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A note on the G-Sarkisov program

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Abstract. The purpose of this note is to prove the G-equivariant Sarkisov program for a connected algebraic group G following the proof of the Sarkisov program by Hacon and McKernan. As a consequence, we obtain a characterisation of connected subgroups of $Bir(Z)$ acting rationally on Z.

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1. Introduction

Let Z be a normal projective and $\mathbb{Q}$-factorial variety. Assume Z is terminal (we refer to Definition 2.1 for the notion of terminality). There is a divisor canonically attached to Z called the canonical divisor and denoted by K_Z. One of the main ideas of birational geometry is that the behaviour of K_Z should determine the geometric properties of Z.

A cornerstone result in this direction is the Cone and Contraction theorem, due to Mori. It describes the curves which have negative intersection with K_Z and says that they can be contracted.

The K_Z*-minimal model program*, or K_Z-MMP, is a sequence of elementary birational maps

$$Z \dashrightarrow Z_1 \dashrightarrow Z_2 \dashrightarrow \dots$$

which are elementary, in the sense that they are defined using the simplest contractions given by the Cone and Contraction theorem, called *extremal contractions*. The process is expected to end and there are two possible outcomes depending on the geometry of Z. Either the outcome is a variety X with nef canonical bundle, that is, such that $K_X \cdot C \geq 0$ for every curve $C \subseteq X$; or X is a Mori fibre space, that is, there is a fibration $\phi\colon X \to T$ such that $\rho(X) = \rho(T) + 1$ and $-K_X$ restricted to the fibres of ϕ is ample.

Due to the development of the Minimal Model Program, the objects considered are, more than varieties, *pairs*. A pair (Z, Φ) is the data of a normal variety Z and a $\mathbb{Q}$-Weil divisor Φ such that $K_Z + \Phi$, called the *log-canonical divisor*, is $\mathbb{Q}$-Cartier. The use of pairs originally comes from compactifications. By Hironaka's desingularization theorem, if Z^0 is a non-compact smooth variety, there is a smooth compact variety $Z \supseteq Z^0$ such that $Z \setminus Z^0$ is a simple normal crossing divisor. The datum of Z^0 is then replaced with $(Z, Z \setminus Z^0)$. This is the reason why the Weil divisor Φ is called the *boundary*. Pairs have many advantages. One of them is that they make the canonical divisor *functorial*. For instance, it is not true that for a hypersurface $Y \subseteq Z$ we have $K_Z|_Y = K_Y$, but it is true for pairs and log-canonical divisors. Indeed, there are two boundaries Φ_Z on Z and Φ_Y on Y such that $(K_Z + \Phi_Z)|_Y = K_Z + \Phi_Y$, it is sufficient to set $\Phi_Z = Y$ and $\Phi_Y = 0$.

More generally, if $f : Y \to Z$ is a morphism, it is not true that $K_Y = f^* K_Z$, but it is in many cases for log-canonical divisors. For instance, if $f : Y \to Z$ is a $\mathbb{P}^n$-bundle, it is possible to find boundaries Φ_Z on Z and Φ_Y on Y such that $K_Z + \Phi_Y = f^*(K_Z + \Phi_Z)$.

Moreover, pairs are a crucial tool in the proof of many important results, like Shokurov's base-point-freeness, where it is necessary to be able to modify/perturb the coefficients of Φ, even in the case $\Phi = 0$.

Another nice feature of pairs is that they encode singularities. If Z is normal but not smooth, it is possible to define the canonical divisor as a Weil divisor. Indeed, on the smooth locus Z^{sm} of Z we have $K_{Z^{sm}} = \sum a_i D_i$ and, since $codim_Z(Z \setminus Z^{sm}) \geq 2$, there is a unique Weil divisor on Z whose restriction is $K_{Z^{sm}}$, namely $K_Z = \sum a_i \overline{D_i}^{Zar}$. The divisor K_Z is in general not $\mathbb{Q}$-Cartier and Φ encodes the lack of "Cartier-ness" of K_Z. Moreover there is a way of measuring the singularities of (Z, Φ) by looking at the coefficients of Φ. We refer to Definition 2.1 for the notion of klt, which is a singularity notion defined in this way. The Cone and Contraction theorem holds for a klt pair and we can run the $K_Z + \Phi$*–minimal model program*.

It is still an open problem whether the $K_Z + \Phi$-MMP stops in general, even when $\Phi = 0$, but there is an important class of varieties for which it is known to terminate, the ones which are covered by curves which have negative intersection with $K_Z + \Phi$. We say in this case that $K_Z + \Phi$ is *not pseudoeffective*. Varieties with this property are uniruled. In this case, after a finite number of steps, we find a Mori fibre space by [BCHM].

The outcome is nevertheless far from being unique. The Sarkisov program describes the relation between two different Mori fibre spaces that are outcomes of two MMP on the same variety:

Theorem 1.1 ([HM], [Cor]). *Suppose that $\phi\colon X \to S$ and $\psi\colon Y \to T$ are two Mori fibre spaces. Then X and Y are birational if and only if they are related by a sequence of Sarkisov links.*

A Sarkisov link between $\phi\colon X \to S$ and $\psi\colon Y \to T$ is a diagram of one of the four following types:

I

$$\begin{tikzcd} X' \arrow[r, dashed] \arrow[d] & Y \arrow[d, "\psi"] \\ X \arrow[d, "\phi"'] & T \arrow[dl] \\ S & \end{tikzcd}$$

II

$$\begin{tikzcd} X' \arrow[r, dashed] \arrow[d] & Y' \arrow[d] \\ X \arrow[d, "\phi"'] & Y \arrow[d, "\psi"] \\ S \arrow[r, equal] & T \end{tikzcd}$$

III

$$\begin{tikzcd} X \arrow[r, dashed] \arrow[d, "\phi"'] & Y' \arrow[d] \\ S \arrow[dr] & Y \arrow[d, "\psi"] \\ & T \end{tikzcd}$$

IV

$$\begin{tikzcd} X \arrow[rr, dashed] \arrow[d, "\phi"'] & & Y \arrow[d, "\psi"] \\ S \arrow[dr] & & T \arrow[dl] \\ & R & \end{tikzcd}$$

The vertical arrows in the diagrams are extremal contractions; the arrows to X or Y are extremal divisorial contractions. The horizontal dotted arrows are isomorphisms in codimension one which are a composition of flops with respect to a suitable boundary. A flop is a composition of birational maps $h_2^{-1} \circ h_1\colon X_1 \dashrightarrow X_2$ with $h_i\colon X_i \to \overline{X}$ such that the exceptional loci of h_1 and h_2 have codimension at least two, $\rho(X_i) = \rho(\overline{X}) + 1$ and there are divisors D_i on X_i such that (X_i, D_i) is klt and $K_{X_i} + D_i$ has zero intersection with every curve contracted by ϕ_i.

Theorem 1.1 is another case where pairs are used in a fundamental way. Indeed, if X_1/S_1 and X_2/S_2 are two Mori fibre spaces birational to each other, there is a variety Z and two boundaries Θ_1 and Θ_2 on Z such that $K_Z + \Theta_i$ is semiample on Z and induces a morphism $Z \to X_i$. Roughly speaking, the possible birational models are encoded by a combinatorial object (a polytope) in terms of boundaries on Z.

The purpose of this note is to give a proof of Theorem 1.1 in the G-equivariant setting. We prove that two Mori fibre spaces that are outcomes of two G-equivariant MMP on the same G-pair are related by a sequence of G-equivariant Sarkisov links.

Theorem 1.2. *Let G be a connected algebraic group. Let (Z, Φ) be a klt pair such that $G < Aut^0(Z)$ and G leaves Φ invariant. Let $\phi\colon X \to S$ and $\psi\colon Y \to T$ be two Mori fibre spaces obtained from (Z, Φ) via a G-equivariant MMP. Then X and Y are related by a sequence of G-equivariant Sarkisov links and in every such link the horizontal dotted arrows are compositions of G-equivariant flops with respect to a suitable boundary.*

In dimension 3 and for $\Phi = 0$ the interested reader can check that the proof in [Cor] is "G-equivariant".

As a direct consequence we obtain the following characterisation of subgroups of $Bir(W)$ that are maximal among the connected groups acting rationally on W.

Corollary 1.3. *Let W be an uniruled variety and let G be a connected algebraic group acting rationally on W. Then G is maximal among the connected groups acting rationally on W if and only if $G = Aut^0(X)$ where $\phi\colon X \to S$ is a Mori fibre space and for every Mori fibre space $\psi\colon Y \to T$ which is related to ϕ by a finite sequence of G-Sarkisov links we have $G = Aut^0(Y)$.*

We have the following application to subgroups of the Cremona group.

Corollary 1.4. *Let G be a connected algebraic group acting rationally on $\mathbb{P}^n$. Then G is maximal among the connected groups acting rationally on $\mathbb{P}^n$ if and only if $G = Aut^0(X)$ where $\phi\colon X \to S$ is a rational Mori fibre space and for every Mori fibre space $\psi\colon Y \to T$ which is related to ϕ by a finite sequence of G-Sarkisov links we have $G = Aut^0(Y)$.*

Theorem 1.2 is an essential ingredient in [BFT, Theorem E], the classification of automorphism groups of rational Mori fibre spaces of dimension 3. This result implies that there is a finite number of conjugacy classes of maximal connected algebraic subgroups of $Bir(\mathbb{P}^3)$ acting rationally on $\mathbb{P}^3$.

2. MMP and group actions

We refer to [KM] for the definitions of singularities of pairs, of MMP and of Mori fibre space and to [DL] for the definition of rational and regular action.

We give now the definition of pairs and terminal and klt singularities:

Definition 2.1. A pair (Z, Φ) is the data of a normal variety Z and a $\mathbb{Q}$-divisor Φ such that $K_Z + \Phi$ is $\mathbb{Q}$-Cartier. A pair is said to be kawamata log terminal, or klt, if for any birational morphism $\mu\colon \widetilde{Z} \to Z$ with $\widetilde{Z}$ smooth we have $K_{\widetilde{Z}} = \mu^*(K_Z + \Phi) + \sum a_E E$ with $a_E > -1$ for all E.

A variety Z is said to be terminal if for any birational morphism $\mu\colon \widetilde{Z} \to Z$ with $\widetilde{Z}$ smooth we have $K_{\widetilde{Z}} = \mu^* K_Z + \sum a_E E$ with $a_E > 0$ for all $E \subseteq Exc(\mu)$.

Definition 2.2. We call a pair (Z, Φ) a *G-pair* if G acts on Z regularly and for all $g \in G$ we have $g \cdot \Phi = \Phi$.

Remark 2.3. The pair $(Z,0)$ is a G-pair for every subgroup G of $Aut(Z)$.

Remark 2.4. In the next lemma we prove that if (Z,Φ) is a G-pair with G connected, then every MMP on (Z,Φ) is a G-MMP. The situation is thus very different from the one where the group G is finite. Indeed, in the connected case, the action of G on the polyhedral part of $NE(X)$ is trivial, while a finite group can act in a non-trivial way on $NE(X)$ and one has to consider the G-invariant part $NE(X)^G$.

An easy example which illustrates well this difference is $(\mathbb{P}^1\times\mathbb{P}^1,0)$ with the action of $\mathbb{Z}/2\mathbb{Z}$ exchanging the two factors. We have $NE(\mathbb{P}^1\times\mathbb{P}^1)^{\mathbb{Z}/2\mathbb{Z}} = \mathbb{R}_+[l_1+l_2]$ where l_1 and l_2 are fibres of the projections. The only possible $\mathbb{Z}/2\mathbb{Z}$-invariant contraction is $\mathbb{P}^1\times\mathbb{P}^1\to\{p\}$. If we consider instead the diagonal action of $PGL(2)$, we have $NE(\mathbb{P}^1\times\mathbb{P}^1)^{PGL(2)}=\mathbb{R}_+[l_1]+\mathbb{R}_+[l_2]$ and there are two extremal contractions, the two projections onto the two factors. We refer to [KM] and [C$^+$] for a discussion of the case where G is finite, but also to the classical [Man] and to the work of Prokhorov (e.g., [Pro]).

Let G be a connected algebraic group. If we have a klt G-pair (Z,Φ), we can run a G-equivariant MMP on (Z,Φ), that is, an MMP where all the birational maps, divisorial contractions and flips, are compatible with the action of the group.

Lemma 2.5. *Let G be a connected group and (Z,Φ) a G-pair. Then any MMP on (Z,Φ) is G-equivariant.*

Proof. Indeed let $Z \dashrightarrow Z_1$ be the first step of the MMP. If it is an extremal contraction, then by the Blanchard's lemma [BSU, Prop 4.2.1] there is an induced action on Z_1 making the map G-equivariant. Equivalently, a connected group acts trivially on the extremal rays contained in the $(K_Z+\Phi)$-negative part of the Mori cone, that are discrete. Then the extremal ray corresponding to the contraction $Z\to Z_1$ is G-invariant and so is the locus spanned by it.

Assume that the first step is a flip given by the composition of two small contractions $\mu\colon Z\to Y$ and $\mu^+\colon Z_1\to Y$. By the discussion above, there is an action on Y making μ G-equivariant. Moreover, $Z_1\cong Proj_Y\bigoplus_m \mu_*\mathcal{O}_X(m(K_Z+\Phi))$. Since $K_Z+\Phi$ is G-invariant, the group G acts on $\mathcal{O}_X(m(K_Z+\Phi))$ for m sufficiently divisible, and subsequently on Z_1. $\square$

If $K_Z+\Phi$ is not pseudoeffective, by [BCHM] the outcome of such an MMP is a Mori fibre space $\phi\colon X\to S$ with a regular action of G on X.

Definition 2.6.

- A *Mori fibre space* is the data of a fibration $\phi\colon X \to S$ such that X is terminal, $\rho(X) - \rho(S) = 1$ and a boundary Δ on X such that (X, Δ) is klt and $-(K_X + \Delta)$ is ϕ-ample.
- We will call two Mori fibre spaces $\phi\colon X \to S$ and $\psi\colon Y \to T$ *Sarkisov related* if X and Y are outcomes of running the $K_Z + \Phi$-MMP, for the same $\mathbb{Q}$-factorial klt pair (Z, Φ).
- If G is a connected group, a *G-Mori fibre space* is a Mori fibre space with a regular action of a group G.
- We say that two G-Mori fibre spaces $\phi\colon X \to S$ and $\psi\colon Y \to T$ are *G-Sarkisov related* if X and Y are results the G-equivariant MMP on (Z, Φ), for the same $\mathbb{Q}$-factorial klt G-pair (Z, Φ).

The following remark will be useful in the next section.

Remark 2.7. If there is a commutative diagram

$$\begin{tikzcd} & Z \arrow[dl, dashed, "f_1"'] \arrow[dr, dashed, "f_2"] & \\ X_1 \arrow[dr, "g_1"'] & & X_2 \arrow[dl, "g_2"] \\ & Y & \end{tikzcd}$$

where f_i is birational and G-equivariant for $i = 1, 2$ and g_i is a morphism for $i = 1, 2$, then, by the Blanchard's lemma [BSU, Prop 4.2.1] there are two actions of G on Y. These actions coincide. Indeed, they coincide on an open set contained in the image of the open set where $f_2 \circ f_1^{-1}$ is an isomorphism.

Remark 2.8. If G is connected and X is uniruled, or if G is rationally connected, then all the complete linear systems are G-invariant, although the individual divisors might not be left invariant by the action of G.

3. Proof of Theorem 1.2

In this section we present a proof of Theorem 1.2. Let Z be a projective variety. From now on, we assume the setup of [HM, Section 3].

Definition 3.1. [BCHM] Let V be a finite dimensional affine subspace of the real vector space $WDiv_{\mathbb{R}}(Z)$ of Weil divisors on Z, which is defined over the rationals, and $A \geq 0$ an ample $\mathbb{Q}$-divisor on Z.

$$\begin{aligned}\mathcal{L}_A(V) &= \{\Theta = A + B \mid B \in V,\ B \geq 0,\ K_Z + \Theta \text{ is log canonical}\},\\ \mathcal{E}_A(V) &= \{\Theta \in \mathcal{L}_A(V) \mid K_Z + \Theta \text{ is pseudo-effective}\}.\end{aligned}$$

As in [HM], we assume that there exists $B_0 \in V$ such that $K_Z + \Theta_0 = K_Z + A + B_0$ is big and klt. Given a rational contraction $f\colon Z \dashrightarrow X$, define

$$\mathcal{A}_{A,f}(V) = \{\Theta \in \mathcal{E}_A(V) \mid f \text{ is the ample model of } (Z, \Theta)\}.$$

In addition, let $\mathcal{C}_{A,f}(V)$ denote the closure of $\mathcal{A}_{A,f}(V)$. We refer to [HM, Definition 3.1] for the definition of ample model. We recall the following result from [HM] for the benefit of the reader.

Theorem 3.2. [HM, Theorem 3.3] *There is a natural number m and there are $f_i\colon Z \dashrightarrow X_i$ rational contractions $1 \leq i \leq m$ with the following properties:*

(1) *$\{\mathcal{A}_i = \mathcal{A}_{A,f_i} \mid 1 \leq i \leq m\}$ is a partition of $\mathcal{E}_A(V)$. $\mathcal{A}_i$ is a finite union of relative interiors of rational polytopes. If f_i is birational, then $\mathcal{C}_i = \mathcal{C}_{A,f_i}$ is a rational polytope.*

(2) *If $1 \leq i \leq m$ and $1 \leq j \leq m$ are two indices such that $\mathcal{A}_j \cap \mathcal{C}_i \neq \varnothing$, then there is a contraction morphism $f_{ij}\colon X_i \to X_j$ and a factorisation $f_j = f_{ij} \circ f_i$.*

Now suppose in addition that V spans the Néron–Severi group of Z.

(3) *Pick $1 \leq i \leq m$ such that a connected component $\mathcal{C}$ of $\mathcal{C}_i$ intersects the interior of $\mathcal{L}_A(V)$. The following are equivalent:*
   - *$\mathcal{C}$ spans V.*
   - *f_i is birational and X_i is $\mathbb{Q}$-factorial.*

The following is a G-equivariant version of [HM, Lemma 4.1].

Lemma 3.3. *Let G be a connected group. Let $\phi\colon X \to S$ and $\psi\colon Y \to T$ be two G-Sarkisov related G-Mori fibre spaces corresponding to two $\mathbb{Q}$-factorial klt projective G-pairs (X, Δ) and (Y, Γ). Then we may find a smooth projective variety Z with a regular action of G, two birational G-equivariant contractions $f\colon Z \dashrightarrow X$ and $g\colon Z \dashrightarrow Y$, a klt G-pair (Z, Φ), an ample $\mathbb{Q}$-divisor A on Z and a two-dimensional rational affine subspace V of $WDiv_{\mathbb{R}}(Z)$ such that*

(1) *if $\Theta \in \mathcal{L}_A(V)$ then $\Theta - \Phi$ is ample,*

(2) *$\mathcal{A}_{A,\phi\circ f}$ and $\mathcal{A}_{A,\psi\circ g}$ are not contained in the boundary of $\mathcal{L}_A(V)$,*

(3) *V satisfies (1-4) of Theorem 3.2,*

(4) *$\mathcal{C}_{A,f}$ and $\mathcal{C}_{A,g}$ are two dimensional, and*

(5) *$\mathcal{C}_{A,\phi\circ f}$ and $\mathcal{C}_{A,\psi\circ g}$ are one dimensional.*

Proof. By assumption we may find a $\mathbb{Q}$-factorial klt G-pair (Z,Φ) such that $f\colon Z \dashrightarrow X$ and $g\colon Z \dashrightarrow Y$ are both outcomes of the G-equivariant MMP on (Z,Φ). Let $p\colon W \to Z$ be any G-equivariant log resolution of (Z,Φ) which resolves the indeterminacy of f and g. Such pair exists for instance by [Kol, Proposition 3.9.1, Theorem 3.35, Theorem 3.36]. The rest of the proof goes as in [HM, Lemma 4.1]. □

Proof of Theorem 1.2. The proof follows the same lines as [HM, Theorem 1.3] but instead of choosing (Z,Φ) as in [HM, Lemma 4.1] we choose the G-pair given by Lemma 3.3.

We prove now that any X_i as in Theorem 3.2 carries a regular action of G making f_i G-equivariant. Since $\Theta \in \mathcal{L}_A(V)$ implies $\Theta-\Phi$ is ample, by Theorem 3.2(3) and [HM, Lemma 3.6], for every X_i corresponding to a full-dimensional polytope of $\mathcal{A}_i$, the variety X_i is the result of an MMP on (Z,Φ). By Lemma 2.5 this MMP is G-equivariant and therefore there is a regular action of G on X_i.

Let X_j be a variety corresponding to a non full-dimensional polytope $\mathcal{A}_j$. Let $\mathcal{A}_i$ be a full-dimensional polytope such that $\mathcal{C}_i \cap \mathcal{A}_j \neq \varnothing$. By Theorem 3.2 there is a surjective morphism $f_{ij}\colon X_i \to X_j$. By the Blanchard's lemma [BSU, Prop 4.2.1] there is an action of G on X_j making the morphism f_{ij} G-equivariant. By Remark 2.7 this action does not depend on the choice of i.

The links given by [HM, Theorem 3.7] are G-equivariant. Indeed the maps appearing in the links are either

- morphisms of the form f_{ij} as in Theorem 3.2, and those are G-equivariant by the discussion above; or
- flops (with respect to a suitable boundary) of the form $f_{ij}\circ f_{kj}^{-1}$, and those are again G-equivariant. □

4. Proof of Corollary 1.3

Definition 4.1. A connected subgroup $G < Bir(Z)$ is not maximal among the connected groups acting rationally on Z if there is a connected subgroup of $Bir(Z)$ acting rationally on Z such that $G \subsetneq H$. It is *maximal among the connected groups acting rationally on Z* otherwise. We will say *maximal* for short.

Proof of Corollary 1.3. By a theorem of Weil [Wei] (see also [Kra]) there is a birational model $\widetilde{W}$ of W, such that G acts regularly on $\widetilde{W}$. We then run a G-equivariant MMP on $\widetilde{W}$ (see Lemma 2.5) and by [BDPP] and [BCHM] the result is a G-Mori fibre space $\phi\colon X \to S$.

We prove now that G is maximal if and only if for every Mori fibre space $\psi\colon Y \to T$ which is related to ϕ by a finite sequence of G-Sarkisov links we have $G = Aut^0(Y)$.

Assume that G is maximal. Let $X/S \dashrightarrow Y/T$ be a composition of G-Sarkisov links and let $\varphi\colon X \dashrightarrow Y$ be the corresponding birational map. Then $\varphi G \varphi^{-1} \subseteq Aut^0(Y)$ and if G is maximal $\varphi G \varphi^{-1} = Aut^0(Y)$.

Assume that G is not maximal and let H be a connected subgroup of $Bir(Z)$ acting rationally on Z and such that $G \subsetneq H$. By a theorem of Weil [Wei] there is a birational model $\widehat{W}$ of $\widetilde{W}$, such that H acts regularly on $\widehat{W}$. We run an H-equivariant MMP on $\widehat{W}$ and obtain a Mori fibre space $Y \to T$ such that $H < Aut^0(Y)$. This MMP is also G-equivariant. Therefore $X \to S$ and $Y \to T$ are G-Sarkisov related. By Theorem 1.2, there is a finite sequence of G-Sarkisov links $X \to S \dashrightarrow Y \to T$. □

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