) \quad \mathbf{C}_c\{z\} = \{f \in \mathbf{C}\{z\} \text{ with } |A_n(f)| \leq c^{n+1} \text{ for all } n\}.$$

Then  $\mathbf{C}_c\{z\} \subset \mathbf{C}_{c'}\{z\}$  for  $c \leq c'$ , and  $\mathbf{C}\{z\}$  is the increasing union of all  $\mathbf{C}_c\{z\}$ .

**Lemma 5.5.**  *$\mathbf{C}_c\{z\}$  is compact, and contained in  $\mathcal{A}_r$  for all  $r < c^{-1}$ . Every compact subset  $\Lambda \subset \mathbf{C}\{z\}$  is contained in some  $\mathbf{C}_c\{z\}$ .*

By compactness, the topology on  $\mathbf{C}_c\{z\}$  induced by  $\mathcal{A}_r$  and by  $\mathbf{C}\{z\}$  agree.

*Proof.* From Lemma 5.3, we deduce that  $\mathbf{C}_c\{z\}$  is closed in  $\mathbf{C}\{z\}$ . If  $f \in \mathbf{C}_c\{z\}$  and  $r < c^{-1}$ , then

$$(5.7) \quad \|f\|_{\mathcal{A}_r} \leq \sum_n c^{n+1} r^n \leq \frac{c}{1 - cr}$$

This means that  $\mathbf{C}_c\{z\}$  is a bounded subset in  $\mathcal{A}_r$ . Since the inclusion  $\mathcal{A}_r \rightarrow \mathcal{A}_s$  is compact for  $r > s$ ,  $\mathbf{C}_c\{z\}$  has compact closure in  $\mathcal{A}_s$ , hence in  $\mathbf{C}\{z\}$ . Since  $\mathbf{C}_c\{z\}$  is closed, it is compact.

To prove the second assertion, assume by contradiction that there is a compact subset  $\Lambda \subset \mathbf{C}\{z\}$  such that for every integer  $m > 0$  there exists  $f_m \in \Lambda \setminus \mathbf{C}_m\{z\}$ .

By definition, there is an index  $n_m \geq 0$  with  $|A_{n_m}(f_m)| > m^{n_m+1}$ . By Lemma 5.4, each individual coefficient is continuous and thus bounded on the compact  $\Lambda$ . It follows that  $n_m$  goes to  $+\infty$  as  $m$  does. We can thus assume, passing to a subsequence if necessary, that the  $n_m$ 's are pairwise distinct.

Set  $\theta_{n_m} = (\frac{1}{m})^{n_m}$ , and  $\theta_n = 0$  if  $n$  is not one of the indices  $n_m$ . Then  $\theta_n^{1/n}$  converges towards 0 as  $n$  goes to  $+\infty$ , meaning that the power series  $\sum_n \theta_n z^n$  has infinite convergence radius. By Lemma 5.4, the map  $f \mapsto \Theta(f) = \sum_n \theta_n |A_n(f)|$  is continuous on  $\mathbf{C}\{z\}$  and thus bounded on our compact set  $\Lambda$ . On the other hand we have

$$\Theta(f_m) \geq \theta_{n_m} |A_{n_m}(f_m)| \geq m.$$

This yields the desired contradiction.  $\square$

**Remark 5.6.** Given  $r > 0$ , introduce

$$(5.8) \quad \mathcal{B}_r = \left\{ f \in \mathbf{C}\{z\} \mid \text{rad}(f) \geq r, \sup_{\mathbb{D}_r} |f| \leq \frac{1}{r} \right\}.$$

This is the closure in  $\mathbf{C}\{z\}$  of a ball in  $\mathcal{A}_r$  and is therefore compact (Remark 5.1). There are functions  $c_1, c_2, r_1$ , and  $r_2 : \mathbf{R}_+^* \rightarrow \mathbf{R}_+^*$  such that

$$\mathbf{C}_{c_1(r)}\{z\} \subset \mathcal{B}_r \subset \mathbf{C}_{c_2(r)}\{z\} \text{ and } \mathcal{B}_{r_1(c)} \subset \mathbf{C}_c\{z\} \subset \mathcal{B}_{r_2(c)}.$$

It follows that one could equivalently state the results of this section in terms of the filtration  $(\mathcal{B}_r)_{r>0}$  instead of  $(\mathbf{C}_c\{z\})_{c>0}$ .

The following corollary allows us to view the final topology on  $\mathbf{C}\{z\}$  as the weak topology associated to the filtration by the compact sets  $\mathbf{C}_c\{z\}$ .

**Corollary 5.7.** *A subset  $F \subset \mathbf{C}\{z\}$  is closed if and only if for all  $c > 0$ ,  $F \cap \mathbf{C}_c\{z\}$  is closed. A map  $F : \mathbf{C}\{z\} \rightarrow X$  to a topological space is continuous if and only if its restriction to  $\mathbf{C}_c\{z\}$  is continuous for all  $c > 0$ .*

*Proof.* Clearly, it suffices to prove the first assertion. If  $F$  is closed, so is  $F \cap \mathbf{C}_c\{z\}$ . Assume conversely that  $F \cap \mathbf{C}_c\{z\}$  is closed for all  $c > 0$ , and let us prove that  $F$  is closed. By definition of the final topology, we need to prove that given  $r > 0$ , its preimage  $j_r^{-1}(F)$  under the inclusion  $j_r : \mathcal{A}_r \rightarrow \mathbf{C}\{z\}$  is closed. It suffices to prove that for any  $R > 0$ , its intersection with the ball  $B_{\mathcal{A}_r}(R)$  is closed in  $\mathcal{A}_r$ . Since  $B_{\mathcal{A}_r}(R)$  has compact closure, there exists  $c > 0$  such that  $B_{\mathcal{A}_r}(R) \subset \mathbf{C}_c\{z\}$ . Since  $F \cap \mathbf{C}_c\{z\}$  is closed,  $B_{\mathcal{A}_r}(R) \cap j_r^{-1}(F) = B_{\mathcal{A}_r}(R) \cap j_r^{-1}(F \cap \mathbf{C}_c\{z\})$  is a closed subset of  $\mathcal{A}_r$  which concludes the proof.  $\square$

Although one can show that the topology on  $\mathbf{C}\{z\}$  is not metrizable, each space  $\mathbf{C}_c\{z\}$  is a metric space. Being compact, the topology on  $\mathbf{C}_c\{z\}$  can be described in many equivalent ways:

**Proposition 5.8.** *Let  $c$  be a positive real number. Let  $(f_m)$  be a sequence in  $\mathbf{C}_c\{z\}$  and let  $f_\infty$  be an element of  $\mathbf{C}_c\{z\}$ . The following are equivalent:*

- (1)  $(f_m)$  converges to  $f_\infty$  in  $\mathbf{C}_c\{z\}$ ;
- (2) for some (any)  $r < c^{-1}$ ,  $(f_m)$  converges uniformly toward  $f_\infty$  on  $\overline{\mathbb{D}}_r$ ;
- (3)  $(f_m)$  converges toward  $f_\infty$  uniformly on every compact subset of  $\mathbb{D}_{c^{-1}}$ ;
- (4) for every index  $n$ ,  $A_n(f_m)$  converges toward  $A_n(f_\infty)$ .

*Proof.* As seen before,  $\mathbf{C}_c\{z\}$  is contained in  $\mathcal{A}_r$  for all  $r < c^{-1}$  and the topology induced by  $\|\cdot\|_{\mathcal{A}_r}$  agrees with the topology induced by  $\mathbf{C}_c\{z\}$ . This proves the equivalence of the first three assertions.

To prove the equivalence with the last assertion, consider the map  $\Phi : \mathbf{C}_c\{z\} \rightarrow [0, 1]^\mathbb{N}$  defined by  $\Phi(f) = (\frac{A_n(f)}{c^{n+1}})_{n \in \mathbb{N}}$ , where  $[0, 1]^\mathbb{N}$  is endowed with the product topology. This map being continuous and injective, it is a homeomorphism to its image, and the result follows.  $\square$

**5.4. The final topology on a field with an absolute value.** In this section, we explain that the final topology induced by the filtration  $\mathbf{C}_c\{z\}$  makes sense for every field  $\mathbf{k}$  with an absolute value  $|\cdot|$ ; but the results based on Montel theorem (Remark 5.1) may fail for fields  $\mathbf{k} \neq \mathbf{C}$ .

Let  $\mathbf{k}$  be a complete field  $\mathbf{k}$  for some absolute value  $|\cdot| : \mathbf{k} \rightarrow \mathbf{R}_+$ . By Ostrowski's Theorem,  $\mathbf{k}$  is either  $\mathbf{R}$  or  $\mathbf{C}$ , or the absolute value is non-archimedean:  $|x + y| \leq \max(|x|, |y|)$  for all  $x, y \in \mathbf{k}$ . The algebra  $\mathbf{k}\{z\}$  of convergent power series is filtrated by the family of subsets

$$(5.9) \quad \mathbf{k}_c\{z\} = \{f \in \mathbf{k}\{z\} \text{ with } |A_n(f)| \leq c^{n+1} \text{ for all } n\}$$

for  $c > 0$ . We endow  $\mathbf{k}_c\{z\}$  with the product topology, via the embedding  $f \in \mathbf{k}_c\{z\} \rightarrow (A_n(f))_{n \in \mathbb{N}} \in \mathbf{k}^\mathbb{N}$ : a sequence  $(f_k)_{k \in \mathbb{N}}$  of elements of  $\mathbf{k}_c\{z\}$  converges to  $f_\infty \in \mathbf{k}_c\{z\}$  if and only if  $A_n(f_k) \rightarrow A_n(f_\infty)$  for all  $n$ . For  $c \leq c'$ ,  $\mathbf{k}_c\{z\}$  is closed in  $\mathbf{k}_{c'}\{z\}$  and the inclusion is a homeomorphism to its image. We then endow  $\mathbf{k}\{z\}$  with the topology associated to this filtration: a subset  $F \subset \mathbf{k}\{z\}$  is closed if and only if  $F \cap \mathbf{k}_c\{z\}$  is closed in  $\mathbf{k}_c\{z\}$ . Equivalently, a map  $\varphi : \mathbf{k}\{z\} \rightarrow X$  to a topological space is continuous if and only if its restriction to  $\mathbf{k}_c\{z\}$  is continuous for every  $c > 0$ . By construction, the maps  $f \mapsto A_n(f)$  are continuous on  $\mathbf{k}\{z\}$ . Proposition 5.8 shows that, when  $\mathbf{k} = \mathbf{C}$ , this topology agrees with the final topology defined in Section 5.1.

If  $\mathbf{k}$  is locally compact, each  $\mathbf{k}_c\{z\}$  is compact. In general, since  $\mathbf{k}_c\{z\}$  is a countable product of complete metric spaces, we get:

**Proposition 5.9.** *If  $\mathbf{k}$  is a complete field, then  $\mathbf{k}_c\{z\}$  is a metrizable complete space. In particular, it is a Baire space.*

On the other hand,  $\mathbf{k}\{z\}$  is not a Baire space since it is a countable union of  $\mathbf{k}_c\{z\}$ , each of which is closed and has an empty interior.

**5.5. The topological group of germs of diffeomorphisms.** Any  $f \in \text{Diff}(\mathbf{k}, 0)$  can be written as  $f = \lambda(z + z^2 \tilde{f})$  for some  $\tilde{f} \in \mathbf{k}\{z\}$  or equivalently as

$$(5.10) \quad f = \lambda(z + \tilde{a}_2 z^2 + \cdots + \tilde{a}_k z^k + \dots)$$

for some  $\lambda \in \mathbf{k}^*$  and  $\tilde{a}_n \in \mathbf{k}$ . Thus, we define the maps  $\tilde{A}_n : \text{Diff}(\mathbf{k}, 0) \rightarrow \mathbf{k}$  by

$$(5.11) \quad \tilde{A}_n(f) = A_n(f)/A_1(f) = \tilde{a}_n.$$

Given two real numbers  $c > 0$  and  $\lambda_0 > 1$ , we define the two subsets

$$(5.12) \quad \text{Diff}_c(\mathbf{k}, 0) = \{f \in \text{Diff}(\mathbf{k}, 0) ; |\tilde{A}_n(f)| \leq c^{n-1} \text{ for all } n\}$$

and

$$(5.13) \quad \text{Diff}_{\lambda_0, c}(\mathbf{k}, 0) = \left\{ f \in \text{Diff}_c(\mathbf{k}, 0) ; \frac{1}{\lambda_0} \leq |A_1(f)| \leq \lambda_0 \right\}$$

Observe that if we denote by  $m_\alpha : z \mapsto \alpha z$  the multiplication by some scalar  $\alpha \in \mathbf{k}^*$  then we have

$$(5.14) \quad m_\alpha \text{Diff}_c(\mathbf{k}, 0) m_\alpha^{-1} = \text{Diff}_{c\alpha}(\mathbf{k}, 0)$$

and

$$(5.15) \quad m_\alpha \text{Diff}_{\lambda_0, c}(\mathbf{k}, 0) m_\alpha^{-1} = \text{Diff}_{\lambda_0, c\alpha}(\mathbf{k}, 0)$$

**Lemma 5.10.** *A map  $\varphi : \text{Diff}(\mathbf{k}, 0) \rightarrow X$  to a topological space is continuous if and only if it is continuous in restriction to  $\text{Diff}_c(\mathbf{k}, 0)$  (or equivalently to  $\text{Diff}_{\lambda_0, c}(\mathbf{k}, 0)$ ) for every  $c > 0$  and  $\lambda_0 > 1$ .*

*Proof.* It suffices to check the continuity of  $\varphi$  on the open set  $U_{\lambda_0} = \{f ; \frac{1}{\lambda_0} < |A_1(f)| < \lambda_0\}$  for all  $\lambda_0 > 1$ . By definition of the final topology, it suffices to check its continuity on  $U_{\lambda_0} \cap \mathbf{k}_c\{z\}$  for every  $c > 1$ . But  $U_{\lambda_0} \cap \mathbf{k}_c\{z\}$  is a subset of  $\text{Diff}_{\lambda_0, c'}(\mathbf{k}, 0)$  as soon as  $c' \geq \max(\lambda_0, c^3)$ ; since we know that  $\varphi$  is continuous on  $\text{Diff}_{\lambda_0, c'}(\mathbf{k}, 0)$ , this proves the lemma.  $\square$

**Proposition 5.11.** *If  $\mathbf{k}$  is a complete field, then  $\text{Diff}_{\lambda_0, c}(\mathbf{k}, 0)$  and  $\text{Diff}_c(\mathbf{k}, 0)$  are complete metric spaces. In particular, they are Baire spaces.*

*Proof.* By definition,  $\text{Diff}_{c, \lambda_0}(\mathbf{k}, 0)$  is homeomorphic to a countable product of closed subsets of  $\mathbf{k}$ ; so,  $\mathbf{k}$  being complete, its topology is induced by a complete metric. Since  $\mathbf{k}^*$  is homeomorphic to the closed subset  $\{(x, y) | xy - 1\} \subset \mathbf{k}^2$ , the same argument applies to  $\text{Diff}_c(\mathbf{k}, 0)$ .  $\square$

**Theorem 5.12.** *Let  $(\mathbf{k}, |\cdot|)$  be a field with a complete absolute value. With the final topology,  $\text{Diff}(\mathbf{k}, 0)$  is a topological group.*

**Lemma 5.13.** *For every real number  $c > 1$ , there exists a real number  $c' > 1$  such that the following holds: if  $f$  and  $g$  are in  $\text{Diff}_{c, c}(\mathbf{k}, 0)$ , then  $f \circ g$  and  $f^{-1}$  lie in  $\text{Diff}_{c', c'}(\mathbf{k}, 0)$ .*

*Proof.* Let  $f = \lambda(z + \sum_{n \geq 2} \tilde{a}_n z^n)$ ,  $g = \mu(z + \sum_{n \geq 2} \tilde{b}_n z^n)$  with  $|\tilde{a}_n|, |\tilde{b}_n| \leq c^{n-1}$  and  $|\lambda|, |\mu|, |\lambda|^{-1}, |\mu|^{-1} \leq c$ . Let  $F = c(z + \sum_{n \geq 2} c^{n-1} z^n) = \frac{cz}{1-cz} \in \mathbf{R}\{z\}$  so that the absolute value of the coefficients of  $f$  and  $g$  are bounded by the coefficients of  $F$ . Then the absolute value of the coefficients of  $f \circ g = \sum_{n \geq 1} a_n (\sum_{m \geq 1} b_m z^m)^n$  are bounded by the coefficients of  $F \circ F = \frac{cz(1-cz)}{1-cz-c^2z}$ . Since  $F \circ F$  has positive convergence radius, there exists  $c' \geq c^2$  such that  $\tilde{A}_n(F \circ F) \leq c'^{n-1}$  for all  $n \geq 2$ . The first assertion follows.

We now prove the second assertion. Let  $f = \lambda(z + \sum_{n \geq 2} \tilde{a}_n z^n)$ , and let  $f^{-1} = \lambda^{-1}(z + \sum_{n \geq 2} \tilde{b}_n z^n)$ . The inversion formula from Section 2.1 gives

$$\begin{aligned} |\tilde{b}_n| &\leq \frac{|\lambda|}{|\lambda|^n} \sum_{k_1, k_2, \dots} \frac{(n+1) \cdots (n-1+k_1+k_2+\dots)}{k_1! k_2! \dots} \cdot |\tilde{a}_2|^{k_1} |\tilde{a}_3|^{k_2} \dots \\ &\leq c^{n-1} \sum_{k_1, k_2, \dots} \frac{(n+1) \cdots (n-1+k_1+k_2+\dots)}{k_1! k_2! \dots} (c)^{k_1} (c)^{2k_2} \dots \\ &= c^{2n-2} \sum_{k_1, k_2, \dots} \frac{(n+1) \cdots (n-1+k_1+k_2+\dots)}{k_1! k_2! \dots}. \end{aligned}$$

Thus, we have to bound the quantity

$$K_n := \sum_{k_1, k_2, \dots} \frac{(n+1) \cdots (n-1+k_1+k_2+\dots)}{k_1! k_2! \dots}.$$

But the numbers  $K_n$  are the coefficients of the power series expansion of the reciprocal diffeomorphism  $g^{-1}$  of

$$g(z) = z - z^2 - z^3 - z^4 \dots = z \left( 2 - \frac{1}{1-z} \right) = \frac{z-2z^2}{1-z}.$$

In close form, we obtain

$$g^{-1}(y) = \frac{(1+y)}{4} - \frac{1}{4}\sqrt{1-6y+y^2}.$$

Since  $g^{-1}$  has positive convergence radius, there exists  $c_0$  such that for all  $n \geq 2$ ,  $K_n \leq c_0^{n-1}$  hence  $|\tilde{b}_n| \leq (c_0 c^2)^{n-1}$  and the result follows.  $\square$

*Proof of Theorem 5.12.* By definition of the topology, given  $c > 0$ , one only needs to check the continuity of the group laws in restriction to  $\text{Diff}_{c,c}(\mathbf{k}, 0)$ .

Since  $A_n(f \circ g)$  and  $A_n(f^{-1})$  are given by polynomials in the coefficients  $\tilde{A}_i(f)$ ,  $\tilde{A}_i(g)$ ,  $A_1(f)^{\pm 1}$ ,  $A_1(g)^{\pm 1}$ , the maps  $(f, g) \mapsto A_n(f \circ g)$  and  $f \mapsto A_n(f^{-1})$  are continuous on  $\text{Diff}_{c,c}(\mathbf{k}, 0)$ . By Lemma 5.13 there exists  $c'$  such that for all  $f, g \in \text{Diff}_{c,c}(\mathbf{k}, 0)$ ,  $f \circ g$  and  $f^{-1}$  lie in  $\text{Diff}_{c',c'}(\mathbf{k}, 0)$ . Since the topology on  $\text{Diff}_{c',c'}(\mathbf{k}, 0)$  is the product topology, the continuity of the coefficients implies the continuity of the group laws.  $\square$

**5.6. Other topologies.** First, we would like to point out that there are other reasonable and useful topologies on  $\mathbf{C}\{z\}$ , but for which the group laws are not continuous. This is the case for the so-called *Takens topology* [MRR, BT]; this is the topology induced by the distance

$$(5.16) \quad \text{dist}(f, g) = \sup_n |A_n(f) - A_n(g)|^{1/n}.$$

Note that in particular the convergence radius of  $f - g$  is large if  $f$  and  $g$  are close to each other in the Takens topology, and this implies that the right translation  $R_f: g \mapsto g \circ f$  is not continuous if the radius of convergence of  $f$  is finite. Indeed, a small perturbation  $g(z) + \epsilon z$  is mapped to  $R_f(g + \epsilon z) = g \circ f + \epsilon f$ , and the difference  $\epsilon f$  is not small in the Takens topology because its radius of convergence does not depend on  $\epsilon$ .

We comment now on another important topology on  $\text{Diff}(\mathbf{C}, 0)$ , but for which the Baire property fails. Let  $\text{Jets}_\ell(\mathbf{C}, 0)$  be the group of  $\ell$ -jets of diffeomorphisms  $a_1 z + \dots + a_\ell z^\ell \pmod{(z^{\ell+1})}$ , with  $a_1 \neq 0$ ; it can be considered as a solvable algebraic group and thus as a solvable complex Lie group. Let

$$(5.17) \quad j_\ell: \text{Diff}(\mathbf{C}, 0) \rightarrow \text{Jets}_\ell(\mathbf{C}, 0)$$

denote the homomorphism that maps a power series  $f = \sum_n a_n z^n$  to  $\sum_{n=1}^\ell a_n z^n$ . We can then define a topology on  $\text{Diff}(\mathbf{C}, 0)$  (resp. on  $\text{Diff}(\mathbf{C}, 0)$ ): the weakest topology for which all projections  $j_\ell$  are continuous. With this topology,  $\text{Diff}(\mathbf{C}, 0)$  is a topological group, because the projections  $j_\ell$  are homomorphisms. Moreover, a sequence  $(f_m)$  converges toward a germ of diffeomorphism  $g$  if and only if the coefficients  $A_n(f_m)$  converge to  $A_n(g)$  for all  $n$ . In other words, this is the

topology of simple convergence on the coefficients. In particular,  $\text{Diff}(\mathbf{C}, 0)$  is not a closed subset of  $\widehat{\text{Diff}}(\mathbf{C}, 0)$  for this topology. With this topology,  $\widehat{\text{Diff}}(\mathbf{C}, 0)$  is a Baire space, but  $\text{Diff}(\mathbf{C}, 0)$  is not (Proposition 5.11 fails if  $\text{Diff}(\mathbf{C}, 0)$  is endowed with this topology).

**5.7. Continuity in the Koenigs linearization Theorem.** A contraction  $f \in \text{Diff}(\mathbf{k}, 0)$  is an element with  $|A_1(f)| < 1$ . In this case, Koenigs theorem says that the unique formal diffeomorphism  $h_f$  tangent to the identity that conjugates  $f$  to the homothety  $z \mapsto A_1(f)z$  has positive convergence radius. The following result shows that  $f \mapsto h_f$  is continuous for the final topology on the set of contractions

$$(5.18) \quad \text{Cont}(\mathbf{k}, 0) = \{f \in \text{Diff}(\mathbf{k}, 0) \mid |A_1(f)| < 1\}.$$

**Theorem 5.14.** *Let  $\mathbf{k}$  be a field with a complete non-trivial absolute value. For every germ  $f \in \text{Cont}(\mathbf{k}, 0)$ , the unique formal diffeomorphism  $h_f$  such that*

$$h_f(f(z)) = A_1(f) \cdot h_f(z) \quad \text{and} \quad A_1(h_f) = 1$$

*has positive convergence radius, and the map*

$$h : f \in \text{Cont}(\mathbf{k}, 0) \mapsto h_f \in \text{Diff}(\mathbf{k}, 0)$$

*is continuous for the final topology. The coefficients of  $h_f$  are polynomial functions with integer coefficients in the variables  $A_i(f)$  and  $(A_1(f)^j - 1)^{-1}$ , for  $i, j \geq 1$ .*

When  $\mathbf{k} = \mathbf{C}$ , it is shown in [Mil, Chapter 8] that  $h_f$  is convergent and its coefficients depend holomorphically on  $f$ . Theorem 5.14 is just a variation on this classical result.

*Proof.* We refer to [Sie] for the real and complex cases, and to [HY] for the non-archimedean ones.

The coefficients of  $h_f$  can be computed inductively and turn out to be polynomials with integer coefficients in the variables  $A_i(f)$  and  $(A_1(f)^j - 1)^{-1}$ , for  $i, j \geq 1$  (see for instance [Sie, Eq. 4]). If  $|A_1(f)| \leq \alpha$  for some  $\alpha$  in the interval  $[0, 1[$ , then

$$(5.19) \quad |(A_1(f))^n - 1| \geq 1 - \alpha$$

for all  $n > 0$ . By [Sie, Theorem 1] and [HY, Theorem 1] in the archimedean and non-archimedean cases respectively,  $h_f$  is convergent and for all  $c, \lambda > 1$ , there exists  $c'$  such that  $h_f \in \text{Diff}_{c'}(\mathbf{k}, 0)$  if  $f \in \text{Diff}_{\lambda, c}(\mathbf{k}, 0)$ .

The topology on  $\text{Diff}_{c'}(\mathbf{C}, 0)$  is the product topology on the coefficients. Since the coefficients of  $h_f$  are continuous functions of  $f$ , it follows that the restriction of  $f \mapsto h_f$  to  $\text{Diff}_{\lambda, c}(\mathbf{C}, 0)$  is continuous. By definition of the final topology, this proves the continuity of  $h$ .  $\square$

## 6. A large irreducible component of the representation variety

This section describes our second proof strategy for Theorem A. For simplicity, we consider only the fundamental group of a closed orientable surface of genus 2, but we work over any field  $\mathbf{k}$  with a complete absolute value  $|\cdot|$ .

**6.1. An irreducible set of representations.** Using the presentation

$$(6.1) \quad \Gamma_2 = \langle a, b, \bar{a}, \bar{b} \mid [a, b] = [\bar{a}, \bar{b}] \rangle,$$

we get an identification

$$(6.2) \quad \text{Hom}(\Gamma_2; \text{Diff}(\mathbf{k}, 0)) = \{(f, g, \bar{f}, \bar{g}) \in \widehat{\text{Diff}}(\mathbf{k}, 0)^4 \mid [f, g] = [\bar{f}, \bar{g}]\}.$$

Let  $\mathbb{X} \subset \text{Hom}(\Gamma_2, \text{Diff}(\mathbf{k}, 0))$  be the set of representations  $\rho : \Gamma_2 \rightarrow \text{Diff}(\mathbf{k}, 0)$  such that  $\rho(a)$  is tangent to  $\text{Id}$  and  $\rho(b)$  is a contraction. As in Equation (5.18), we denote by  $\text{Cont}(\mathbf{k}, 0)$  the set of contractions. For  $c > 0$ , we let  $\mathbb{X}_c = \mathbb{X} \cap \text{Diff}_c(\mathbf{C}, 0)$ . Set

$$(6.3) \quad \mathcal{R} = \text{Cont}(\mathbf{k}, 0) \times \text{Diff}(\mathbf{k}, 0) \times \text{Diff}(\mathbf{k}, 0),$$

$$(6.4) \quad \mathcal{R}(c) = \text{Cont}_c(\mathbf{k}, 0) \times \text{Diff}_c(\mathbf{k}, 0) \times \text{Diff}_c(\mathbf{k}, 0),$$

and denote by  $\pi : \mathbb{X} \rightarrow \mathcal{R}$  the projection

$$(6.5) \quad \pi(\rho) = (\rho(b), \rho(\bar{a}), \rho(\bar{b}))$$

**Proposition 6.1.** *The map  $\pi$  is a homeomorphism for the final topology, and its inverse*

$$\pi^{-1} : (g, \bar{f}, \bar{g}) \mapsto (f, g, \bar{f}, \bar{g})$$

*is a polynomial map, in the following sense: for each  $n \in \mathbf{N}^*$ , the map  $(g, \bar{f}, \bar{g}) \mapsto A_n(f)$  is polynomial in (finitely many of) the variables  $A_k(g)$ ,  $A_k(\bar{f})$ ,  $A_k(\bar{g})$ ,  $A_1(g)^{-1}$ ,  $A_1(\bar{f})^{-1}$ ,  $A_1(\bar{g})^{-1}$  and  $(A_1(g)^k - 1)^{-1}$  ( $k \geq 1$ ).*

*Proof.* The projection  $\pi$  is continuous because both  $\mathbb{X}$  and  $\mathcal{R}$  come with the topology induced by the same topology on  $\text{Diff}(\mathbf{k}, 0)$ .

Consider a triple  $(g, \bar{f}, \bar{g}) \in \text{Cont}(\mathbf{k}, 0) \times \text{Diff}(\mathbf{k}, 0) \times \text{Diff}(\mathbf{k}, 0)$ . Since  $[\bar{f}, \bar{g}]$  is tangent to the identity, the germs  $g$  and  $[\bar{f}, \bar{g}] \circ g$  have the same derivative  $\lambda = A_1(g)$  at 0. Since  $|\lambda| < 1$ , we can apply Koenigs Theorem 5.14: we get two germs  $h_1$  and  $h_2 \in \text{Diff}(\mathbf{k}, 0)$  tangent to the identity such that

$$(6.6) \quad h_1 \circ g \circ h_1^{-1} = m_\lambda \quad \text{and} \quad h_2 \circ ([\bar{f}, \bar{g}] \circ g) \circ h_2^{-1} = m_\lambda$$

where  $m_\lambda(z) = \lambda z$  is the multiplication by  $\lambda$ . Then, the map  $f := h_2^{-1} \circ h_1$  conjugates  $g$  to  $[\bar{f}, \bar{g}] \circ g$  so

$$f \circ g \circ f^{-1} = [\bar{f}, \bar{g}] \circ g \quad \text{and} \quad [f, g] = [\bar{f}, \bar{g}].$$

This means that one can define the preimage  $\pi^{-1}(g, \bar{f}, \bar{g}) \in \text{Hom}(\Gamma_2, \text{Diff}(\mathbf{k}, 0))$  by the 4-tuple  $(f, g, \bar{f}, \bar{g})$ : the fact that  $\pi^{-1} \circ \pi = \text{Id}_{\mathcal{R}}$  follows from uniqueness in Koenigs Theorem.

The continuity of  $\pi^{-1}$  is a consequence of the continuity of the conjugacy in Koenigs Theorem 5.14 and of the continuity of the map  $(g, \bar{f}, \bar{g}) \mapsto [\bar{f}, \bar{g}] \circ g$ . The fact that  $A_n(f)$  is polynomial in the given variables is a direct consequence of the corresponding fact in Koenigs Theorem, and the fact that group operations are polynomial mappings.  $\square$

We denote the inverse map  $\pi^{-1}$  by  $\Phi$ :

$$(6.7) \quad \forall s \in \mathcal{R}, \quad \Phi_s = \pi^{-1}(s).$$

Thus, if  $s = (g, \bar{f}, \bar{g})$ , then  $\Phi_s$  is the morphism  $\Gamma_2 \rightarrow \text{Diff}(\mathbf{k}, 0)$  such that  $\Phi_s(b) = g$ ,  $\Phi_s(\bar{a}) = \bar{f}$ ,  $\Phi_s(\bar{b}) = \bar{g}$ , and  $\Phi_s(a)$  is the unique germ of diffeomorphism  $f$  which is tangent to the identity and satisfies the relation  $[f, g] = [\bar{f}, \bar{g}]$ . To conclude the proof, our goal now is to prove that for every  $c > 0$ , the family of morphisms  $\Phi_s$ , for  $s \in \mathcal{R}(c)$ , satisfies the assumptions of Lemma 3.1. Proposition 5.11 shows that  $\mathcal{R}(c)$  is a Baire space. The following corollary proves the irreducibility of  $\mathcal{R}(c)$ .

**Corollary 6.2.** *For any  $w \in \Gamma_2$ , denote by  $\mathcal{R}(c)_w \subset \mathcal{R}(c)$  the set of homomorphisms in  $\mathcal{R}(c)$  that kill  $w$ . Then either  $\mathcal{R}(c)_w = \mathcal{R}(c)$  or  $\mathcal{R}(c)_w$  is a closed subset of  $\mathcal{R}(c)$  with empty interior.*

*Proof.* Since the functions  $s \in \mathcal{R} \mapsto A_k(\Phi_s(w)) - A_k(\text{Id})$  are continuous,  $\mathcal{R}(c)_w$  is closed. Now, assume that  $\mathcal{R}(c)_w \neq \mathcal{R}(c)$ : there exists  $k \geq 1$  and a point  $s = (g_0, \bar{f}_0, \bar{g}_0)$  in  $\mathcal{R}(c)$  such that  $A_k(\Phi_s(w)) \neq A_k(\text{Id})$ . According to Proposition 6.1 the map  $s \mapsto A_k(\varphi_s(w)) - A_k(\text{Id})$  is a polynomial function in finitely many of

- the coefficients  $A_n(g_0)$ ,  $A_n(\bar{f}_0)$  and  $A_n(\bar{g}_0)$  ( $n \geq 1$ ),
- the inverses  $A_1(g_0)^{-1}$ ,  $A_1(\bar{f}_0)^{-1}$ ,  $A_1(\bar{g}_0)^{-1}$  and  $(A_1(g_0)^k - 1)^{-1}$  ( $k \geq 1$ )

(note that  $A_1(g_0)$ ,  $A_1(\bar{f}_0)$ ,  $A_1(\bar{g}_0)$  and  $(A_1(g_0)^k - 1)$  do not vanish on  $\mathcal{R}$ ). Our assumption says that this function does not vanish identically on  $\mathcal{R}(c)$ . Assume that  $\mathcal{R}(c)_w$  contains a non-empty open subset  $\mathcal{U}$ , and choose a point  $s = (g_1, \bar{f}_1, \bar{g}_1)$  in  $\mathcal{U}$ . If  $\mathbf{k} = \mathbf{R}$  or  $\mathbf{C}$ , we denote by  $B_{\mathbf{k}}$  the interval  $[0, 1] \subset \mathbf{R}$ ; in the non-archimedean case we set  $B_{\mathbf{k}} = \{t \in \mathbf{k}, |t| \leq 1\}$ . Then, we consider the convex combination

$$(6.8) \quad s_t = \left( g_t = tf_1 + (1-t)f_0, \overline{f}_t = t\overline{f}_1 + (1-t)\overline{f}_0, \overline{g}_t = t\overline{g}_1 + (1-t)\overline{g}_0 \right)$$

with  $t$  in  $B_{\mathbf{k}}$ . According to Lemma 6.3 below,  $g_t$ ,  $\overline{f}_t$  and  $\overline{g}_t$  are in  $\mathcal{R}(c')$  for some  $c' \geq c$ , and  $t \mapsto s_t$  is continuous; thus  $\{t ; s_t \in \mathcal{U}\}$  is an open neighborhood of 1.

The function  $t \mapsto A_n(\varphi_{s_t}(w)) - A_n(\text{Id})$  does not vanish for  $t = 0$ , it is the restriction of a rational function of the variable  $t$  to the interval  $[0, 1]$ , and it vanishes identically on the open set  $\{t ; s_t \in \mathcal{U}\}$ . This is a contradiction, which shows that the interior of  $\mathcal{R}(c)_w$  is empty.  $\square$

Let  $B_{\mathbf{k}}$  be the interval  $[0, 1] \subset \mathbf{R}$  if  $\mathbf{k} = \mathbf{R}$  or  $\mathbf{C}$ , or the ball  $\{t \in \mathbf{k}, |t| \leq 1\}$  in the non-archimedean case.

**Lemma 6.3.** *Let  $f_0 \in \text{Diff}_{c_0}(\mathbf{k}, 0)$  and  $f_1 \in \text{Diff}_{c_1}(\mathbf{k}, 0)$ , and for  $t \in \mathbf{k}$ , let  $f_t = (1-t)f_0 + tf_1$ . Let  $p \in \mathbf{k}$  be the value of  $t$  (if any) such that  $f'_p(0) = 0$ .*

*If  $c_0 \leq c_1$  and  $c_0|f'_0(0)| \leq c_1|f'_1(0)|$ , then for all  $t \in B_{\mathbf{k}} \setminus \{p\}$ ,  $f_t \in \text{Diff}_{c_1}(\mathbf{k}, 0)$ .*

*Proof.* Denote  $\lambda_0 = |f'_0(0)|$  and  $\lambda_1 = |f'_1(0)|$ . By assumption, for all  $n \geq 2$ ,  $|A_n(f_0)| \leq \lambda_0 c_0^{n-1}$  and  $|A_n(f_1)| \leq \lambda_1 c_1^{n-1}$ .

Consider first the case  $\mathbf{k} = \mathbf{R}$  or  $\mathbf{C}$ . Since  $\lambda_0 c_0 \leq \lambda_1 c_1$ , we get for all  $t \in [0, 1]$ ,

$$(6.9) \quad |A_n(f_t)| \leq (1-t)\lambda_0 c_0^{n-1} + t\lambda_1 c_1^{n-1}$$

$$(6.10) \quad \leq (1-t)\lambda_1 c_1 c_0^{n-2} + t\lambda_1 c_1^{n-1} \leq \lambda_1 c_1^{n-1}.$$

This shows that  $f_t \in \text{Diff}_{c_1}(\mathbf{k})$  as soon as  $f'_t(0) \neq 0$ .

In the non-archimedean case, one has  $|t - 1| \leq 1$  for  $t \in B_{\mathbf{k}}$ . Similarly, we get

$$(6.11) \quad |A_n(f_t)| \leq \max\{|1-t|\lambda_0 c_0^{n-1}, |t|\lambda_1 c_1^{n-1}\}$$

$$(6.12) \quad \leq \max\{\lambda_1 c_1 c_0^{n-2}, \lambda_1 c_1^{n-1}\} \leq \lambda_1 c_1^{n-1}.$$

This shows that  $f_t \in \text{Diff}_{c_1}(\mathbf{k})$  as soon as  $f'_t(0) \neq 0$ . Then, the continuity follows from the continuity of the coefficients  $t \mapsto A_n(f_t)$ .  $\square$

**6.2. Separation.** To conclude the proof, we fix  $c > 0$  and prove that  $\mathcal{R}(c)$  satisfies the separation condition of Lemma 3.1. We thus fix  $g \in \Gamma_2 \setminus \{1\}$  and show that  $\mathcal{R}(c)$  contains a representation that does not kill  $g$ . Write the orientable surface group of genus 2 as  $\Gamma_2 = \langle a, b, \overline{a}, \overline{b} \mid [a, b] = [\overline{a}, \overline{b}] \rangle$ , and let  $p : \Gamma_2 \rightarrow \langle a, b \rangle$  be the morphism fixing  $a$  and  $b$  and sending  $\overline{a}$  and  $\overline{b}$  to  $a$  and  $b$  respectively. Let  $\tau : \Gamma_2 \rightarrow \Gamma_2$  be the Dehn twist around the curve  $c = [a, b]$ , i.e., the automorphism that fixes  $a, b$  and sends  $\overline{a}$  and  $\overline{b}$  to  $c\overline{a}c^{-1}$  and  $c\overline{b}b^{-1}$

respectively. According to Proposition 1.2, there exists a positive integer  $n_0$  such that  $p \circ \tau^N(g) \neq 1$  for all  $N \geq n_0$ .

Apply Theorem 3.4 to get a pair  $f_1, f_2$  of germs of diffeomorphisms generating a free group  $\langle f_1, f_2 \rangle$  of rank 2 and satisfying  $f'_1(0) > 1$  and  $f'_2(0) > 1$ . Define a morphism  $\rho : \langle a, b \rangle \rightarrow \text{Diff}(\mathbf{k}, 0)$  by  $\rho(a) = [f_1, f_2]$  and  $\rho(b) = f_2^{-1}$ . Then  $\rho$  is injective,  $\rho(a)$  is tangent to the identity, and  $\rho(b)$  is a contraction. Set  $\rho_N := \rho \circ p \circ \tau^N$ . For  $N \geq n_0$   $\rho_N(g) \neq 1$ . Thus,  $\pi(\rho_N)$  lies in  $\mathcal{R} \setminus \mathcal{R}_g$ ; but it might not lie in  $\mathcal{R}(c)$ .

Let  $c_N > 0$  be such that  $\pi(\rho_N) \in \mathcal{R}(c_N)$ . Given  $\alpha \in \mathbf{k}^*$ , let  $\text{ad}_\alpha$  be the inner automorphism of  $\text{Diff}(\mathbf{k}, 0)$  given by  $f \mapsto m_\alpha \circ f \circ m_\alpha^{-1}$ . As noticed in Equation (5.12), we have  $\text{ad}_\alpha(\text{Diff}_{c_N}(\mathbf{k}, 0)) = \text{Diff}_{\alpha c_N}(\mathbf{k}, 0)$ . Thus, the representation  $\rho'_N = \text{ad}_\alpha \circ \rho_N$  satisfies  $\pi(\rho'_N) \in \mathcal{R}(c)$  if  $\alpha$  is sufficiently small. Since  $\rho'_N(g) \neq 1$  this concludes that  $\mathcal{R}(c)$  satisfies the separation condition of Lemma 3.1.

**6.3. Conclusion.** The family of representations  $\Phi_s$ , with  $s \in \mathcal{R}(c)$  satisfies the Baire property, the irreducibility property, and the separation property of Lemma 3.1. This lemma implies that a generic element of  $s \in \mathcal{R}(c)$  gives an embedding  $\Phi_s : \Gamma_2 \rightarrow \text{Diff}(\mathbf{k}, 0)$ , proving Theorem A for the group  $\Gamma_2$ .

## Part III

### 7. A $p$ -adic proof

**7.1. Free groups with integer coefficients.** A theorem of White [Whi] shows that the homeomorphisms of  $\mathbf{R}$  defined by  $f : z \mapsto z + 1$  and  $g : z \mapsto z^3$  generate a free group. Conjugating the maps  $f$  and  $gfg^{-1}$  by  $z \mapsto \frac{1}{3z}$ , as in [Gla]<sup>1</sup>, one gets two formal diffeomorphisms

$$(7.1) \quad \begin{aligned} f_0(z) &= \frac{z}{1+3z} = \sum_{n=1}^{\infty} (-3)^{n-1} z^n \\ g_0(z) &= \frac{z}{(1+(3z)^3)^{1/3}} = \sum_{n=0}^{\infty} \left( \begin{array}{c} \frac{-1}{3} \\ n \end{array} \right) 3^{3n} z^{3n+1} \end{aligned}$$

that generate a non-abelian free group  $\langle f_0, g_0 \rangle \subset \widehat{\text{Diff}}(\mathbf{Q}, 0) \subset \widehat{\text{Diff}}(\mathbf{k}, 0)$ . It is remarkable that  $f_0$  and  $g_0$  are tangent to the identity at the origin and have integer coefficients:

<sup>1</sup>This conjugacy is called the “Wilson trick” in [Gla].

**Theorem 7.1.** *The group  $\widehat{\text{Diff}}(\mathbf{Q}, 0)$  contains a non-abelian free group, all of whose elements are tangent to the identity and have integer coefficients.*

Thus, one can produce an explicit free group in  $\widehat{\text{Diff}}(\mathbf{k}, 0)$  for every field  $\mathbf{k}$  of characteristic 0. In characteristic  $p > 0$ , Szegedy proved that almost every pair of elements in the Nottingham group  $\widehat{\text{Diff}}(\mathbf{Z}/p\mathbf{Z}, 0)$  generates a free group [Sze].

**7.2. Subgroups of  $\text{Diff}(\mathbf{Q}_p, 0)$ .** In this section,  $p$  is a prime number, and  $\mathbf{Q}_p$  is the field of  $p$ -adic numbers, with its absolute value  $|\cdot|$  normalized by  $|p| = 1/p$ .

Let  $G_p$  denote the set of elements  $f = \sum_{n \geq 1} a_n z^n$  in  $\text{Diff}(\mathbf{Q}_p, 0)$  such that

$$(7.2) \quad a_n \in \mathbf{Z}_p \quad \forall n \quad \text{and} \quad |a_1| = 1.$$

Every element  $f \in G_p$  satisfies  $\text{rad}(f) \geq 1$ . The ultrametric inequality and the Inversion formula show that  $G_p$  is a subgroup of  $\text{Diff}(\mathbf{Q}_p, 0)$ . With the product topology on coefficients (as in Section 5.6), it is a compact topological group, and the morphism  $j_\ell: G_p \rightarrow \text{Jets}_\ell(\mathbf{Q}_p, 0)$  is continuous for every integer  $\ell \geq 1$ . The kernel of  $j_\ell$  will be denoted  $G_{p,\ell}$ .

From Theorem 7.1, we know that  $G_p$  contains a free group of rank two generated by two germs  $f_0$  and  $g_0$  whose coefficients are in  $\mathbf{Z}$ .

**Corollary 7.2.** *Let  $p$  be a prime number and  $\ell$  be a positive integer. The group  $G_{p,\ell}$  contains a non-abelian free group. The group  $G_p$  contains a free group  $\langle f, g \rangle$  of rank 2 such that  $A_1(f)$  is a transcendental number while  $g$  is tangent to the identity up to order  $\ell$ .*

*Proof.* Start with a non-abelian free group  $F$  in  $G_p$ . Since the group of jets  $\text{Jets}_\ell(\mathbf{Q}_p, 0)$  is solvable, the restriction of  $j_\ell$  to  $F$  is not injective. Its kernel is a free group (as any subgroup of  $F$ ), and if  $\ell$  is large it is not cyclic. Thus, the kernel is a non-abelian free group. This proves the first statement.

Set  $\mathcal{R} = \{t \in \mathbf{Z}_p ; |t| = 1\}$ . Now, take a pair of generators  $f_0$  and  $g_0$  of a free group of rank 2 in  $G_{p,\ell}$ , and for  $t \in \mathcal{R}$  consider the family of representations  $\rho_t: \mathbf{F}_2 = \langle a, b \rangle \rightarrow G_p$  defined by  $\rho_t(a) = m_t \circ f_0$  and  $\rho_t(b) = g_0$  (here, as usual,  $m_t(z) = tz$ ). If  $w$  is an element of  $\mathbf{F}_2$ , and  $n$  is a positive integer, then  $A_n(\rho_t(w))$  is a polynomial function in  $t$  and  $1/t$  (see Section 2.1). If  $w \neq 1$ , there is an integer  $n \geq 1$  such that  $A_n(\rho_1(w)) \neq A_n(\text{Id})$ . Thus, the set  $\mathcal{R}_w \subset \mathcal{R}$  of parameters  $s$  such that  $\rho_s(w) = \text{Id}$  is finite, the union  $\cup_{w \neq 1} \mathcal{R}_w$  is at most countable, and there are transcendental numbers in its complement. For such a parameter  $t$ ,  $\rho_t$  is injective and  $A_1(\rho_t(a)) = t$  is transcendental.  $\square$

Now, we apply the result of [BGSS] described in Section 1.2 to get:

**Theorem 7.3.** *Let  $p$  be a prime number. Let  $\Gamma_2 = \langle a, b, \bar{a}, \bar{b} \mid [a, b] = [\bar{a}, \bar{b}] \rangle$  be the fundamental group of a closed orientable surface of genus 2. Then*

- (1) *For every integer  $\ell \geq 1$ , the group  $\Gamma_2$  embeds in the compact group  $G_{p,\ell}$ .*
- (2) *There is an embedding  $\rho: \Gamma_2 \rightarrow G_p$  such that  $\rho(a)'(0) = \rho(\bar{a})'(0)$  is a transcendental number while  $\rho(b)$  and  $\rho(\bar{b})$  are tangent to the identity up to order  $\ell$ .*

**7.3. Back to complex coefficients.** The field  $\mathbf{Q}_p$ , and thus the ring  $\mathbf{Z}_p$ , embeds (although not continuously) into  $\mathbf{C}$ ; such an embedding induces an embedding, coefficient by coefficient, of  $\mathbf{Z}_p[[z]]$  into  $\mathbf{C}[[z]]$ . Thus, the surface groups constructed in Theorem 7.3 provide surface groups in  $\widehat{\text{Diff}}(\mathbf{C}, 0)$ . This construction does not preserve the convergence of power series, but it preserves the order of tangency to  $\text{Id}$ . Since there are transcendental complex numbers with modulus  $< 1$ , we obtain:

**Corollary 7.4.** *Let  $\ell$  be a positive integer. There is an embedding  $\rho: \Gamma_2 \rightarrow \widehat{\text{Diff}}(\mathbf{C}, 0)$  such that  $|\rho(a)'(0)| = |\rho(\bar{a})'(0)| < 1$  while  $\rho(b)$  and  $\rho(\bar{b})$  are tangent to the identity up to order  $\ell$ .*

We can now prove the following version of Theorem A. This will be our third and last proof of it.

**Theorem 7.5.** *There is an embedding of  $\Gamma_2$  in  $\text{Diff}(\mathbf{C}, 0)$  such that  $|\rho(a)'(0)| = |\rho(\bar{a})'(0)| < 1$  while  $\rho(b)$  and  $\rho(\bar{b})$  are tangent to the identity up to order  $\ell$ .*

*Proof.* The first step is to choose a sequence  $\mathcal{C} = (a_1, a_2, a_3, \dots)$  of complex numbers such that

- (a) the set  $\{a_1, a_2, \dots\}$  is algebraically free: if  $m \geq 1$  and  $P \in \mathbf{Z}[x_1, \dots, x_m]$ , and if  $P(a_1, \dots, a_m) = 0$ , then  $P = 0$ ;
- (b)  $|a_n| \leq 2^{-n}$  for all  $n \geq 1$ .

Such a sequence exists because  $\mathbf{C}$  is uncountable. Concrete examples can be obtained from the Lindemann-Weierstrass theorem (see also [Wal] for the constructions of von Neumann, Perron, Kneser, and Durand of uncountably many, algebraically free complex numbers). We shall consider the  $a_i$  as indeterminates for the field of rational functions  $\mathbf{Q}(a_1, a_2, \dots)$ . Armed with such a set we consider the following three formal diffeomorphisms

(7.3)

$$g = a_1 z + \sum_{i=1}^{\infty} a_{3i+1} z^{i+1}, \quad \bar{f} = z + \sum_{i=\ell}^{\infty} a_{3i+2} z^{i+1}, \quad \bar{g} = z + \sum_{i=\ell}^{\infty} a_{3i+3} z^{i+1}.$$

From the decay relation (b), these three power series have a positive radius of convergence. Since  $|a_1| \leq 1/2$ , the Koenigs linearization theorem gives a unique element  $\bar{f} \in \text{Diff}(\mathbf{C}, 0)$  with  $\bar{f}'(0) = 1$  such that

$$(7.4) \quad f g f^{-1} = [\bar{f}, \bar{g}]g.$$

The four elements  $(f, g, \bar{f}, \bar{g})$  determine a representation  $\varphi$  of  $\Gamma_2$  into  $\text{Diff}(\mathbf{C}, 0)$ .

Let us prove that this representation is faithful. Fix a non-trivial element  $w$  of  $\Gamma_2$ , and write it as a word in  $a, b, \bar{a}, \bar{b}$  and their inverses. For every integer  $n$ , the coefficient  $A_n(\varphi(w))$  is a polynomial function  $Q_{w,n}$  in the variables  $a_n$  (for  $n \geq 1$ ),  $a_1^{-1}$ , and the  $(a_1^k - 1)^{-1}$  (for  $k \geq 1$ ) with integer coefficients.

Now, take a faithful representation  $\rho: \Gamma_2 \rightarrow \widehat{\text{Diff}}(\mathbf{C}, 0)$  that satisfies the conclusion of Corollary 7.4. There is an integer  $n \geq 1$  such that  $A_n(\rho(w)) \neq A_n(\text{Id})$ . This implies that  $Q_{w,n} \neq A_n(\text{Id})$  when we specialize the indeterminates  $a_i$  to the coefficients of the generators  $\rho(\bar{a})$ ,  $\rho(b)$ , and  $\rho(\bar{b})$ . Since  $Q_{w,n} \neq A_n(\text{Id})$ ,  $\varphi(w) \neq \text{Id}$  and  $\varphi$  is the identity.  $\square$

## Part IV

### 8. Complements and open questions

**8.1. Takens' theorem and smooth diffeomorphisms.** To conclude this chapter, we mention the following result which allows to realize *any* faithful representation of a surface group in the group of formal germs as a group of  $C^\infty$  germs. Note that the  $p$ -adic method provides many embeddings of surface groups in  $\widehat{\text{Diff}}(\mathbf{R}, 0)$  (see Corollary 7.4).

Recall that  $\Gamma_g$  denotes the fundamental group of the closed orientable surface of genus  $g$ .

**Theorem C.** *Let  $\hat{\rho}: \Gamma_g \rightarrow \widehat{\text{Diff}}(\mathbf{R}, 0)$  be a faithful representation of the surface group  $\Gamma_g$  in the group of formal diffeomorphisms in one real variable. Then, there exists a faithful representation  $\rho: \Gamma_g \rightarrow \text{Diff}^\infty(\mathbf{R}, 0)$  into the group of germs of  $C^\infty$ -diffeomorphisms such that the Taylor expansion of  $\rho(w)$  coincides with  $\hat{\rho}(w)$  for every  $w \in \Gamma_g$ .*

The proof will be a consequence of the following result (this theorem is easily derived from the Sternberg linearization theorem and Theorem 2 of [Tak]):

**Theorem 8.1** (Sternberg [Ste], Takens, [Tak]). *Let  $f, g: (\mathbf{R}, 0) \rightarrow (\mathbf{R}, 0)$  be two germs of  $\mathcal{C}^\infty$ -diffeomorphisms, and let  $\hat{f}$  and  $\hat{g}$  denote their Taylor expansions. Suppose that  $f$  is not flat to the identity, that is  $\hat{f} \neq \text{Id}$ . Then, if  $\hat{f}$  and  $\hat{g}$  are conjugate by a formal diffeomorphism  $\hat{h}$ , there exists a germ of  $\mathcal{C}^\infty$ -diffeomorphism  $h: (\mathbf{R}, 0) \rightarrow (\mathbf{R}, 0)$  such that*

- the Taylor expansion of  $h$  coincides with  $\hat{h}$ ;
- $h$  conjugates  $f$  to  $g$ .

*Proof of Theorem C.* Denote by  $\hat{a}_i$ ,  $\hat{b}_i$ ,  $1 \leq i \leq g$  the images of the standard generators of  $\Gamma_g$  by the representation  $\rho$ ; they satisfy the relation

$$(8.1) \quad \hat{a}_1 \circ \hat{b}_1 \circ \hat{a}_1^{-1} = \left( \prod_{j=2}^g [\hat{a}_j, \hat{b}_j] \right) \circ \hat{b}_1.$$

By the theorem of Borel and Peano, one can find germs of diffeomorphisms  $b_1$  and  $a_j$ ,  $b_j$ ,  $j \geq 2$ , whose respective Taylor expansions coincide with  $\hat{b}_1$ ,  $\hat{a}_j$ , and  $\hat{b}_j$  respectively. Then, Theorem 8.1 provides a germ of diffeomorphism  $a_1$  such that  $a_1 \circ b_1 \circ a_1^{-1} = (\prod_{j=2}^g [a_j, a_j]) \circ b_1$ . Thus, one gets a representation  $\rho$  of  $\Gamma_2$  into  $\text{Diff}^\infty(\mathbf{R}, 0)$  with Taylor expansion equal to  $\hat{\rho}$ . Since the initial representation  $\hat{\rho}$  is injective, so is  $\rho$ .  $\square$

**8.2. Conjugacy classes.** Two subgroups  $\Gamma$  and  $\Gamma'$  of  $\text{Diff}(\mathbf{C}, 0)$  are *topologically conjugate* if there is a germ of homeomorphism  $\varphi: (\mathbf{C}, 0) \rightarrow (\mathbf{C}, 0)$  such that  $\varphi \circ \Gamma \circ \varphi^{-1} = \Gamma'$ , and are *formally conjugate* if there is a formal diffeomorphism  $\hat{\varphi}$  such that  $\hat{\varphi} \circ \Gamma \circ \hat{\varphi}^{-1} = \Gamma'$ . A germ of homeomorphism  $\varphi$  is *anti-holomorphic* if its complex conjugate  $z \mapsto \overline{\varphi(z)}$  is holomorphic.

**Theorem 8.2** (Nakai, Cerveau-Moussu). *Let  $\Gamma$  and  $\Gamma'$  be two subgroups of  $\text{Diff}(\mathbf{C}, 0)$  which are not solvable.*

- (1) *If  $\varphi$  is a local homeomorphism that conjugates  $\Gamma$  to  $\Gamma'$ , then  $\varphi$  is holomorphic, or anti-holomorphic.*
- (2) *If  $\hat{\varphi}$  is a formal conjugacy between  $\Gamma$  and  $\Gamma'$ , then  $\hat{\varphi}$  converges and is therefore a holomorphic conjugacy.*

Thus, (the images of) two embeddings of  $\Gamma_g$  in  $\text{Diff}(\mathbf{C}, 0)$  are topologically or formally conjugate if and only if they are analytically conjugate.

### 8.3. Two questions.

**8.3.1.** It would be interesting to exhibit an embedding  $\alpha$  of the group  $\Gamma_g$ ,  $g \geq 2$ , into the group of analytic diffeomorphisms of the circle  $\mathbf{R}/\mathbf{Z}$  fixing the origin  $o \in \mathbf{R}/\mathbf{Z}$ . If such an embedding exists, the suspension of this representation  $\alpha$  gives a compact manifold  $M_\alpha$  of dimension 3 that fibers over  $\Sigma_g$ , together with a foliation  $\mathcal{F}_\alpha$  of co-dimension 1 which is transverse to the fibration  $\pi: M_\alpha \rightarrow \Sigma_g$  and whose monodromy is given by  $\tau$ . The fixed point gives a compact leaf of  $\mathcal{F}_\alpha$  with holonomy given by the same representation  $\tau$ .

**Question.** Does there exist an embedding of  $\Gamma_2$  into the group of analytic diffeomorphisms of the circle fixing the origin ?

This question was the original motivation of Cerveau and Ghys when they asked for a proof of Theorem A (see [Cer]).

**Remark 8.3.** According to Theorem 7.3, there is an embedding  $\rho$  of  $\Gamma_2$  in  $\text{Diff}(\mathbf{Q}_p, 0)$  such that  $\rho(a)'(0)$  and  $\rho(\bar{a})'(0)$  have modulus 1 while  $\rho(b)$  and  $\rho(\bar{b})$  are tangent to the identity. Conjugate  $\rho$  by the homothety  $a \mapsto p^N z$  for some positive integer  $N$ . If  $N$  is large enough, the coefficients  $a_n$ ,  $n \geq 2$ , of all elements of  $\rho(\Gamma_2)$  have norm  $< 1$ , and the ultrametric inequality shows that  $\rho(\Gamma_2)$  preserves the open disks  $\{z \in \mathbf{C}_p ; |z| < 1 - \epsilon\}$  for every  $\epsilon > 0$ . Thus, it preserves arbitrary thin annuli  $\{z \in \mathbf{C}_p ; 1 - \epsilon|z| \leq 1\}$ . (Here  $\mathbf{C}_p$  is the completion of the algebraic closure of  $\mathbf{Q}_p$ .)

A related, but a priori simpler question is: does there exist an embedding of  $\Gamma_2$  into the group of increasing, real analytic diffeomorphisms of  $[0, 1]$  fixing 0 and 1 ? Here, we demand that the diffeomorphisms extend to germs of real analytic diffeomorphisms on neighbourhoods of 0 and 1. If we replace real analytic diffeomorphisms by  $\mathcal{C}^\infty$  diffeomorphisms, interesting examples have been constructed in [MS]. We refer to the introduction of [MS] for a description of the difficulties in trying to apply the strategy of [BGSS]: this is related to the question of deciding when a diffeomorphism  $f$  of  $[0, 1]$  is contained in the flow of a smooth vector field, hence to Mather's invariant (see [EBN, Yoc]).

**8.3.2.** The derived subgroup of  $\text{Diff}(\mathbf{C}, 0)$  is the kernel of the morphism  $j_1: f \mapsto f'(0)$ :

**Theorem 8.4.** *Let  $\mathbf{k}$  be a complete, non-discrete valued field. An element  $f$  of  $\text{Diff}(\mathbf{k}, 0)$  is a commutator if and only if  $f'(0) = 1$ . All higher terms of the lower central series coincide with the kernel of  $j_1: \text{Diff}(\mathbf{k}, 0) \rightarrow \text{Jets}_1(\mathbf{k}, 0)$ .*

*Proof.* If  $f$  is a commutator  $[g, h]$  then  $f'(0) = 1$ . If  $f'(0) = 1$ , compose  $f$  with the homothety  $m_\lambda(z) = \lambda z$  for some  $\lambda \in \mathbf{k}^*$  of norm  $|\lambda| \neq 1$ , and

apply Koenigs linearization theorem to find an element  $h \in \text{Diff}(\mathbf{k}, 0)$  such that  $m_\lambda \circ f = h \circ m_\lambda h^{-1}$  and  $h'(0) = 1$ . Then  $f = [h, m_\lambda]$ . This proves that the derived subgroup of  $\text{Diff}(\mathbf{k}, 0)$  is the kernel of  $j_1$ ; since  $h$  is in the kernel of  $j_1$ , all subsequent terms of the lower central series coincide with the derived subgroup.  $\square$

Now, consider the upper central series. The first terms are  $\text{Diff}(\mathbf{C}, 0)$  and its derived subgroup  $\text{Diff}(\mathbf{k}, 0)^{(1)}$ . Then comes

$$(8.2) \quad \text{Diff}(\mathbf{k}, 0)^{(2)} := [\text{Diff}(\mathbf{k}, 0)^{(1)}, \text{Diff}(\mathbf{k}, 0)^{(1)}].$$

The group of jets of order 3 which are tangent to identity, i.e., jets of the form  $j(z) = z + a_2 z^2 + a_3 z^3$  modulo  $z^4$ , is an abelian group; at the level of formal germs, it is known that the kernel of  $j_3$  in  $\widehat{\text{Diff}}(\mathbf{k}, 0)^{(1)}$  coincides with the derived subgroup  $\widehat{\text{Diff}}(\mathbf{k}, 0)^{(2)}$  (see [Cam], §3, for the description of the upper central series of  $\widehat{\text{Diff}}(\mathbf{k}, 0)$ ). We don't know if a similar statement holds for germs of diffeomorphisms:

**Question.** Does the kernel of  $j_3$  coincide with the second derived subgroup of  $\text{Diff}(\mathbf{C}, 0)^{(2)}$ ? More generally, what is the upper central series of  $\text{Diff}(\mathbf{C}, 0)$ ?

## 9. Appendix: Free groups

The following theorem, and its proof, are strongly inspired by [MRR]. The proof given in [MRR] is somewhat difficult because it makes use of a topology on  $\text{Diff}(\mathbf{C}, 0)$  which is not compatible with the group law. We adapt the same proof, without reference to such a topology.

**Theorem 9.1.** *Let  $(\mathbf{k}, |\cdot|)$  be a complete, non-discrete valued field. Let  $f$  and  $g$  be elements of  $\text{Diff}(\mathbf{k}, 0)$  of infinite order. Let  $w$  be a non-trivial element of the free group  $\mathbf{F}_2$ . Then, there is a polynomial germ of diffeomorphism  $h$  such that  $w(hfh^{-1}, g) \neq \text{Id}$ .*

*If  $w = a^{n_\ell} b^{n_{\ell-1}} \cdots a^{n_2} b^{n_1}$ , one can choose  $h$  of the form  $z + \epsilon z^2 P(z)$  with an arbitrarily small  $\epsilon$  and a polynomial function  $P \in \mathbf{k}[z]$  such that  $\deg(P) \leq (2\ell)!$  and  $|P(x)| \leq 1$  for all  $x \in \mathbb{D}_1$ .*

Before proving this result, let us introduce some vocabulary and notation. Write  $w$  as a reduced word in the generators  $a$  and  $b$  of the free group:

$$(9.1) \quad w = a^{n_\ell} b^{n_{\ell-1}} \cdots a^{n_2} b^{n_1}$$

where the  $n_i$  are in  $\mathbf{Z} \setminus \{0\}$ , except maybe if  $n_1$  or  $n_\ell$  is zero, but conjugating  $w$  by a power of  $a$ , we only need to consider the case  $n_1 n_\ell \neq 0$ . Set  $N = \max |n_i|$ .

Let  $h$  be an element of  $\text{Diff}(\mathbf{k}, 0)$ , and set  $f_h = h^{-1} \circ f \circ h$ . Let  $r > 0$  be smaller than the convergence radius of  $h$ ,  $f$ ,  $g$  and their inverses. Choose  $R > 0$  such that all these germs, and all their compositions of length  $\leq 3N\ell$  map  $\mathbb{D}_R$  inside  $\mathbb{D}_r$ . If  $z$  is a point in  $\mathbb{D}_R$ , then its orbit under the action of  $f_h$  and  $g$  stays in  $\mathbb{D}_r$  for all compositions of these germs given by words of length  $\leq N\ell$  in  $\mathbf{F}_2$ ; in this situation, we say that the orbit of  $z$  is *well defined* up to length  $N\ell$ . In particular, if we look at the composition  $w(f_h, g)$ , and pick a point  $z$  in  $\mathbb{D}_R$ , we get a sequence of points

$$(9.2) \quad z_0 = z, z_1 = g^{n_1}(z_0), z_2 = f_h^{n_2}(z_1), \dots, z_\ell = w(f_h, g)(z_0).$$

To prove the theorem, we construct a triple  $(h, R, z)$  such that the orbit of  $z$  is well defined and the  $z_i$  are pairwise distinct; in particular,  $z_\ell \neq z_0$  and  $w(f_h, g) \neq \text{Id}$ .

*Proof.* We do a recursion on the length  $\ell$ , proving the existence of a triple  $(h, R, z)$  such that the  $z_i$  are pairwise distinct for  $0 \leq i \leq \ell$ . Since  $f$  and  $g$  have infinite order, the union of all fixed points of  $f^m$  and  $g^m$  in  $\mathbb{D}_r$  for  $-N \leq m \leq N$  is a finite set  $F$ . For  $j = 1$ , we just pick a point  $z_0$  sufficiently near the origin with  $z_1 := g^{n_1}(z_0) \neq z_0$ ; the only constraint is to take  $z_0$  in the complement of  $F$ . The points  $z_0$  and  $z_1$  will be kept fixed in the recursion.

Assume that a polynomial germ of diffeomorphism  $h_k$  has been constructed, in such a way that (a) the points  $z_0, z_1, z_2, \dots, z_{2k}$ , and  $z_{2k+1}$  are pairwise distinct (we just initialized the recursion for  $k = 0$ ), and (b)  $h_k(z) = z + \epsilon_k R_k(z)$  for some small  $\epsilon_k \in \mathbf{k}$  and some element  $R_k \in \mathbf{k}[z]$  of degree  $\leq (2k)!$  which is divisible by  $z^2$ . Consider a polynomial germ

$$(9.3) \quad P_k(z) = z + \eta_k z^2 \prod_{j=0}^{2k} (z - z_j)$$

with a small  $\eta_k \in \mathbf{k}$ ; then

- $P_k$  fixes  $z_j$  for all  $j \leq 2k$ ,
- $P_k(z_{2k+1}) = a_k \eta_k + b_k$  for some pair  $(a_k, b_k) \in \mathbf{k}^2$  with  $a_k \neq 0$ ,
- as  $\eta_k$  goes to 0, the radius of convergence of  $P_k$  and its inverse  $P_k^{-1}$  go to infinity.

If we compose  $h_k$  with  $P_k$  then  $H = h_k \circ P_k$  is a new polynomial germ such that the orbit of  $z_0$  under  $f_H$  and  $g$  gives the same sequence  $z_0, z_1, \dots$ , up to  $z_{2k+1}$ . The next point is

$$(9.4) \quad z_{2k+2} = f_H^{n_{2k+2}}(z_{2k+1})$$

and we want to exclude the possibility  $z_{2k+2} \in \{z_0, \dots, z_{2k+1}\}$ ; since

$$(9.5) \quad z_{2k+2} = (P_k^{-1} \circ f_{h_k}^{n_{2k+2}} \circ P_k)(z_{2k+1})$$

we want to avoid the inclusion

$$(9.6) \quad f_{h_k}^{n_{2k+2}}(P_k(z_{2k+1})) \subset P_k\{z_0, \dots, z_{2k+1}\},$$

and for that we just need to choose the parameter  $\eta_k$  in the definition of  $P_k$  in such a way that  $f_{h_k}^{n_{2k+2}}(P_k(z_{2k+1}))$  is not in  $\{z_0, \dots, z_{2k}\}$  and  $P_k(z_{2k+1})$  is not a fixed point of  $f_{h_k}^{n_{2k+2}}$ . These constraints are satisfied for all small non-zero values of  $\eta_k$  because  $f_{h_k}^{n_{2k+2}}$  is not the identity and the coefficient  $a_k$  in  $P_k(z_{2k+1}) = a_k \eta_k + b_k$  is not zero.

The next point is  $z_{2k+3} = g^{n_{2k+3}}(z_{2k+2})$  and we want it to be disjoint from  $\{z_0, \dots, z_{2k+1}, z_{2k+2}\}$ . For this, we do a second perturbation of the conjugacy. Let

$$(9.7) \quad Q_k(z) = z + \beta_k z^2 \prod_{j=0}^{2k+1} (z - z_j)$$

with a small  $\beta_k \in \mathbf{k}$ ; then

- $Q_k$  fixes  $z_j$  for all  $j \leq 2k + 1$ ,
- $Q_k(z_{2k+2}) = c_k \beta_k + d_k$  for some pair  $(c_k, d_k) \in \mathbf{k}^2$  with  $c_k \neq 0$ ,
- as  $\beta_k$  goes to 0, the radius of convergence of  $Q_k$  and its inverse  $Q_k^{-1}$  go to infinity.

Now, we set  $h_{k+1} = Q_k \circ H$ . This does not change the sequence  $z_i$  for  $0 \leq i \leq 2k + 1$ , but the last point  $z_{2k+2}$  is replaced by  $c_k \beta_k + d_k$ . Since  $g^{n_{2k+3}} \neq \text{Id}$  and  $c_k \neq 0$  any non-zero, small enough value of  $\beta_k$  assures that  $z_{2k+3} \notin \{z_0, \dots, z_{2k+1}, z_{2k+2}\}$ .

To sum up, if we set  $h_{k+1} = Q_k \circ P_k \circ h_k$  then the sequence  $z_0, \dots, z_{2k+3}$  is now made of pairwise distinct points. Moreover, when the parameters  $\eta_k$  and  $\beta_k$  go to zero, the germ  $Q_k \circ P_k$  and its inverse converge uniformly to the identity on the disk  $\mathbb{D}_{2R}$ , so we can assume that the orbit of  $z_0$  is well defined for all composition of  $h_{k+1}$ ,  $f$ ,  $g$ , and their inverses of length  $\leq 3N\ell$ . The germ  $Q_k \circ P_k$  is equal to  $z + S_k(z)$  where  $S_k$  is divisible by  $z^2$  and  $\deg(S_k) \leq (2k+1) \times (2k+2)$ . Thus,

$$(9.8) \quad h_{k+1}(z) = z + P_{k+1}(z)$$

where  $z^2$  divides  $P_{k+1}$  and

$$(9.9) \quad \deg(P_{k+1}) \leq \deg(P_k) \times (2k+1) \times (2k+2) \leq (2k+2)!$$

This proves the recursion and finishes the proof of the theorem.  $\square$

**Theorem 9.2.** *Let  $(\mathbf{k}, |\cdot|)$  be a complete, non-discrete valued field. If  $f$  and  $g$  are elements of  $\text{Diff}(\mathbf{k}; 0)$  of infinite order, there exists an element  $h$  of  $\text{Diff}(\mathbf{k}; 0)$  such that  $f_h := h \circ f \circ h^{-1}$  and  $g$  generate a free group of rank 2. One can choose  $h$  such that  $h'(0) = 1$ .*

Note that Theorem 3.4 is a direct corollary of that result; one just need to start with  $f = \lambda_1 z$  or  $\lambda_1 z + z^2$  if  $\lambda_1$  is a root of unity, and similarly for  $g$ .

*Proof.* Denote by  $a_n$  and  $b_n$  the coefficients of  $f$  and  $g$  respectively. Let  $L \subset \mathbf{k}$  be the field generated by the  $a_n$  and  $b_n$ . Since  $\mathbf{k}$  is not discrete, it has no isolated point; being complete with no isolated point, it is uncountable (a simple consequence of Baire's theorem [Oxt]), and it follows that its transcendental degree over  $L$  is infinite: it contains an infinite sequence  $(c_i)$  of algebraically independent numbers (over the prime field of  $\mathbf{k}$ , see [Lan, Chapter VIII]). We can moreover assume that all  $c_i$  are in the unit disk. Set  $h_0(z) = \sum_{n \geq 1} c_n z^n$ .

Consider a non-trivial element  $w$  of  $\mathbf{F}_2$ . The  $N$ -th coefficient function

$$(9.10) \quad h \mapsto A_N(w(h \circ f \circ h^{-1}, g))$$

is a polynomial function on  $\widehat{\text{Diff}}(\mathbf{k}, 0)$  in the sense of Section 2.1; this means that it is a polynomial function in the coefficients of  $h$  and  $A_1(h)^{-1}$  (here,  $f$  and  $g$  are fixed). If  $A_N(w(h_0 \circ f \circ h_0^{-1}, g))$  vanishes (resp. is equal to 1), then  $A_N(w(h \circ f \circ h^{-1}, g)) = 0$  (resp. 1) for all formal diffeomorphisms  $h$ , because the  $c_i$  are algebraically independent over  $\mathbf{k}$ . Thus, Theorem 9.1 implies that  $w(h_0 \circ f \circ h_0^{-1}, g) \neq \text{Id}$ , and this shows that  $f_{h_0} := h_0 \circ f \circ h_0^{-1}$  and  $g$  generate a free group of rank 2.

In this argument, we could start with  $h_0 = z + \sum_{n \geq 2} c_n z^n$ , because we can choose the germ  $h$  in Theorem 9.1 with the additional constraint  $h'(0) = 1$ .  $\square$

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## References

- [Bau] G. BAUMSLAG, On generalised free products. *Math. Z.* **78** (1962), 423–438. Zbl 0104.24402 MR 0140562
- [BCLN] M. BERTHIER, D. CERVEAU and A. LINS NETO, Sur les feuilletages analytiques réels et le problème du centre. *J. Differential Equations* **131** (1996), 244–266. Zbl 0886.58090 MR 1419014
- [BGSS] E. BREUILLARD, T. GELANDER, J. SOUTO and P. STORM, Dense embeddings of surface groups. *Geom. Topol.* **10** (2006), 1373–1389. Zbl 1132.22011 MR 2255501
- [Bru1] A. BRUDNYI, Some algebraic aspects of the center problem for ordinary differential equations. *Qual. Theory Dyn. Syst.* **9** (2010), 9–28. Zbl 1202.37011 MR 2737360
- [Bru2] —— Subgroups of the group of formal power series with the big powers condition. Preprint arXiv:1908.04918 (2019).
- [BT] H. W. BROER and F. M. TANGERMAN, From a differentiable to a real analytic perturbation theory, applications to the Kupka Smale theorems. *Ergodic Theory Dynam. Systems* **6** (1986), 345–362. Zbl 0582.58019 MR 0863199
- [Cam] R. CAMINA, The Nottingham group. In *New horizons in pro- $p$  groups*, vol. 184 of *Progr. Math.* Birkhäuser Boston, Boston, MA (2000), pp. 205–221. Zbl 0977.20020 MR 1765121
- [Cer] D. CERVEAU, Quelques problèmes en géométrie feuilletée pour les 60 années de l’IMPA. *Bull. Braz. Math. Soc. (N.S.)* **44** (2013), 653–679. Zbl 1333.37061 MR 3167127
- [CG] C. CHAMPETIER and V. GUIRARDEL, Limit groups as limits of free groups. *Israel J. Math.* **146** (2005), 1–75. Zbl 1103.20026 MR 2151593
- [EBN] H. EYNARD-BONTEMPS and A. NAVAS, Mather invariant, distortion, and conjugates for diffeomorphisms of the interval. Preprint, arXiv:1912.09305 (2019), 1–59.
- [Gla] A. M. W. GLASS, The ubiquity of free groups. *Math. Intelligencer* **14** (1992), 54–57. Zbl 0791.20001 MR 1184319
- [HY] M. HERMAN and J.-C. Yoccoz, Generalizations of some theorems of small divisors to non-Archimedean fields. In *Geometric dynamics (Rio de Janeiro, 1981)*, vol. 1007 of *Lecture Notes in Math.* Springer, Berlin (1983). pp. 408–447. Zbl 0528.58031 MR 0730280
- [IP] Y. S. ILYASHENKO and A. S. PYARTLI, The monodromy group at infinity of a generic polynomial vector field on the complex projective plane. *Russian J. Math. Phys.* **2** (1994), 275–315. Zbl 0915.34005 MR 1330871
- [Jen] S. A. JENNINGS, Substitution groups of formal power series. *Canadian J. Math.* **6** (1954), 325–340. Zbl 0058.02201 MR 0061610
- [Koe] G. KOENIGS, Nouvelles recherches sur les équations fonctionnelles. *Ann. Sci. École Norm. Sup. (3)* **2** (1885), 385–404. Zbl 17.0370.02 MR 1508770

[Lan] S. LANG, *Algebra*, vol. 211 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, third ed. (2002). Zbl 0984.00001 MR 1878556

[Les] J. LESLIE, On the group of real analytic diffeomorphisms of a compact real analytic manifold. *Trans. Amer. Math. Soc.* **274** (1982), 651–669. Zbl 0513.58017 MR 0675073

[Lyn] R. C. LYNDON, The equation  $a^2b^2 = c^2$  in free groups. *Michigan Math. J.* **6** (1959), 89–95. <http://projecteuclid.org/euclid.mmj/1028998143> Zbl 0084.02803 MR 0103218

[Mil] J. MILNOR, *Dynamics in One Complex Variable*. Vol. 160 of *Annals of Mathematics Studies*. Princeton University Press, Princeton, NJ, third ed. (2006). Zbl 1085.30002 MR 2193309

[MRR] J.-F. MATTEI, J. C. REBELO and H. REIS, Generic pseudogroups on  $(\mathbb{C}, 0)$  and the topology of leaves. *Compos. Math.* **149** (2013), 1401–1430. Zbl 1314.32033 MR 3103071

[MS] L. MARQUIS and J. SOUTO, Surface groups of diffeomorphisms of the interval. *Israel J. Math.* **227** (2018), 379–396. Zbl 06976582 MR 3846328

[Ols] A. Y. OL'SHANSKIĬ, On residualing homomorphisms and  $G$ -subgroups of hyperbolic groups. *Internat. J. Algebra Comput.* **3** (1993), 365–409. MR 1250244

[Oxt] J. C. OXToby, *Measure and Category*. Vol. 2 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, Berlin, second ed. (1980). A survey of the analogies between topological and measure spaces. Zbl 0435.28011 MR 0584443

[Sie] C. L. SIEGEL, Iteration of analytic functions. *Ann. of Math.* (2) **43** (1942), 607–612. Zbl 0061.14904 MR 0007044

[Ste] S. STERNBERG, On the structure of local homeomorphisms of euclidean  $n$ -space. *Amer. J. Math.* **80** (1958), 623–631. Zbl 0083.31406 MR 0096854

[Sze] B. SZEGEDY, Almost all finitely generated subgroups of the Nottingham group are free. *Bull. London Math. Soc.* **37** (2005), 75–79. Zbl 1073.20022 MR 2105821

[Tak] F. TAKENS, Normal forms for certain singularities of vector fields. *Ann. Inst. Fourier* **23** (1973), 163–195. Zbl 0266.34046 MR 0365620

[Wal] M. WALDSCHMIDT, Independance algébrique des nombres de liouville. *Lecture Notes in Math.* **1415** (1990), 225–235. Zbl 0729.11035 MR 1044117

[Whi] S. WHITE, The group generated by  $x \mapsto x + 1$  and  $x \mapsto x^p$  is free. *J. Algebra* **118** (1988), 408–422. Zbl 0662.20024 MR 0969681

[Yoc] J.-C. Yoccoz, Centralisateurs et conjugaison différentiable des difféomorphismes du cercle. 231 (1995). pp. 89–242. Petits diviseurs en dimension 1. MR 1367354

Serge CANTAT, Univ Rennes, CNRS,  
IRMAR – UMR 6625, F-35000 Rennes, France  
*e-mail:* serge.cantat@univ-rennes1.fr

Dominique CERVEAU, Univ Rennes, CNRS,  
IRMAR – UMR 6625, F-35000 Rennes, France  
*e-mail:* dominique.cerveau@univ-rennes1.fr

Vincent GUIRARDEL, Univ Rennes, CNRS,  
IRMAR – UMR 6625, F-35000 Rennes, France  
*e-mail:* vincent.guirardel@univ-rennes1.fr

Juan SOUTO, Univ Rennes, CNRS,  
IRMAR – UMR 6625, F-35000 Rennes, France  
*e-mail:* juan.souto@univ-rennes1.fr