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## A note on the $G$ -Sarkisov program

Enrica FLORIS

**Abstract.** The purpose of this note is to prove the  $G$ -equivariant Sarkisov program for a connected algebraic group  $G$  following the proof of the Sarkisov program by Hacon and McKernan. As a consequence, we obtain a characterisation of connected subgroups of  $Bir(Z)$  acting rationally on  $Z$ .

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**Keywords.** Sarkisov program, Cremona group.

### 1. Introduction

Let  $Z$  be a normal projective and  $\mathbb{Q}$ -factorial variety. Assume  $Z$  is terminal (we refer to Definition 2.1 for the notion of terminality). There is a divisor canonically attached to  $Z$  called the canonical divisor and denoted by  $K_Z$ . One of the main ideas of birational geometry is that the behaviour of  $K_Z$  should determine the geometric properties of  $Z$ .

A cornerstone result in this direction is the Cone and Contraction theorem, due to Mori. It describes the curves which have negative intersection with  $K_Z$  and says that they can be contracted.

The  $K_Z$ -minimal model program, or  $K_Z$ -MMP, is a sequence of elementary birational maps

$$Z \dashrightarrow Z_1 \dashrightarrow Z_2 \dashrightarrow \dots$$

which are elementary, in the sense that they are defined using the simplest contractions given by the Cone and Contraction theorem, called *extremal contractions*. The process is expected to end and there are two possible outcomes depending on the geometry of  $Z$ . Either the outcome is a variety  $X$  with nef canonical bundle, that is, such that  $K_X \cdot C \geq 0$  for every curve  $C \subseteq X$ ; or  $X$  is a Mori fibre space, that is, there is a fibration  $\phi: X \rightarrow T$  such that  $\rho(X) = \rho(T) + 1$  and  $-K_X$  restricted to the fibres of  $\phi$  is ample.

Due to the development of the Minimal Model Program, the objects considered are, more than varieties, *pairs*. A pair  $(Z, \Phi)$  is the data of a normal variety  $Z$  and a  $\mathbb{Q}$ -Weil divisor  $\Phi$  such that  $K_Z + \Phi$ , called the *log-canonical divisor*, is  $\mathbb{Q}$ -Cartier. The use of pairs originally comes from compactifications. By Hironaka's desingularization theorem, if  $Z^0$  is a non-compact smooth variety, there is a smooth compact variety  $Z \supseteq Z^0$  such that  $Z \setminus Z^0$  is a simple normal crossing divisor. The datum of  $Z^0$  is then replaced with  $(Z, Z \setminus Z^0)$ . This is the reason why the Weil divisor  $\Phi$  is called the *boundary*. Pairs have many advantages. One of them is that they make the canonical divisor *functorial*. For instance, it is not true that for a hypersurface  $Y \subseteq Z$  we have  $K_Z|_Y = K_Y$ , but it is true for pairs and log-canonical divisors. Indeed, there are two boundaries  $\Phi_Z$  on  $Z$  and  $\Phi_Y$  on  $Y$  such that  $(K_Z + \Phi_Z)|_Y = K_Y + \Phi_Y$ , it is sufficient to set  $\Phi_Z = Y$  and  $\Phi_Y = 0$ .

More generally, if  $f: Y \rightarrow Z$  is a morphism, it is not true that  $K_Y = f^*K_Z$ , but it is in many cases for log-canonical divisors. For instance, if  $f: Y \rightarrow Z$  is a  $\mathbb{P}^n$ -bundle, it is possible to find boundaries  $\Phi_Z$  on  $Z$  and  $\Phi_Y$  on  $Y$  such that  $K_Y + \Phi_Y = f^*(K_Z + \Phi_Z)$ .

Moreover, pairs are a crucial tool in the proof of many important results, like Shokurov's base-point-freeness, where it is necessary to be able to modify/perturb the coefficients of  $\Phi$ , even in the case  $\Phi = 0$ .

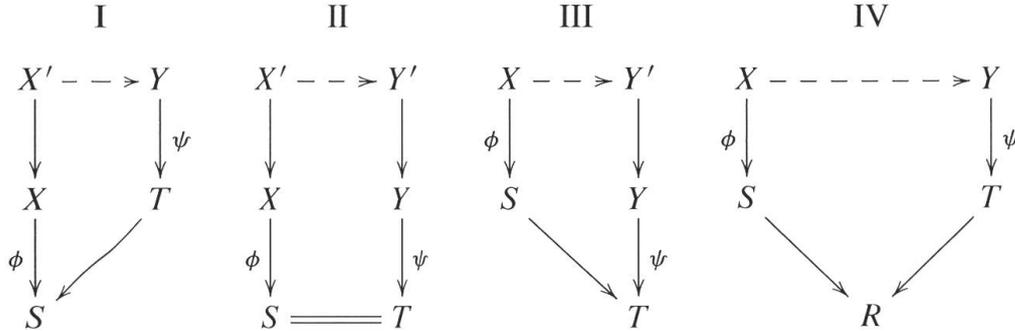
Another nice feature of pairs is that they encode singularities. If  $Z$  is normal but not smooth, it is possible to define the canonical divisor as a Weil divisor. Indeed, on the smooth locus  $Z^{sm}$  of  $Z$  we have  $K_{Z^{sm}} = \sum a_i D_i$  and, since  $\text{codim}_Z(Z \setminus Z^{sm}) \geq 2$ , there is a unique Weil divisor on  $Z$  whose restriction is  $K_{Z^{sm}}$ , namely  $K_Z = \sum a_i \overline{D}_i^{Zar}$ . The divisor  $K_Z$  is in general not  $\mathbb{Q}$ -Cartier and  $\Phi$  encodes the lack of "Cartier-ness" of  $K_Z$ . Moreover there is a way of measuring the singularities of  $(Z, \Phi)$  by looking at the coefficients of  $\Phi$ . We refer to Definition 2.1 for the notion of klt, which is a singularity notion defined in this way. The Cone and Contraction theorem holds for a klt pair and we can run the  $K_Z + \Phi$ -minimal model program.

It is still an open problem whether the  $K_Z + \Phi$ -MMP stops in general, even when  $\Phi = 0$ , but there is an important class of varieties for which it is known to terminate, the ones which are covered by curves which have negative intersection with  $K_Z + \Phi$ . We say in this case that  $K_Z + \Phi$  is *not pseudoeffective*. Varieties with this property are uniruled. In this case, after a finite number of steps, we find a Mori fibre space by [BCHM].

The outcome is nevertheless far from being unique. The Sarkisov program describes the relation between two different Mori fibre spaces that are outcomes of two MMP on the same variety:

**Theorem 1.1** ([HM], [Cor]). *Suppose that  $\phi: X \rightarrow S$  and  $\psi: Y \rightarrow T$  are two Mori fibre spaces. Then  $X$  and  $Y$  are birational if and only if they are related by a sequence of Sarkisov links.*

A Sarkisov link between  $\phi: X \rightarrow S$  and  $\psi: Y \rightarrow T$  is a diagram of one of the four following types:



The vertical arrows in the diagrams are extremal contractions; the arrows to  $X$  or  $Y$  are extremal divisorial contractions. The horizontal dotted arrows are isomorphisms in codimension one which are a composition of flops with respect to a suitable boundary. A flop is a composition of birational maps  $h_2^{-1} \circ h_1: X_1 \dashrightarrow X_2$  with  $h_i: X_i \rightarrow \bar{X}$  such that the exceptional loci of  $h_1$  and  $h_2$  have codimension at least two,  $\rho(X_i) = \rho(\bar{X}) + 1$  and there are divisors  $D_i$  on  $X_i$  such that  $(X_i, D_i)$  is klt and  $K_{X_i} + D_i$  has zero intersection with every curve contracted by  $\phi_i$ .

Theorem 1.1 is another case where pairs are used in a fundamental way. Indeed, if  $X_1/S_1$  and  $X_2/S_2$  are two Mori fibre spaces birational to each other, there is a variety  $Z$  and two boundaries  $\Theta_1$  and  $\Theta_2$  on  $Z$  such that  $K_Z + \Theta_i$  is semiample on  $Z$  and induces a morphism  $Z \rightarrow X_i$ . Roughly speaking, the possible birational models are encoded by a combinatorial object (a polytope) in terms of boundaries on  $Z$ .

The purpose of this note is to give a proof of Theorem 1.1 in the  $G$ -equivariant setting. We prove that two Mori fibre spaces that are outcomes of two  $G$ -equivariant MMP on the same  $G$ -pair are related by a sequence of  $G$ -equivariant Sarkisov links.

**Theorem 1.2.** *Let  $G$  be a connected algebraic group. Let  $(Z, \Phi)$  be a klt pair such that  $G < \text{Aut}^0(Z)$  and  $G$  leaves  $\Phi$  invariant. Let  $\phi: X \rightarrow S$  and  $\psi: Y \rightarrow T$  be two Mori fibre spaces obtained from  $(Z, \Phi)$  via a  $G$ -equivariant MMP. Then  $X$  and  $Y$  are related by a sequence of  $G$ -equivariant Sarkisov links and in every such link the horizontal dotted arrows are compositions of  $G$ -equivariant flops with respect to a suitable boundary.*

In dimension 3 and for  $\Phi = 0$  the interested reader can check that the proof in [Cor] is “ $G$ -equivariant”.

As a direct consequence we obtain the following characterisation of subgroups of  $Bir(W)$  that are maximal among the connected groups acting rationally on  $W$ .

**Corollary 1.3.** *Let  $W$  be an uniruled variety and let  $G$  be a connected algebraic group acting rationally on  $W$ . Then  $G$  is maximal among the connected groups acting rationally on  $W$  if and only if  $G = Aut^0(X)$  where  $\phi: X \rightarrow S$  is a Mori fibre space and for every Mori fibre space  $\psi: Y \rightarrow T$  which is related to  $\phi$  by a finite sequence of  $G$ -Sarkisov links we have  $G = Aut^0(Y)$ .*

We have the following application to subgroups of the Cremona group.

**Corollary 1.4.** *Let  $G$  be a connected algebraic group acting rationally on  $\mathbb{P}^n$ . Then  $G$  is maximal among the connected groups acting rationally on  $\mathbb{P}^n$  if and only if  $G = Aut^0(X)$  where  $\phi: X \rightarrow S$  is a rational Mori fibre space and for every Mori fibre space  $\psi: Y \rightarrow T$  which is related to  $\phi$  by a finite sequence of  $G$ -Sarkisov links we have  $G = Aut^0(Y)$ .*

Theorem 1.2 is an essential ingredient in [BFT, Theorem E], the classification of automorphism groups of rational Mori fibre spaces of dimension 3. This result implies that there is a finite number of conjugacy classes of maximal connected algebraic subgroups of  $Bir(\mathbb{P}^3)$  acting rationally on  $\mathbb{P}^3$ .

## 2. MMP and group actions

We refer to [KM] for the definitions of singularities of pairs, of MMP and of Mori fibre space and to [DL] for the definition of rational and regular action.

We give now the definition of pairs and terminal and klt singularities:

**Definition 2.1.** A pair  $(Z, \Phi)$  is the data of a normal variety  $Z$  and a  $\mathbb{Q}$ -divisor  $\Phi$  such that  $K_Z + \Phi$  is  $\mathbb{Q}$ -Cartier. A pair is said to be kawamata log terminal, or klt, if for any birational morphism  $\mu: \tilde{Z} \rightarrow Z$  with  $\tilde{Z}$  smooth we have  $K_{\tilde{Z}} = \mu^*(K_Z + \Phi) + \sum a_E E$  with  $a_E > -1$  for all  $E$ .

A variety  $Z$  is said to be terminal if for any birational morphism  $\mu: \tilde{Z} \rightarrow Z$  with  $\tilde{Z}$  smooth we have  $K_{\tilde{Z}} = \mu^*K_Z + \sum a_E E$  with  $a_E > 0$  for all  $E \subseteq Exc(\mu)$ .

**Definition 2.2.** We call a pair  $(Z, \Phi)$  a  $G$ -pair if  $G$  acts on  $Z$  regularly and for all  $g \in G$  we have  $g \cdot \Phi = \Phi$ .

**Remark 2.3.** The pair  $(Z, 0)$  is a  $G$ -pair for every subgroup  $G$  of  $Aut(Z)$ .

**Remark 2.4.** In the next lemma we prove that if  $(Z, \Phi)$  is a  $G$ -pair with  $G$  connected, then every MMP on  $(Z, \Phi)$  is a  $G$ -MMP. The situation is thus very different from the one where the group  $G$  is finite. Indeed, in the connected case, the action of  $G$  on the polyhedral part of  $NE(X)$  is trivial, while a finite group can act in a non-trivial way on  $NE(X)$  and one has to consider the  $G$ -invariant part  $NE(X)^G$ .

An easy example which illustrates well this difference is  $(\mathbb{P}^1 \times \mathbb{P}^1, 0)$  with the action of  $\mathbb{Z}/2\mathbb{Z}$  exchanging the two factors. We have  $NE(\mathbb{P}^1 \times \mathbb{P}^1)^{\mathbb{Z}/2\mathbb{Z}} = \mathbb{R}_+[l_1 + l_2]$  where  $l_1$  and  $l_2$  are fibres of the projections. The only possible  $\mathbb{Z}/2\mathbb{Z}$ -invariant contraction is  $\mathbb{P}^1 \times \mathbb{P}^1 \rightarrow \{p\}$ . If we consider instead the diagonal action of  $PGL(2)$ , we have  $NE(\mathbb{P}^1 \times \mathbb{P}^1)^{PGL(2)} = \mathbb{R}_+[l_1] + \mathbb{R}_+[l_2]$  and there are two extremal contractions, the two projections onto the two factors. We refer to [KM] and [C<sup>+</sup>] for a discussion of the case where  $G$  is finite, but also to the classical [Man] and to the work of Prokhorov (e.g., [Pro]).

Let  $G$  be a connected algebraic group. If we have a klt  $G$ -pair  $(Z, \Phi)$ , we can run a  $G$ -equivariant MMP on  $(Z, \Phi)$ , that is, an MMP where all the birational maps, divisorial contractions and flips, are compatible with the action of the group.

**Lemma 2.5.** *Let  $G$  be a connected group and  $(Z, \Phi)$  a  $G$ -pair. Then any MMP on  $(Z, \Phi)$  is  $G$ -equivariant.*

*Proof.* Indeed let  $Z \dashrightarrow Z_1$  be the first step of the MMP. If it is an extremal contraction, then by the Blanchard's lemma [BSU, Prop 4.2.1] there is an induced action on  $Z_1$  making the map  $G$ -equivariant. Equivalently, a connected group acts trivially on the extremal rays contained in the  $(K_Z + \Phi)$ -negative part of the Mori cone, that are discrete. Then the extremal ray corresponding to the contraction  $Z \rightarrow Z_1$  is  $G$ -invariant and so is the locus spanned by it.

Assume that the first step is a flip given by the composition of two small contractions  $\mu: Z \rightarrow Y$  and  $\mu^+: Z_1 \rightarrow Y$ . By the discussion above, there is an action on  $Y$  making  $\mu$   $G$ -equivariant. Moreover,  $Z_1 \cong Proj_Y \bigoplus_m \mu_* \mathcal{O}_X(m(K_Z + \Phi))$ . Since  $K_Z + \Phi$  is  $G$ -invariant, the group  $G$  acts on  $\mathcal{O}_X(m(K_Z + \Phi))$  for  $m$  sufficiently divisible, and subsequently on  $Z_1$ .  $\square$

If  $K_Z + \Phi$  is not pseudoeffective, by [BCHM] the outcome of such an MMP is a Mori fibre space  $\phi: X \rightarrow S$  with a regular action of  $G$  on  $X$ .

**Definition 2.6.**

- A *Mori fibre space* is the data of a fibration  $\phi: X \rightarrow S$  such that  $X$  is terminal,  $\rho(X) - \rho(S) = 1$  and a boundary  $\Delta$  on  $X$  such that  $(X, \Delta)$  is klt and  $-(K_X + \Delta)$  is  $\phi$ -ample.
- We will call two Mori fibre spaces  $\phi: X \rightarrow S$  and  $\psi: Y \rightarrow T$  *Sarkisov related* if  $X$  and  $Y$  are outcomes of running the  $K_Z + \Phi$ -MMP, for the same  $\mathbb{Q}$ -factorial klt pair  $(Z, \Phi)$ .
- If  $G$  is a connected group, a  *$G$ -Mori fibre space* is a Mori fibre space with a regular action of a group  $G$ .
- We say that two  $G$ -Mori fibre spaces  $\phi: X \rightarrow S$  and  $\psi: Y \rightarrow T$  are  *$G$ -Sarkisov related* if  $X$  and  $Y$  are results the  $G$ -equivariant MMP on  $(Z, \Phi)$ , for the same  $\mathbb{Q}$ -factorial klt  $G$ -pair  $(Z, \Phi)$ .

The following remark will be useful in the next section.

**Remark 2.7.** If there is a commutative diagram

$$\begin{array}{ccc}
 & Z & \\
 f_1 \swarrow & & \searrow f_2 \\
 X_1 & & X_2 \\
 g_1 \searrow & & \swarrow g_2 \\
 & Y &
 \end{array}$$

where  $f_i$  is birational and  $G$ -equivariant for  $i = 1, 2$  and  $g_i$  is a morphism for  $i = 1, 2$ , then, by the Blanchard's lemma [BSU, Prop 4.2.1] there are two actions of  $G$  on  $Y$ . These actions coincide. Indeed, they coincide on an open set contained in the image of the open set where  $f_2 \circ f_1^{-1}$  is an isomorphism.

**Remark 2.8.** If  $G$  is connected and  $X$  is uniruled, or if  $G$  is rationally connected, then all the complete linear systems are  $G$ -invariant, although the individual divisors might not be left invariant by the action of  $G$ .

### 3. Proof of Theorem 1.2

In this section we present a proof of Theorem 1.2. Let  $Z$  be a projective variety. From now on, we assume the setup of [HM, Section 3].

**Definition 3.1.** [BCHM] Let  $V$  be a finite dimensional affine subspace of the real vector space  $WDiv_{\mathbb{R}}(Z)$  of Weil divisors on  $Z$ , which is defined over the rationals, and  $A \geq 0$  an ample  $\mathbb{Q}$ -divisor on  $Z$ .

$$\begin{aligned} \mathcal{L}_A(V) &= \{\Theta = A + B \mid B \in V, B \geq 0, K_Z + \Theta \text{ is log canonical}\}, \\ \mathcal{E}_A(V) &= \{\Theta \in \mathcal{L}_A(V) \mid K_Z + \Theta \text{ is pseudo-effective}\}. \end{aligned}$$

As in [HM], we assume that there exists  $B_0 \in V$  such that  $K_Z + \Theta_0 = K_Z + A + B_0$  is big and klt. Given a rational contraction  $f: Z \dashrightarrow X$ , define

$$\mathcal{A}_{A,f}(V) = \{\Theta \in \mathcal{E}_A(V) \mid f \text{ is the ample model of } (Z, \Theta)\}.$$

In addition, let  $\mathcal{C}_{A,f}(V)$  denote the closure of  $\mathcal{A}_{A,f}(V)$ . We refer to [HM, Definition 3.1] for the definition of ample model. We recall the following result from [HM] for the benefit of the reader.

**Theorem 3.2.** [HM, Theorem 3.3] *There is a natural number  $m$  and there are  $f_i: Z \dashrightarrow X_i$  rational contractions  $1 \leq i \leq m$  with the following properties:*

- (1)  $\{\mathcal{A}_i = \mathcal{A}_{A,f_i} \mid 1 \leq i \leq m\}$  is a partition of  $\mathcal{E}_A(V)$ .  $\mathcal{A}_i$  is a finite union of relative interiors of rational polytopes. If  $f_i$  is birational, then  $\mathcal{C}_i = \mathcal{C}_{A,f_i}$  is a rational polytope.
- (2) If  $1 \leq i \leq m$  and  $1 \leq j \leq m$  are two indices such that  $\mathcal{A}_j \cap \mathcal{C}_i \neq \emptyset$ , then there is a contraction morphism  $f_{ij}: X_i \rightarrow X_j$  and a factorisation  $f_j = f_{ij} \circ f_i$ .

Now suppose in addition that  $V$  spans the Néron–Severi group of  $Z$ .

- (3) Pick  $1 \leq i \leq m$  such that a connected component  $\mathcal{C}$  of  $\mathcal{C}_i$  intersects the interior of  $\mathcal{L}_A(V)$ . The following are equivalent:
  - $\mathcal{C}$  spans  $V$ .
  - $f_i$  is birational and  $X_i$  is  $\mathbb{Q}$ -factorial.

The following is a  $G$ -equivariant version of [HM, Lemma 4.1].

**Lemma 3.3.** *Let  $G$  be a connected group. Let  $\phi: X \rightarrow S$  and  $\psi: Y \rightarrow T$  be two  $G$ -Sarkisov related  $G$ -Mori fibre spaces corresponding to two  $\mathbb{Q}$ -factorial klt projective  $G$ -pairs  $(X, \Delta)$  and  $(Y, \Gamma)$ . Then we may find a smooth projective variety  $Z$  with a regular action of  $G$ , two birational  $G$ -equivariant contractions  $f: Z \dashrightarrow X$  and  $g: Z \dashrightarrow Y$ , a klt  $G$ -pair  $(Z, \Phi)$ , an ample  $\mathbb{Q}$ -divisor  $A$  on  $Z$  and a two-dimensional rational affine subspace  $V$  of  $WDiv_{\mathbb{R}}(Z)$  such that*

- (1) if  $\Theta \in \mathcal{L}_A(V)$  then  $\Theta - \Phi$  is ample,

- (2)  $\mathcal{A}_{A,\phi \circ f}$  and  $\mathcal{A}_{A,\psi \circ g}$  are not contained in the boundary of  $\mathcal{L}_A(V)$ ,
- (3)  $V$  satisfies (1-4) of Theorem 3.2,
- (4)  $\mathcal{C}_{A,f}$  and  $\mathcal{C}_{A,g}$  are two dimensional, and
- (5)  $\mathcal{C}_{A,\phi \circ f}$  and  $\mathcal{C}_{A,\psi \circ g}$  are one dimensional.

*Proof.* By assumption we may find a  $\mathbb{Q}$ -factorial klt  $G$ -pair  $(Z, \Phi)$  such that  $f: Z \dashrightarrow X$  and  $g: Z \dashrightarrow Y$  are both outcomes of the  $G$ -equivariant MMP on  $(Z, \Phi)$ . Let  $p: W \rightarrow Z$  be any  $G$ -equivariant log resolution of  $(Z, \Phi)$  which resolves the indeterminacy of  $f$  and  $g$ . Such pair exists for instance by [Kol, Proposition 3.9.1, Theorem 3.35, Theorem 3.36]. The rest of the proof goes as in [HM, Lemma 4.1].  $\square$

*Proof of Theorem 1.2.* The proof follows the same lines as [HM, Theorem 1.3] but instead of choosing  $(Z, \Phi)$  as in [HM, Lemma 4.1] we choose the  $G$ -pair given by Lemma 3.3.

We prove now that any  $X_i$  as in Theorem 3.2 carries a regular action of  $G$  making  $f_i$   $G$ -equivariant. Since  $\Theta \in \mathcal{L}_A(V)$  implies  $\Theta - \Phi$  is ample, by Theorem 3.2(3) and [HM, Lemma 3.6], for every  $X_i$  corresponding to a full-dimensional polytope of  $\mathcal{A}_i$ , the variety  $X_i$  is the result of an MMP on  $(Z, \Phi)$ . By Lemma 2.5 this MMP is  $G$ -equivariant and therefore there is a regular action of  $G$  on  $X_i$ .

Let  $X_j$  be a variety corresponding to a non full-dimensional polytope  $\mathcal{A}_j$ . Let  $\mathcal{A}_i$  be a full-dimensional polytope such that  $\mathcal{C}_i \cap \mathcal{A}_j \neq \emptyset$ . By Theorem 3.2 there is a surjective morphism  $f_{ij}: X_i \rightarrow X_j$ . By the Blanchard's lemma [BSU, Prop 4.2.1] there is an action of  $G$  on  $X_j$  making the morphism  $f_{ij}$   $G$ -equivariant. By Remark 2.7 this action does not depend on the choice of  $i$ .

The links given by [HM, Theorem 3.7] are  $G$ -equivariant. Indeed the maps appearing in the links are either

- morphisms of the form  $f_{ij}$  as in Theorem 3.2, and those are  $G$ -equivariant by the discussion above; or
- flops (with respect to a suitable boundary) of the form  $f_{ij} \circ f_{kj}^{-1}$ , and those are again  $G$ -equivariant.  $\square$

#### 4. Proof of Corollary 1.3

**Definition 4.1.** A connected subgroup  $G < \text{Bir}(Z)$  is not maximal among the connected groups acting rationally on  $Z$  if there is a connected subgroup of  $\text{Bir}(Z)$  acting rationally on  $Z$  such that  $G \subsetneq H$ . It is *maximal among the connected groups acting rationally on  $Z$*  otherwise. We will say *maximal* for short.

*Proof of Corollary 1.3.* By a theorem of Weil [Wei] (see also [Kra]) there is a birational model  $\widetilde{W}$  of  $W$ , such that  $G$  acts regularly on  $\widetilde{W}$ . We then run a  $G$ -equivariant MMP on  $\widetilde{W}$  (see Lemma 2.5) and by [BDPP] and [BCHM] the result is a  $G$ -Mori fibre space  $\phi: X \rightarrow S$ .

We prove now that  $G$  is maximal if and only if for every Mori fibre space  $\psi: Y \rightarrow T$  which is related to  $\phi$  by a finite sequence of  $G$ -Sarkisov links we have  $G = \text{Aut}^0(Y)$ .

Assume that  $G$  is maximal. Let  $X/S \dashrightarrow Y/T$  be a composition of  $G$ -Sarkisov links and let  $\varphi: X \dashrightarrow Y$  be the corresponding birational map. Then  $\varphi G \varphi^{-1} \subseteq \text{Aut}^0(Y)$  and if  $G$  is maximal  $\varphi G \varphi^{-1} = \text{Aut}^0(Y)$ .

Assume that  $G$  is not maximal and let  $H$  be a connected subgroup of  $\text{Bir}(Z)$  acting rationally on  $Z$  and such that  $G \subsetneq H$ . By a theorem of Weil [Wei] there is a birational model  $\widehat{W}$  of  $\widetilde{W}$ , such that  $H$  acts regularly on  $\widehat{W}$ . We run an  $H$ -equivariant MMP on  $\widehat{W}$  and obtain a Mori fibre space  $Y \rightarrow T$  such that  $H < \text{Aut}^0(Y)$ . This MMP is also  $G$ -equivariant. Therefore  $X \rightarrow S$  and  $Y \rightarrow T$  are  $G$ -Sarkisov related. By Theorem 1.2, there is a finite sequence of  $G$ -Sarkisov links  $X \rightarrow S \dashrightarrow Y \rightarrow T$ .  $\square$

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## References

- [BCHM] C. BIRKAR, P. CASCINI, C. D. HACON and J. MCKERNAN, Existence of minimal models for varieties of log general type. *J. Amer. Math. Soc.* **23** (2010), 405–468. Zbl1210.14019 MR 2601039
- [BDPP] S. BOUCKSOM, J.-P. DEMAILLY, M. P  AUN and T. PETERNELL, The pseudo-effective cone of a compact K  hler manifold and varieties of negative Kodaira dimension. *J. Algebraic Geom.* **22** (2013), 201–248. Zbl1267.32017 MR 3019449
- [BFT] J. BLANC, A. FANELLI and R. TERPEREAU, Connected algebraic groups acting on 3-dimensional mori fibrations. arXiv:1912.11364

- [BSU] M. BRION, P. SAMUEL and V. UMA, *Lectures on the Structure of Algebraic Groups and Geometric Applications*. Vol. 1 of CMI Lecture Series in Mathematics. Hindustan Book Agency, New Delhi; Chennai Mathematical Institute (CMI), Chennai (2013). Zbl1326.14001 MR 3088271
- [C<sup>+</sup>] D. CHAN, K. CHAN, L. DE THANHOFFER DE VÖLCSEY, C. INGALLS, K. JABBUSCH, S. J. KOVÁCS, R. KULKARNI, B. LERNER, B. NANAYAKKARA, S. OKAWA and M. VAN DEN BERGH, The minimal model program for b-log canonical divisors and applications. arXiv:1707.00834 (2017).
- [Cor] A. CORTI, Factoring birational maps of threefolds after Sarkisov. *J. Algebraic Geom.* **4** (1995), 223–254. Zbl0866.14007 MR 1311348
- [DL] A. DUBOULOZ and A. LIENDO, Rationally integrable vector fields and rational additive group actions. *Internat. J. Math.* **27** (2016), 1650060, 19. Doi 10.1142/S0129167X16500609. Zbl1362.14015 MR 3530281
- [HM] C. D. HACON and J. MCKERNAN, The Sarkisov program. *J. Algebraic Geom.* **22** (2013), 389–405. Doi 10.1090/S1056-3911-2012-00599-2. Zbl1267.14024 MR 3019454
- [KM] J. KOLLÁR and S. MORI, *Birational geometry of algebraic varieties*, vol. 134 of Cambridge Tracts in Mathematics. Cambridge University Press, Cambridge (1998). Doi 10.1017/CBO9780511662560. With the collaboration of C. H. Clemens and A. Corti, Translated from the 1998 Japanese original. Zbl0926.14003 MR 1658959
- [Kol] J. KOLLÁR, *Lectures on Resolution of Singularities*, vol. 166 of Annals of Mathematics Studies. Princeton University Press, Princeton, NJ (2007). Zbl1113.14013 MR 2289519
- [Kra] H. KRAFT, Regularization of rational group actions. arXiv:1808.08729 (2018).
- [Man] J. I. MANIN, Rational surfaces over perfect fields. II. *Mat. Sb. (N.S.)* **72** (1967), 161–192. Zbl0182.23701 MR 0225781
- [Pro] Y. PROKHOROV, *G*-Fano threefolds, II. *Adv. Geom.* **13** (2013), 419–434. Doi 10.1515/advgeom-2013-0009. Zbl1291.14025 MR 3100918
- [Wei] A. WEIL, On algebraic groups of transformations. *Amer. J. Math.* **77** (1955), 355–391. Doi 10.2307/2372535. Zbl0065.14201 MR 0074083

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