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Autor: Allcock, Daniel / Gal, Itamar / Mark, Alice
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The Conway–Sloane calculus for 2-adic lattices

Daniel ALLCOCK, Itamar GAL and Alice MARK

Abstract. We motivate and explain the system introduced by Conway and Sloane for working with quadratic forms over the 2-adic integers, and prove its validity. Their system is far better for actual calculations than earlier methods, and has been used for many years, but no proof has been published before now.

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1. Introduction

Our goal in this paper is to explain the system that Conway and Sloane developed for working with lattices (quadratic forms) over the ring of 2-adic integers $\mathbb{Z}_2$. Algorithms were already known for determining when two lattices were isometric, and for finding a canonical form for each one. But these were clumsy. In his influential book on quadratic forms, Cassels even wrote about 2-adic integral canonical forms: "only the masochist is invited to read the rest of this section" [Cas, §8.4]. To this day, 2-adic lattices retain their reputation for complexity.

But the 2-adic part of a lattice over $\mathbb{Z}$ is its most important part. Many questions about $\mathbb{Z}$-lattices reduce to p-adic versions of the same questions, where p varies over the primes. For example, consider the question of whether one $\mathbb{Z}$-lattice is isometric to another. We restrict to the case of rank ≥ 3 and some fixed indefinite signature, because then it is (almost) true that an isometry exists if and only if one exists p-adically for each p. Most questions about p-adic lattices are easy for odd p, including this isomorphism problem. So all the real work takes place at $p = 2$. Other examples of questions with this same flavor are whether a lattice represents a given number, or whether one lattice admits another as a direct summand (or as a primitive sublattice). See Section 2 for a little more on this larger picture.

The Conway–Sloane calculus [CS, ch. 15] is much simpler than previous approaches to 2-adic lattices, for example the original papers on invariants and canonical forms by Pall [Pal] and Jones [Jon]. It is widely used in modern applications, for example [All1, BEF, HM, Tur]. The innovation of Conway and Sloane was to introduce "oddity fusion" and "sign walking" operations, which are notationally simple and generate all equivalences. Strangely, their formal statement of results (their Theorem 10) completely avoids these operations. So it has the same unwieldy feel as the papers of Pall and Jones just mentioned. Proofs of their theorem appear in [Xu] and in Bartels' unpublished dissertation [Bar]. But the literature contains no treatment of the calculus as it is actually used. We hope to make it more accessible. All is new here are the "givers" and "receivers" of Section 4, and the "signways" of Section 6. In particular, we use signways to correct an error in their formulation of canonical forms.

Here is a fairly detailed overview of the calculus. Our goal is to show what it looks like and what it involves, rather than to explain it properly. For that, see the formal development beginning in Section 3.

Unimodular lattices. The first step in all approaches to $\mathbb{Z}_2$-lattices is to classify the unimodular ones. Conway and Sloane indicate them by symbols like $L = 1_2^{+2}$ or 1_3^{-3} or 1_{II}^{-4}. The main number 1 says that L is unimodular (over $\mathbb{Z}_2$). If L is even, which is to say that all norms are even, then the subscript is II. Otherwise, L is diagonalizable and the subscript is the *oddity* $o(L)$ of L, meaning the sum mod 8 of the diagonal terms in any diagonalization. As defined in this paper, the oddity is not a lattice invariant in general. But for unimodular lattices it is, because for them it coincides with the 2-signature, which *is* an invariant. See section 3 for more discussion. The superscript is not a signed number, but rather a sign and a separate nonnegative integer. The integer is $\dim L$. The sign is $+$ or $-$ according to whether $\det(L) \equiv \pm 1$ or ± 3 mod 8. The sign, dimension and oddity turn out to determine the isometry class of L. We prove this in Theorem 5.1.

An example of the notation: the lattices with diagonal inner product matrices $\langle 1, -1, 3\rangle$, $\langle -1, -1, -3\rangle$ and $\langle 3, 3, -3\rangle$ are all unimodular, with determinant ± 3 (up to squares), hence sign $-$. They also have dimension 3 and oddity $3 \in \mathbb{Z}/8$. So they are isometric to each other, and we write 1_3^{-3} to represent their isometry class. We built these lattices by starting with the symbol 1_3^{-3} and choosing three terms inside $\langle \dots \rangle$ to have product ± 3 and sum 3 (both mod 8). In this way it is always easy to construct representative lattices for any symbol $1_{\cdots}^{\cdots}$.

The symbols behave cleanly under direct sum: signs multiply and dimensions and subscripts add. For subscripts this means addition in $\mathbb{Z}/8$, together with the special rule $\mathrm{II} + t = t$. For example, $1_2^{+2} \oplus 1_3^{-3} \oplus 1_{\mathrm{II}}^{-4} \cong 1_5^{+9}$.

Jordan decompositions. A general $\mathbb{Z}_2$-lattice can be expressed as a direct sum, where the terms are got by rescaling unimodular lattices by distinct powers of 2. This is called a Jordan decomposition and the terms are called Jordan constituents. Conway and Sloane use symbols like 1_{II}^{+2}, 2_2^{-2}, 4_1^{+3} and 64_{II}^{-2} to indicate them. These lattices are got from the unimodular lattices with the same decorations, namely 1_{II}^{+2}, 1_2^{-2}, 1_1^{+3} and 1_{II}^{-2}, by scaling inner products by 1, 2, 4 and 64 respectively. The *scale* of each term means this scaling factor. The *type* is I or II according to whether the unimodular lattice is odd or even. A general $\mathbb{Z}_2$-lattice is a direct sum of such terms, for example

(1.1) $$1_{\mathrm{II}}^2\, 2_4^{-2} 4_{-1}^3\, 16_1^1\, 32_{\mathrm{II}}^2\, 64_{\mathrm{II}}^{-2}\, 128_{-1}^1 256_1^1 512_{\mathrm{II}}^{-4}$$

where we have suppressed + signs in superscripts and $\oplus$ symbols between the terms. A *Jordan symbol* means an expression like (1.1), describing a Jordan decomposition. We will use this example many times: it is complicated enough to illustrate many phenomena.

There are two main ways that the case of p an odd prime is simpler than the $p = 2$ case. The first is that the unimodular classification is simpler: one needs no subscripts. The second is that the Jordan decomposition is unique up to isometry. So when p is odd, understanding a p-adic lattice amounts to a writing down something like (1.1) without subscripts. Equivalences between distinct Jordan decompositions are the subtle part of 2-adic lattice theory. Conway and Sloane introduced *oddity fusion* and *sign walking* to organize these equivalences.

Oddity fusion. An example of nonuniqueness of Jordan decomposition is

(1.2) $$2_4^{-2} 4_{-1}^3 \cong 2_2^{-2} 4_1^3 \cong 2_{-2}^{-2} 4_5^3$$

These are the same except for the oddities (subscripts) of the terms, and in all three cases the sum of the oddities is 3 mod 8. This illustrates a general phenomenon called oddity fusion: when the scales of a sequence of Jordan constituents are consecutive powers of 2, and the subscripts are oddities rather than "II", then those constituents "share" their oddities.

To express this more formally we say two terms of type I are in the same *compartment* if the terms at all intermediate scales also have type I. In example (1.1) there are three compartments: one consisting of the terms of scales 2 and 4, one consisting of the term of scale 16, and one consisting of the terms of scales 128 and 256. (The scale 8 term is unwritten because it is 0-dimensional. But it has type II, hence separates 2 and 4 from 16.) Usually one indicates the compartments with brackets, for example

(1.3) $$1_{\mathrm{II}}^2\, [2_4^{-2} 4_{-1}^3][16_1^1]\, 32_{\mathrm{II}}^2\, 64_{\mathrm{II}}^{-2}\, [128_{-1}^1 256_1^1] 512_{\mathrm{II}}^{-4}$$

The brackets are usually omitted for a compartment consisting of a single term, so here we would omit the brackets that enclose 16^1_1.

"Oddity fusion" means that two Jordan symbols J, J', which are the same except for the subscripts in a compartment, represent isometric lattices if the sum of the oddities over that compartment in J is equal to the corresponding sum in J'. Therefore we record this sum (the compartment's oddity) rather than the oddities of the individual terms. We attach it as a subscript to the closing bracket. For example, we write $[2^{-2}4^3]_3$ rather than any of the three Jordan symbols in (1.2). This notation displays less information, while still capturing the isometry class, so it is more canonical. After oddity fusion, our example (1.3) becomes

$$1^{2}_{\mathrm{II}}\,[2^{-2}4^{3}]_{3}\,16^{1}_{1}\,32^{2}_{\mathrm{II}}\,64^{-2}_{\mathrm{II}}\,[128^{1}\,256^{1}]_{0}\,512^{-4}_{\mathrm{II}} \tag{1.4}$$

Most of the simplicity of the Conway–Sloane approach comes from the use of oddity fusion. We call a symbol like (1.4) a *2-adic symbol.*

Sign walking. Oddity fusion does not generate all equivalences between 2-adic Jordan decompositions. For example, (1.4) turns out to be isometric to each of

$$\underbrace{1^{-2}_{\mathrm{II}}\,[2^{2}}\,4^{3}]_{-1}\,16^{1}_{1}\,32^{2}_{\mathrm{II}}\,64^{-2}_{\mathrm{II}}\,[128^{1}\,256^{1}]_{0}\,512^{-4}_{\mathrm{II}} \tag{1.5}$$

$$1^{2}_{\mathrm{II}}\,[\underbrace{2^{2}4^{-3}}]_{-1}\,16^{1}_{1}\,32^{2}_{\mathrm{II}}\,64^{-2}_{\mathrm{II}}\,[128^{1}\,256^{1}]_{0}\,512^{-4}_{\mathrm{II}} \tag{1.6}$$

$$1^{2}_{\mathrm{II}}\,[2^{-2}\,\underbrace{4^{-3}]_{-1}\,16^{-1}_{-3}}\,32^{2}_{\mathrm{II}}\,64^{-2}_{\mathrm{II}}\,[128^{1}\,256^{1}]_{0}\,512^{-4}_{\mathrm{II}} \tag{1.7}$$

In each case we have negated the signs of two nearby scales of (1.4), and changed by 4 the oddity of each compartment involved. The underbrackets indicate the terms whose signs were changed. In (1.5) and (1.6) the only compartment of (1.4) involved was $[2^{-2}4^3]_3$, so we changed its oddity by 4. In (1.7) the compartment 16^1_1 was also involved, so we also changed its oddity by 4.

The rules for which pairs of terms admit such a *sign walk* are subtle enough that we postpone them to Section 6. But to illustrate the flexibility they provide, we show which terms of our example can interact with each other via some chain of sign walks:

$$\underbrace{1^{2}_{\mathrm{II}}\,[2^{-2}4^{3}]_{3}\,16^{1}_{1}\,32^{2}_{\mathrm{II}}}\,\underbrace{64^{-2}_{\mathrm{II}}\,[128^{1}}\,\underbrace{256^{1}]_{0}\,512^{-4}_{\mathrm{II}}} \tag{1.8}$$

We call these groups of terms *signways*, suggesting highways along which signs can move (or cancel). In the language of Conway and Sloane, the classification of 2-adic lattices amounts to the theorem that sign walking generates all equivalences between 2-adic symbols. (Theorem 6.2.)

Some equivalence relations are like mazes, where it is not clear which "moves" to make when seeking an equivalence between two objects, or perhaps only an arcane recipe for these moves is available. This is the nature of earlier classifications of 2-adic lattices. Happily, sign walking is simple. For any given 2-adic symbol, the sign walks generate an elementary abelian 2-group, acting simply transitively on the 2-adic symbols that are equivalent to it. (See the proof of Theorem 6.3.) In (1.8) this group is $(\mathbb{Z}/2)^4 \times (\mathbb{Z}/2) \times (\mathbb{Z}/2)$. The $(\mathbb{Z}/2)^4$ factor changes signs in the first signway, arbitrarily subject to maintaining the overall sign. The factors $\mathbb{Z}/2$ play the same role for the other signways. The alterations of oddities that accompany any given sign walk are easy to figure out.

One can use sign walking to define a canonical form: walk all the $-$ signs as far left as possible, canceling pairs of such signs when possible. Then all signs will be $+$ except perhaps for the first terms of some of the signways. For (1.8) this canonical form is

$$1_{\mathrm{II}}^{-2}\,[2^2 4^3]_{-1}\,16_1^1\,32_{\mathrm{II}}^2\,64_{\mathrm{II}}^{-2}\,[128^1\,256^{-1}]_4 512_{\mathrm{II}}^4$$

The main virtues of the Conway–Sloane notation are that (i) it allows easy passage between the notation and the lattices, (ii) it behaves well under direct sum and scaling, and duality too, (iii) no more information is displayed than necessary, and (iv) rather than being constrained to a single canonical form, one can easily pass between all possible 2-adic symbols for a particular lattice. See Example 6.5 for an illustration of (iv): we find all the $\mathbb{Z}_2$-lattices whose sum with $\langle 2, 2\rangle$ is isometric to (1.4).

After some (strictly) motivational background in Section 2, we cover some technical preliminaries in Section 3. Then Section 4 defines what we call a *fine decomposition* of a 2-adic lattice and describes some moves between them. In Section 5 we classify the unimodular lattices and introduce oddity fusion. In Section 6 we define 2-adic symbols and prove that sign walking generates all equivalences between them. We also discuss canonical forms and how to define some numerical invariants of 2-adic lattices. The final section is devoted to the proof of Theorem 4.4.

This note developed from part of a course on quadratic forms given by the first author at the University of Texas at Austin, with his lecture treatment greatly improved by the second and third authors.

2. The larger picture

This section is meant to describe how the 2-adic lattice theory fits into the larger theory of integer quadratic forms. It is not needed later in the paper.

A lattice over $\mathbb{Z}$ or the p-adic integers $\mathbb{Z}_p$ means a free module equipped with a symmetric bilinear pairing that takes values in the fraction field $\mathbb{Q}$ or

$\mathbb{Q}_p$. An *isometry* from one such lattice to another means a module isomorphism that preserves inner products. In many situations one wants to understand whether two $\mathbb{Z}$-lattices are isometric. If L is a $\mathbb{Z}$-lattice, then $L \otimes \mathbb{Z}_p$ is a $\mathbb{Z}_p$-lattice. If L' is another $\mathbb{Z}$-lattice, then L, L' are said to lie in the same genus if they have the same signature and $L \otimes \mathbb{Z}_p$ and $L' \otimes \mathbb{Z}_p$ are isometric for all primes p. Isometric $\mathbb{Z}$-lattices obviously lie in the same genus.

The famous Hasse–Minkowski theory of quadratic forms over $\mathbb{Q}$ says that two quadratic spaces over $\mathbb{Q}$ are isomorphic if and only if they are isomorphic over $\mathbb{R}$ and every $\mathbb{Q}_p$. It would be unreasonable to hope for the corresponding result for lattices over $\mathbb{Z}$: that the genus determines the isomorphism class. What is surprising is how close to truth this comes.

Until work of Eichler in the 1950s, it was open whether the genus determines the isomorphism class of an indefinite $\mathbb{Z}$-lattice of dimension ≥ 3. Eichler discovered a subtle equivalence relation, whose equivalence classes are called spinor genera. Each genus consists of finitely many spinor genera, and each spinor genus consists of finitely many isometry classes of lattices. But some mild hypotheses promote both cases of "finitely many" to "one":

Theorem 2.1. *An indefinite genus G of dimension $n \geq 3$ consists of exactly one spinor genus, unless there exists some prime p such that $G \otimes \mathbb{Z}_p$ is p-adically diagonalizable, with the p-power parts of the diagonal terms all being distinct. If G is integral, then this exceptional case can only occur if $p^{\binom{n}{2}} \mid \det G$.*

Theorem 2.2 (Eichler). *An indefinite spinor genus of dimension ≥ 3 consists of exactly one isometry class.*

Note that the rational number $\det G$ is determined by the signature and the p-adic valuations of the p-adic determinants. And the $\mathbb{Z}_p$-lattice $G \otimes \mathbb{Z}_p$ is well-defined, by the definition of a genus. See [CS, Ch. 15, Thm. 19], or the proof of the Corollary to Lemma 3.7 in [Cas, Ch. 10], for Theorem 2.1. See [Eic] or [Cas, Ch. 10, Thm. 1.4] for Theorem 2.2. The restriction to indefinite forms and dimension ≥ 3 is essential: in dimension 2 the spinor genus behaves very differently than in higher dimensions, and for definite forms a genus typically contains many isomorphism classes.

Except in small dimension, lattices with the distinct-powers-of-p property in Theorem 2.1 do not seem to occur in nature. So these two theorems form the basis for our statement in the introduction that for indefinite lattices of dimension ≥ 3, it is "almost" true that genera coincide with isometry classes. Even if a genus (indefinite of rank ≥ 3) does have the distinct-powers-of-p property, it can still consist of a single isometry class, and one can check this. It is just no longer guaranteed.

This almost-correspondence between genera and isomorphism classes is the reason that many questions about $\mathbb{Z}$-lattices reduce to $\mathbb{Z}_p$-lattices. For $p > 2$, a $\mathbb{Z}_p$-lattice has only one isomorphism class of Jordan decomposition. And each Jordan constituent J is characterized by its scale, dimension and sign. In this case there is no subtlety to the isometry classification: to determine whether two p-adic lattices are isometric one just finds Jordan decompositions and compares them. So the $p = 2$ case accounts for most of the isometry analysis.

(For odd p, the sign of J is defined as the Legendre symbol $\left(\frac{d}{p}\right) = \pm 1$, where d is a unit of $\mathbb{Z}_p$ such that $\det J = dp^n$. We always abbreviate this symbol to $\pm$. Although we did not say so in the introduction, when $p = 2$ the sign of J is Kronecker's generalization $\left(\frac{d}{2}\right)$ of the Legendre symbol.)

A second common question about a $\mathbb{Z}$-lattice L is whether a given lattice M occurs a direct summand. When L is the only lattice in its genus, and the signatures of M and L are compatible, this reduces to the question of whether $M \otimes \mathbb{Z}_p$ is a summand of $L \otimes \mathbb{Z}_p$ for all primes p. For $p > 2$ this is easy: $M \otimes \mathbb{Z}_p$ is a summand if and only if each constituent of $M \otimes \mathbb{Z}_p$ either is lower-dimensional than the corresponding constituent of $L \otimes \mathbb{Z}_p$, or else has the same dimension and sign. The corresponding question for $p = 2$ is more subtle – see Example 6.5 for a taste of the required analysis.

A third common question is whether M occurs as a primitive sublattice of L. Under the same conditions as in the previous paragraph, this reduces to the problem of building a suitable candidate for the orthogonal complement of $M \otimes \mathbb{Z}_p$ in $L \otimes \mathbb{Z}_p$, for each prime p. The case of odd p is no longer trivial, but still the $p = 2$ case usually dominates the analysis. See [All2] for an extended calculation of this sort.

3. Preliminaries

Now we begin our formal exposition. Henceforth, an *integer* means an element of the ring $\mathbb{Z}_2$ of 2-adic integers, and we write $\mathbb{Q}_2$ for $\mathbb{Z}_2$'s fraction field. We assume known that two odd elements of $\mathbb{Z}_2$ differ by a square factor if and only if they are congruent mod 8. Every nonzero $x \in \mathbb{Q}_2$ can be written uniquely as $2^a u$ with $a \in \mathbb{Z}$ and u a unit of $\mathbb{Z}_2$. We call u the *odd part* of x.

A *lattice* L means a finite-dimensional free module over $\mathbb{Z}_2$, equipped with a $\mathbb{Q}_2$-valued symmetric bilinear form. We call L *nondegenerate* if the natural map $L \to \mathrm{Hom}(L, \mathbb{Q}_2)$ is injective. In this case, the *dual lattice* $L^* = \mathrm{Hom}(L, \mathbb{Z}_2)$ is naturally identified with the set of vectors in $L \otimes \mathbb{Q}_2$ that have integral inner products with all elements of L. We call L *integral* if all inner products in L are integers. An integral lattice is called *even* if all its elements have even norm (self-inner-product), and *odd* otherwise. If L is integral and nondegenerate, then we regard it as a sublattice of L^*.

The *determinant* $\det L$ means the determinant of the inner product matrix of any basis for L, and is well-defined up to multiplication by squares of units of $\mathbb{Z}_2$. In particular, the odd part of $\det L$ is well-defined mod 8. If L is integral and nondegenerate then $\det L = [L^* : L]$ up to a unit of $\mathbb{Z}_2$. We call L *unimodular* if $L = L^*$; this is equivalent to L being integral with odd determinant.

The *sign* of a unimodular lattice U means the Kronecker symbol $\left(\frac{\det U}{2}\right)$. Recall that this is defined as $+1$ or -1 according to whether $\det U \equiv \pm 1$ or ± 3 mod 8. We will always abbreviate ± 1 to $\pm$. The Kronecker symbol has special properties that are important in quadratic reciprocity. But these play no role in this paper; for us it is just a way to record information about odd numbers mod 8.

Now consider a lattice got by scaling the inner product on a unimodular lattice. We say it has *type* I or II according to whether the unimodular lattice is odd or even. For example, $\langle 2 \rangle$ has type I, although it is an even lattice, because it was got by scaling the odd lattice $\langle 1 \rangle$. On the other hand, $\left(\begin{smallmatrix} 4 & 2 \\ 2 & 4 \end{smallmatrix}\right)$ has type II, because it was got by scaling the even unimodular lattice $\left(\begin{smallmatrix} 2 & 1 \\ 1 & 2 \end{smallmatrix}\right)$.

The last invariant we need is a $\mathbb{Z}/8$-valued invariant of quadratic spaces over $\mathbb{Q}_2$, called the 2-*signature* and written σ_2. It is easy to compute. A not-quite-invariant similar to the 2-signature, called the *oddity*, is defined below. It is even easier to compute, and is what is actually used in the Conway–Sloane calculus.

To compute the 2-signature of a quadratic space V over $\mathbb{Q}_2$, choose any basis for which the inner product is diagonal. Then $\sigma_2(V) \in \mathbb{Z}/8$ is defined as the sum of the odd parts of the diagonal entries, plus 4 for each diagonal entry which is an *antisquare*. Here an antisquare is defined as a 2-adic number of the form $2^{\text{odd}}u$ where $u \equiv \pm 3$ mod 8. The fact that $\sigma_2(V)$ is independent of the choice of basis is surprising. See [CS, Ch. 15, §6.1–6.2] for a proof. Some examples:

$$\sigma_2(\langle 1,3,3,7 \rangle) = 1+3+3+7 \equiv 4 \qquad (\text{mod } 8)$$
$$\sigma_2(\langle 1,3,3,14 \rangle) = 1+3+3+7 \equiv 4$$
$$\sigma_2(\langle 1,3,6,7 \rangle) = 1+3+3+7+4 \equiv 0$$
$$\sigma_2\left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right) = \sigma_2(\langle 1,-1 \rangle) = 1-1 \equiv 0 \tag{3.1}$$
$$\sigma_2\left(\begin{smallmatrix} 2 & 1 \\ 1 & 2 \end{smallmatrix}\right) = \sigma_2(\langle 2,6 \rangle) = 1+3+4 \equiv 0 \tag{3.2}$$

In the last two lines we started with even unimodular lattices, diagonalized them over $\mathbb{Q}_2$, and then computed σ_2. The 2-signature is obviously additive: $\sigma_2(V \oplus V') = \sigma_2(V) \oplus \sigma_2(V')$ for any quadratic spaces V, V' over $\mathbb{Q}_2$.

Conway and Sloane *define* the "oddity" of V to be just another name for $\sigma_2(V)$. But in actual *use*, "oddity" seems to refer to the subscripts used in their calculus, rather than σ_2. Our own experience is that this shift of language is very natural, since the subscripts are what one actually uses. It seems to be their experience too: their statement of oddity fusion [CS, p. 381] hints at this, and this is the only sensible interpretation of "... the total oddity of a compartment must be changed by 4 mod 8, precisely when ..." [CS, p. 382]. (For experts: they are discussing sign walking, and sign walking within a compartment replaces the corresponding set of Jordan constituents by new Jordan constituents, without changing the sublattice they span. Since the "oddity" changes under this, they cannot be referring to σ_2 of the sublattice.)

We resolve this conflict by defining oddity according to actual use. We only define it for lattices got by scaling unimodular lattices, and for lattices which are expressed as direct sums of such lattices. So it is a function not on lattices but on lattices equipped with such a direct sum decomposition. We only speak of the oddity of a lattice when it is understood which decomposition we mean. If L is got by scaling a unimodular lattice U by a power of 2, then the *oddity* $o(L)$ is defined as $\sigma_2(U)$. If L is expressed as a direct sum of rescaled unimodular lattices, then $o(L)$ is defined as the sum of the oddities of the summands. By this definition the oddity is additive: $o(L \oplus M) = o(L) + o(M)$.

The oddity is often the same as the 2-signature; to describe the difference we begin with direct sum decompositions of a 2-adic lattice:

Lemma 3.1. *Every 2-adic lattice is a direct sum of 1-dimensional lattices and copies of the lattices $\left(\begin{smallmatrix}0&1\\1&0\end{smallmatrix}\right)$ and $\left(\begin{smallmatrix}2&1\\1&2\end{smallmatrix}\right)$, scaled by powers of 2.*

Proof. After pulling off some summands $\langle 0\rangle$ it is enough to treat the nondegenerate case. By scaling it is enough to treat the integral case. We use induction on dimension and the fact that any sublattice with odd determinant is a summand. Suppose a nondegenerate intergral lattice L is given; by scaling we may suppose some inner product is odd. A vector of odd norm spans a summand, and then we can appeal to induction. So suppose L is even, and choose two elements with odd inner product. Their inner product matrix $\left(\begin{smallmatrix}\text{even}&\text{odd}\\\text{odd}&\text{even}\end{smallmatrix}\right)$ has determinant $(\text{even})(\text{even}) - (\text{odd})^2 \equiv 3 \bmod 4$. Therefore they span a summand and we can apply induction to the complementary summand.

All that remains to prove is: every 2-dimensional even unimodular lattice U is isomorphic to $\left(\begin{smallmatrix}0&1\\1&0\end{smallmatrix}\right)$ or $\left(\begin{smallmatrix}2&1\\1&2\end{smallmatrix}\right)$. If $U \otimes \mathbb{Q}_2$ has an isotropic vector then so does U. Choosing a primitive one as the first basis vector and using row and column operations proves $U \cong \left(\begin{smallmatrix}0&1\\1&0\end{smallmatrix}\right)$. This argument applies in particular if $\det U$ is (in the square class of) -1. This is because an orthogonal basis for $U \otimes \mathbb{Q}_2$ has isotropic inner product matrix $\left(\begin{smallmatrix}t&0\\0&-t\cdot\text{square}\end{smallmatrix}\right) \cong \left(\begin{smallmatrix}t&0\\0&-t\end{smallmatrix}\right)$.

Now suppose U is anisotropic. We have reduced to the case $\det U \equiv 3 \mod 8$. Therefore U has an inner product matrix of the form $\left(\begin{smallmatrix} 2u & 1 \\ 1 & 2v \end{smallmatrix}\right)$ where u, v are units. And $\det U \equiv 3 \mod 8$ also forces $u \equiv v \mod 8$. The vectors $(1,0)$, $(1,1)$, $(2,1)$ and $(3,1)$ have norms $\equiv 2u$, $6u$, $-2u$ and $-6u \mod 16$. Therefore U has a vector of norm $\equiv 2 \mod 16$. Rescaling it gives a norm 2 lattice vector. Rescaling a supplementary basis vector multiplies the determinant by a square, so we may suppose the determinant is exactly 3. Then a row/column operation lets us take the off-diagonal terms to be 1, after which $\det U = 3$ forces $U \cong \left(\begin{smallmatrix} 2 & 1 \\ 1 & 2 \end{smallmatrix}\right)$. □

Suppose L is a unimodular lattice decomposed as a direct sum corresponding to a diagonalization $\langle d_1, \ldots, d_n \rangle$, with the d_i units in $\mathbb{Z}_2$. By definition, both L's oddity and 2-signature are $d_1 + \cdots + d_n \pmod 8$. Every even unimodular lattice is a sum of copies of summands $\left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$ and $\left(\begin{smallmatrix} 2 & 1 \\ 1 & 2 \end{smallmatrix}\right)$. Our calculations (3.1)–(3.2) shows that such a lattice has 2-signature 0, hence also oddity 0.

Scaling a unimodular lattice by a power of 2 might change the 2-signature, by introducing or eliminating antisquares. But it leaves the oddity alone. For example, scaling $\langle 3 \rangle$ by 2 to get $\langle 6 \rangle$ changes the 2-signature from 3 to 7 but leaves the oddity equal to 3. The general rule is: any direct sum decomposition as in Lemma 3.1 has oddity equal to the sum (mod 8) of the odd parts of the 1-dimensional terms. We will discuss oddity further when we introduce fine decompositions in Section 4 and Jordan decompositions in Section 5.

For unimodular lattices, the dimension, sign, type and oddity turn out to be a complete set of invariants. We prove this in Theorem 5.1. Conway and Sloane express the isometry class of a unimodular lattice as $1_t^{\pm n}$ where $\pm$ is the sign, n is the dimension and t is either the formal symbol II (for even lattices) or the oddity (for odd lattices). In particular, the subscript implicitly records the parity of the lattice. We just saw that all even unimodular lattices have oddity 0, so in this case there is no point recording it.

If q is a power of 2 then (after Theorem 5.1) we will write $q_{\mathrm{II}}^{\pm n}$ or $q_t^{\pm n}$ for the lattice got from $1_{\mathrm{II}}^{\pm n}$ or $1_t^{\pm n}$ by rescaling all inner products by q. For example, 2_{II}^{-2} has inner product matrix $\left(\begin{smallmatrix} 4 & 2 \\ 2 & 4 \end{smallmatrix}\right)$. The subscript records whether the lattice has type I or II, which we recall is the scale-invariant generalization of the oddness/evenness of unimodular lattices. The number q is called the *scale* of the symbol (or lattice). Although one quickly learns the rules, the following table lets one read off the oddity and 2-signature of any sum of scaled unimodular lattices:

	L	$1_t^{\cdots}$ or $2_t^{+\cdots}$	$2_t^{-\cdots}$	$1_{\mathrm{II}}^{\cdots}$ or $2_{\mathrm{II}}^{\cdots}$	
(3.3)	$o(L)$	t	t	0	invariant under scaling by 2
	$\sigma_2(L)$	t	$t+4$	0	invariant under scaling by 4

Except for special cases, we will not use this notation until we have classified the unimodular lattices in Theorem 5.1. The special cases are in dimension 1 and the even case in dimension 2: for q any power of 2 we define

	q_1^{+1}	q_{-1}^{+1}	q_3^{-1}	q_{-3}^{-1}	q_{II}^{+2}	q_{II}^{-2}
as	$\langle q\rangle$	$\langle -q\rangle$	$\langle 3q\rangle$	$\langle -3q\rangle$	$\left(\begin{smallmatrix}0 & q\\ q & 0\end{smallmatrix}\right)$	$\left(\begin{smallmatrix}2q & q\\ q & 2q\end{smallmatrix}\right)$
with oddity	1	-1	3	-3	0	0

These definitions are compatible with the more general notation. We will usually omit the symbol $\oplus$ from direct sums, for example writing $1_{-1}^{+1}\,1_3^{-1}\,4_{\mathrm{II}}^{+2}$ for $1_{-1}^{+1}\oplus 1_3^{-1}\oplus 4_{\mathrm{II}}^{+2}$. To lighten the notation one usually suppresses plus signs in superscripts, for example $1_{-1}^{1}\,1_3^{-1}\,4_{\mathrm{II}}^{2}$, and/or suppresses the dimensions when they are 1, for example $1_{-1}^{+}\,1_3^{-}\,4_{\mathrm{II}}^{2}$. One could suppress even more, such as leaving the subscript blank for summands of type II. But excessive abbreviation is more error-prone than helpful.

4. Fine symbols

In this section we work with a finer decomposition of a lattice than the usual Jordan decomposition. The goal is to establish that certain "moves" between such decompositions do not change the isometry class of the lattice. This will make the corresponding facts for Jordan decompositions in the next section easy to state and prove. Theorem 4.4, proven in section 7, captures the full classification of 2-adic lattices, but in a very clumsy way. The rest of this paper recasts this classification in a simpler form.

By a *fine decomposition* of a lattice L we mean a direct sum decomposition in which each summand (or *term*) has the form $q_{\pm 1}^{1}$, $q_{\pm 3}^{-1}$ or $q_{\mathrm{II}}^{\pm 2}$, with the last case only occurring if every term of that scale has type II. The name reflects the fact that no further decomposition of the summands is possible. By (3.3), the oddity of (this decomposition of) L can be read off as the sum mod 8 of the numerical subscripts. And the 2-signature of L can be got from that by adding 4 for each term $q_{\dots}^{-\cdots}$ with $q = 2^{\text{odd}}$. A fine decomposition always exists, by starting with a decomposition as a sum of $q_t^{\pm 1}$'s and $q_{\mathrm{II}}^{\pm 2}$'s (Lemma 3.1) and applying the next lemma repeatedly.

Lemma 4.1. *If $\varepsilon, \varepsilon'$ are signs then $1_t^{\varepsilon 1}\,1_{\mathrm{II}}^{\varepsilon' 2}$ admits an orthogonal basis.*

Proof. Write M and N for the two summands and consider the three elements of $(M/2M)\oplus(N/2N)$ that lie in neither $M/2M$ nor $N/2N$. Any lifts of them have odd norms and even inner products. Applying row and column operations to their inner product matrix leads to a diagonal matrix with odd diagonal entries. □

In order to discuss the relation between distinct fine decompositions of a given lattice, we introduce the following special language for 1-dimensional lattices only. We call q_1^{+1} and q_{-3}^{-1} "givers" and q_{-1}^{+1} and q_3^{-1} "receivers". (Type II lattices are neither givers nor receivers.) The idea is that a giver can give away two oddity and remain a meaningful symbol ($q_1^+ \to q_{-1}^+$ or $q_{-3}^- \to q_3^-$), while a receiver can accept two oddity. We often use a subscript R or G in place of the oddity, so that 1_G^+ and 1_G^- mean 1_1^+ and 1_{-3}^-, while 1_R^+ and 1_R^- mean 1_{-1}^+ and 1_3^-. Scaling inner products by -3 negates signs and preserves giver/receiver status, while scaling them by -1 preserves signs and reverses giver/receiver status.

A *fine symbol* means a sequence of symbols $q_{\mathrm{II}}^{\pm 2}$ and $q_{R\,\mathrm{or}\,G}^{\pm}$. We replace R and G by numerical subscripts whenever convenient, and regard two symbols as the same if they differ by permuting terms. Two scales are called *adjacent* if they differ by a factor of 2.

Lemma 4.2 (Sign walking). *Consider a fine symbol and two terms of it that satisfy one of the following conditions:*

(0) *they have the same scale;*

(1) *they have adjacent scales and different types;*

(2) *they have adjacent scales and are both givers or both receivers;*

(3) *their scales differ by a factor of* 4 *and they both have type* I.

Consider as well the fine symbol got by negating the signs of these two terms, and in case (2) *also changing both from givers to receivers or vice-versa. Then the two fine symbols represent isometric lattices.*

An alternate name for (3) might be sign jumping. Conway and Sloane informally describe it as a composition of two sign walks of type (1). For example,

$$1_1^1 2_{\mathrm{II}}^{+0} 4_1^1 \quad\to\quad 1_{-3}^{-1} 2_{\mathrm{II}}^{-0} 4_1^1 \quad\to\quad 1_{-3}^{-1} 2_{\mathrm{II}}^{+0} 4_{-3}^{-1}.$$

They also observe that this doesn't really make sense: 2_{II}^{-0} is illegal because the 0-dimensional lattice has determinant 1, hence sign $+$.

Proof. It suffices to prove the following isometries, where $\varepsilon, \varepsilon'$ are signs, X represents R or G, and X' represents R or G:

$$(0) \qquad 1_{\mathrm{II}}^{\varepsilon 2}\, 1_{\mathrm{II}}^{\varepsilon' 2} \cong 1_{\mathrm{II}}^{-\varepsilon 2}\, 1_{\mathrm{II}}^{-\varepsilon' 2} \quad \text{and} \quad 1_X^{\varepsilon}\, 1_{X'}^{\varepsilon'} \cong 1_X^{-\varepsilon}\, 1_{X'}^{-\varepsilon'}$$

$$(1) \qquad 1_{\mathrm{II}}^{\varepsilon 2}\, 2_{X'}^{\varepsilon'} \cong 1_{\mathrm{II}}^{-\varepsilon 2}\, 2_{X'}^{-\varepsilon'} \quad \text{and} \quad 1_{X'}^{\varepsilon'}\, 2_{\mathrm{II}}^{\varepsilon 2} \cong 1_{X'}^{-\varepsilon'}\, 2_{\mathrm{II}}^{-\varepsilon 2}$$

$$(2) \qquad 1_G^{\varepsilon}\, 2_G^{\varepsilon'} \cong 1_R^{-\varepsilon}\, 2_R^{-\varepsilon'}$$

$$(3) \qquad 1_X^{\varepsilon}\, 4_{X'}^{\varepsilon'} \cong 1_X^{-\varepsilon}\, 4_{X'}^{-\varepsilon'}$$

The first part of (0) is trivial except for the assertion $1_{\text{II}}^{+2}\,1_{\text{II}}^{+2}\cong 1_{\text{II}}^{-2}\,1_{\text{II}}^{-2}$. Choose a norm 4 vector x of the right side. Then choose y to have inner product 1 with x. The span of x and y is even of determinant $\equiv -1 \bmod 8$, so it is a copy of 1_{II}^{+2}. Its orthogonal complement must also be even unimodular, hence one of $1_{\text{II}}^{\pm 2}$, hence 1_{II}^{+2} by considering the determinant.

The second part of (0) is best understood using numerical subscripts: we must show $1_t^{\varepsilon}\,1_{t'}^{\varepsilon'}\cong 1_{t+4}^{-\varepsilon}\,1_{t'+4}^{-\varepsilon'}$, i.e., $\langle t,t'\rangle\cong\langle t+4,t'+4\rangle$. To see this, note that the left side represents $t+4t'\equiv t+4 \bmod 8$, that this is odd and therefore corresponds to some direct summand, and the determinants of the two sides are equal. Note that givers and receivers always have oddities congruent to 1 and $-1 \bmod 4$ respectively, so changing a numerical subscript by 4 doesn't alter giver/receiver status. Furthermore, the sign on 1_t^{ε} changes since exactly one of $t, t+4$ lies in $\{\pm 1\}$ and the other in $\{\pm 3\}$, and similarly for $1_{t'}^{\varepsilon'}$. The same argument works for (3), in the form $1_t^{\varepsilon}\,4_{t'}^{\varepsilon'}\cong 1_{t+4}^{-\varepsilon}\,4_{t'+4}^{-\varepsilon'}$.

For the first part of (1) we choose a basis for $1_{\text{II}}^{\varepsilon 2}$ with inner product matrix $\left(\begin{smallmatrix}2 & 1\\ 1 & 0\text{ or }2\end{smallmatrix}\right)$ where the lower right corner depends on ε. Replacing the second basis vector by its sum with a generator of $2_{X'}^{\varepsilon'}$ changes the lower right corner by 2 mod 4. This toggles the 2×2 determinant between -1 and 3 mod 8. Therefore it gives an even unimodular summand of determinant -3 times that of $1_{\text{II}}^{\varepsilon 2}$, hence of sign $-\varepsilon$. Since the overall determinant is an invariant, the determinant of its complement is therefore -3 times that of $2_{X'}^{\varepsilon' 2}$. So the complement is got from $2_{X'}^{\varepsilon' 2}$ by scaling by -3. We observed above that scaling by -3 negates the sign and preserves giver/receiver status, so the complement is $2_{X'}^{-\varepsilon' 2}$. The second part of (1) follows from the first by passing to dual lattices and then scaling inner products by 2. (It is easy to see that the dual lattice has the same symbol with each scale replaced by its reciprocal.)

(2) After rescaling by -3 if necessary to take $\varepsilon=+$, it suffices to prove $1_G^{+}\,2_G^{\varepsilon'}\cong 1_R^{-}\,2_R^{-\varepsilon'}$, i.e., $\langle 1,2\rangle\cong\langle 3,6\rangle$ and $\langle 1,-6\rangle\cong\langle 3,-2\rangle$. In each case one finds a vector on the left side whose norm is odd and appears on the right, and then compares determinants. □

Further equivalences between fine symbols are phrased in terms of "compartments". A *compartment* means a set of type I terms, the set of whose scales forms a sequence of consecutive powers of 2, and which is maximal with these properties. For example in $1_{\text{II}}^{2}\,2_G^{-}\,2_R^{-}\,4_G^{+}\,16_R^{-}$, the set of scales that have type I are $\{2,4,16\}$. These fall into two strings of consecutive powers of 2, namely $\{2,4\}$ and $\{16\}$. So there are two compartments: $2_G^{-}\,2_R^{-}\,4_G^{+}$ and 16_R^{-}.

Lemma 4.3 (Giver permutation and conversion). *Consider a fine symbol and the symbol obtained by one of the following operations. Then the lattices they represent are isometric and have the same oddity.*

(1) *Permute the subscripts G and R within a single compartment.*

(2) *Convert any four G's in a compartment to R's, or vice versa.*

Proof. Giver permutation, meaning operation (1), can be achieved by repeated use of the isomorphisms $1^{\varepsilon}_{G}1^{\varepsilon'}_{R} \cong 1^{\varepsilon}_{R}1^{\varepsilon'}_{G}$ and $1^{\varepsilon}_{G}2^{\varepsilon'}_{R} \cong 1^{\varepsilon}_{R}2^{\varepsilon'}_{G}$ (scaled up or down as necessary). To establish these we first rescale by -3 if necessary, to take $\varepsilon = +$ without loss of generality. This leaves the cases $\langle 1,-1\rangle \cong \langle -1,1\rangle$, $\langle 1,3\rangle \cong \langle -1,-3\rangle$, $\langle 1,-2\rangle \cong \langle -1,2\rangle$ and $\langle 1,6\rangle \cong \langle -1,10\rangle$. One proves each by finding a vector on the left whose norm is odd and appears on the right, and then comparing determinants. To see the invariance of the oddity, imagine the giver giving 2 oddity to the receiver. This converts the giver to a receiver and vice-versa.

For giver conversion, meaning operation (2), the oddity is invariant by the same argument as for giver permutation. It remains to prove the isomorphism of the lattices. We assume first that more than one scale is present in the compartment, so we can choose terms of adjacent scales. Assuming four G's are present in the compartment, we permute a pair of them to our chosen terms, then use sign walking to convert these terms to receivers. This negates both signs. Then we permute these R's away, replacing them by the second pair of G's, and repeat the sign walking. This converts the second pair of G's to R's and restores the original signs.

For the case that only a single scale is present we first treat what will be the essential cases, namely

$$1^{+}_{G}1^{+}_{G}1^{+}_{G}1^{+}_{G} \cong 1^{+}_{R}1^{+}_{R}1^{+}_{R}1^{+}_{R} \quad \text{and} \quad 1^{-}_{G}1^{+}_{G}1^{+}_{G}1^{+}_{G} \cong 1^{-}_{R}1^{+}_{R}1^{+}_{R}1^{+}_{R}$$

That is,

$$\langle 1,1,1,1\rangle \cong \langle -1,-1,-1,-1\rangle \quad \text{and} \quad \langle -3,1,1,1\rangle \cong \langle 3,-1,-1,-1\rangle$$

In the first case we exhibit a suitable basis for the left side, namely $(2,1,1,1)$ and the images of $(-1,2,1,-1)$ under cyclic permutation of the last 3 coordinates. In the second we note that the left side is the orthogonal sum of the span of $(1,0,0,0)$ and $(0,1,1,1)$, which is a copy of $\langle -3,3\rangle$, and the span of $(0,-1,1,0)$ and $(0,0,1,-1)$, which is a copy of 1^{-2}_{II}. Since each of these is isometric to its scaling by -1, so is their direct sum.

Now we treat the general case when only a single scale is present. Suppose there are at least 4 givers. By scaling by a power of 2 it suffices to treat the unimodular case. By sign walking we may change the signs on any even number of them, so we may suppose at most one $-$ is present. (Recall that sign walking between terms of the same scale doesn't affect subscripts G or R.) By the previous paragraph we may convert four G's to R's. Then we reverse the sign walking operations to restore the original signs. □

The following theorem captures the full classification of 2-adic lattices. It is already simpler than the results in [Jon] and [Pal]. But fine symbols package information poorly, and much greater simplification is possible. We will develop this in the next two sections.

Theorem 4.4 (Equivalence of fine symbols). *Two fine symbols represent isometric lattices if and only if they are related by a sequence of sign walking, giver permutation and giver conversion operations.*

Although it is natural to state the theorem here, its proof depends on Theorem 5.1. The first place we use it is to prove Theorem 6.2, so logically the proof could go anywhere in between. But in fact we defer it to Section 7 to avoid breaking the flow of ideas.

5. Jordan symbols

In this section we define and study the Jordan decompositions of a lattice. The main point is that "oddity fusion" neatly wraps up all the giver permutation and conversion operations from the previous section. We begin by classifying the unimodular lattices:

Theorem 5.1 (Unimodular lattices). *A unimodular lattice is characterized by its dimension, type, sign and oddity.*

Recall that for unimodular lattices, the oddity is defined to be the 2-signature. Since the 2-signature is a genuine invariant of lattices, the oddity is a genuine invariant of unimodular lattices. Also, recall from (3.1)–(3.2) that all even unimodular lattices have vanishing 2-signature (hence vanishing oddity).

Proof. Consider unimodular lattices U, U' with the same dimension, type, sign and oddity, and fine symbols F, F' for them. The product of the signs in F equals the sign of U, and similarly for U'. Since U and U' have the same sign, we may use sign walking to make the signs in F the same as in F'. If U, U' are even then the terms in F are now the same as in F', so $U \cong U'$. So suppose U, U' are odd.

By giver permutation, and exchanging F and F' if necessary, we may suppose that all non-matching subscripts are R in F and G in F'. And by giver conversion we may suppose that the number of non-matching subscripts is $k \leq 3$. Since changing a receiver to a giver without changing the sign increases the oddity by two, $o(U') = o(U) + 2k$. Since $o(U') \equiv o(U) \mod 8$ we have $k = 0$. So the terms in F are the same as in F', and $U \cong U'$. □

We now have license to use the notation $q_t^{\pm n}$ and $q_{\mathrm{II}}^{\pm n}$ from Section 3. We say that such a symbol is *legal* if it represents a lattice. The legal symbols are

$$\begin{array}{l} q_{\mathrm{II}}^{+0} \\ q_{\pm 1}^{+1} \quad \text{and} \quad q_{\pm 3}^{-1} \\ q_0^{+2}, \quad q_{\pm 2}^{+2}, \quad q_4^{-2} \quad \text{and} \quad q_{\pm 2}^{-2} \\ q_t^{\pm n} \text{ with } n > 2 \text{ and } t \equiv n \bmod 2 \\ q_{\mathrm{II}}^{\pm n} \text{ with } n \text{ positive and even} \end{array}$$

A good way to mentally organize these is to regard the conditions for dimensions $\neq 1, 2$ as obvious, remember that q_4^2 and q_0^{-2} are illegal, and remember that the subscript of $q_t^{\pm 1}$ determines the sign.

The illegality of 1_4^2 and 1_0^{-2} follows by considering all possible sums $1_t^{\varepsilon 1} 1_{t'}^{\varepsilon' 1}$. When the signs $\varepsilon, \varepsilon'$ are the same, either both subscripts are in $\{\pm 1\}$ or both are in $\{\pm 3\}$, so the total oddity cannot be 4. When the signs are different, one subscript is ± 1 and the other is ± 3, so the total oddity cannot be 0.

This calculation used the simple rules for direct sums of unimodular lattices: signs multiply and dimensions and subscripts add, subject to the special rules $\mathrm{II} + \mathrm{II} = \mathrm{II}$ and $\mathrm{II} + t = t$.

A *Jordan decomposition* of a lattice means a direct sum decomposition whose summands (called *constituents*) are unimodular lattices scaled by distinct powers of 2. By the *Jordan symbol* for the decomposition we mean the list of the symbols (or *terms*) $q_{\mathrm{II}}^{\pm n}$ and $q_t^{\pm n}$ for the summands. An example we will use in this section and the next, and mentioned already in the introduction, is

$$1_{\mathrm{II}}^2\, 2_6^{-2} 4_{-3}^3\, 16_1^1\, 32_{\mathrm{II}}^2\, 64_{\mathrm{II}}^{-2}\, 128_1^1\, 256_{-1}^1\, 512_{\mathrm{II}}^{-4} \tag{5.1}$$

It is sometimes convenient and sometimes annoying to allow trivial (0-dimensional) terms in a Jordan decomposition.

The main difficulty of 2-adic lattices is that a given lattice may have several inequivalent Jordan decompositions. The purpose of the Conway–Sloane calculus is to allow one to move easily between all possible isometry classes of Jordan decompositions. Some of the data in the Jordan symbol remains invariant under these moves. First, if one has two Jordan decompositions for the same lattice L, then each term in one has the same dimension as the term of that scale in the other. (Scaling reduces the general case to the integral case, which follows by considering the structure of the abelian group L^*/L.) Second, the type I or II of the term of any given scale is independent of the Jordan decomposition. (One can show this directly, but we won't need it until after Theorem 6.2, which implies it.) The signs and oddities of the constituents are not usually invariants of L.

We define a *compartment* of a Jordan decomposition just as we did for fine decompositions: a set of type I constituents, whose scales form a sequence of consecutive powers of 2, and which is maximal with these properties. The example above has three compartments: $2_6^{-2}4_{-3}^3$, 16_1^1 and $128_1^1 256_{-1}^1$. The *oddity* of a compartment means the oddity of the direct sum of its Jordan constituents. By the definition of the symbols, the compartment oddity is the sum (mod 8) of the subscripts of those constituents. *Caution: The oddity of a compartment depends on the Jordan decomposition, and is not an isometry invariant of the underlying lattice*. See Lemma 6.1 for how it can change. Despite this non-invariance, the oddity of a compartment is useful:

Lemma 5.2 (Oddity fusion). *Consider a lattice, a Jordan symbol J for it, and the Jordan symbol J' got by reassigning all the subscripts in a compartment, in such a way that that all resulting terms are legal and the compartment's oddity remains unchanged. Then J, J' represent isometric lattices.*

Proof. By discarding the rest of J we may suppose it is a single compartment. The argument is similar to the odd case of Theorem 5.1. We refine J, J' to fine symbols F, F'. By hypothesis, the terms of J' have the same signs as those of J. It follows that for each scale, the product of the signs of F's terms of that scale is the same as the corresponding product for F'. Therefore sign walking between equal-scale terms lets us suppose that the signs in F are the same as in F'. Recall from the proof of Lemma 4.2(0) that this sort of sign walking amounts to the isomorphisms $1_t^{\varepsilon 1} 1_{t'}^{\varepsilon' 1} \cong 1_{t+4}^{-\varepsilon 1} 1_{t'+4}^{-\varepsilon' 1}$, which don't change the compartment's oddity. By Lemma 4.3, giver permutation and conversion also leave the compartment oddity invariant.

By giver permutation and possibly swapping F with F', we may suppose that the non-matching subscripts are R's in F and G's in F'. By giver conversion we may suppose $k \leq 3$ subscripts fail to match, and the assumed equality of oddities shows $k = 0$. Therefore the fine symbols are the same, so the lattices are isometric. □

6. 2-adic symbols

One can translate sign walking between fine symbols to the language of Jordan symbols, but it turns out to be fussier than necessary. Things become simpler once we incorporate oddity fusion into the notation as follows. The 2*-adic symbol* of a Jordan decomposition means the Jordan symbol, except that each compartment is enclosed in brackets and the enclosed terms are stripped of their subscripts, whose sum in $\mathbb{Z}/8$ is attached to the right bracket as a subscript. This is called the compartment's oddity. For our example (5.1) this yields

$$1_{\mathrm{II}}^{2}\,[2^{-2}4^{3}]_{3}[16^{1}]_{1}32_{\mathrm{II}}^{2}\,64_{\mathrm{II}}^{-2}\,[128^{1}\,256^{1}]_{0}512_{\mathrm{II}}^{-4}$$

If a compartment consists of a single term, such as $[16^1]_1$, then one usually omits the brackets:

$$1_{\mathrm{II}}^{2}\,[2^{-2}4^{3}]_{3}16_{1}^{1}\,32_{\mathrm{II}}^{2}\,64_{\mathrm{II}}^{-2}\,[128^{1}\,256^{1}]_{0}512_{\mathrm{II}}^{-4} \tag{6.1}$$

Lemma 5.2 shows that the isometry type of a lattice with given 2-adic symbol is well-defined.

If a compartment has total dimension ≤ 2 then its oddity is constrained by its overall sign in the same way as for an odd unimodular lattice of that dimension. For compartments of dimension 1 this is the same constraint as before. In 2 dimensions, $[1^+2^-]_0$ and $[1^-2^+]_0$ are illegal (cannot come from any fine symbol) because each term $1^+_{\dots}$ or $2^+_{\dots}$ would have ± 1 as its subscript, while each term $1^-_{\dots}$ or $2^-_{\dots}$ would have ± 3 as its subscript. There is no way to choose subscripts summing to 0. The same reasoning shows that $[1^+2^+]_4$ and $[1^-2^-]_4$ are also illegal.

Lemma 6.1 (Sign walking for 2-adic symbols). *Consider the 2-adic symbol of a Jordan decomposition of a lattice, and two nontrivial terms of it that satisfy one of the following:*

(1) *They have adjacent scales and different types;*

(2) *they have adjacent scales and type* I*, and their compartment either has dimension > 2 or compartment oddity ± 2;*

(3) *they have type* I*, their scales differ by a factor of 4, and the term between them is trivial.*

Then the 2-adic symbol got by negating their signs, and changing by 4 the oddity of each compartment that contains at least one of the terms, represents an isometric lattice.

Remark. As in Lemma 4.2, one could also call (3) sign jumping. If the intermediate term were nontrivial of type II, then one could achieve the same effect by two moves (1). If the intermediate term had type I, then one could achieve a *similar* effect by two moves (2). It would not be quite the same, because both moves would affect the same compartment. So its oddity would change by 4 twice, i.e., not at all.

Our example (6.1)

$$
\begin{array}{lll}
 & 1^{2}_{\mathrm{II}}\,[2^{-2}4^{3}]_{3}\,16^{1}_{1}\,32^{2}_{\mathrm{II}}\,64^{-2}_{\mathrm{II}}\,[128^{1}\,256^{1}]_{0}512^{-4}_{\mathrm{II}} & \\
\text{can walk to} & \underbrace{1^{-2}_{\mathrm{II}}\,[2^{2}}\,4^{3}]_{-1}\,16^{1}_{1}\,32^{2}_{\mathrm{II}}\,64^{-2}_{\mathrm{II}}\,[128^{1}\,256^{1}]_{0}512^{-4}_{\mathrm{II}} & \text{by (1)},\\
\text{or} & 1^{2}_{\mathrm{II}}\,\underbrace{[2^{2}4^{-3}]}_{-1}\,16^{1}_{1}\,32^{2}_{\mathrm{II}}\,64^{-2}_{\mathrm{II}}\,[128^{1}\,256^{1}]_{0}512^{-4}_{\mathrm{II}} & \text{by (2)},\\
\text{or} & 1^{2}_{\mathrm{II}}\,[2^{-2}\,\underbrace{4^{-3}]_{-1}16^{-1}_{-3}}\,32^{2}_{\mathrm{II}}\,64^{-2}_{\mathrm{II}}\,[128^{1}\,256^{1}]_{0}512^{-4}_{\mathrm{II}} & \text{by (3)}.
\end{array}
$$

Underbrackets indicate the terms involved in the moves. One can also sign walk between scales 16 and 32, between scales 64 and 128, and between scales 256 and 512. But no sign walk is possible between scales 128 and 256, because that compartment has dimension 2 and oddity $\not\equiv \pm 2 \bmod 8$. More conceptually, a sign walk would give the illegal compartment $[128^{-}256^{-}]_4$.

Proof. Refine the Jordan decomposition to a fine decomposition F, apply the corresponding sign walk operation (1)–(3) from Lemma 4.2 to suitable terms of F, and observe the corresponding change in the 2-adic symbol. In case (2) some care is required because Lemma 4.2 requires both terms of F to be givers or both to be receivers. If the compartment has dimension > 2 then we may arrange this by giver permutation (which preserves compartment oddity by Lemma 4.3). In dimension 2 the hypothesis

$$(\text{compartment oddity}) \equiv \pm 2 \mod 8$$

rules out the case that one is a giver and one a receiver, since givers and receivers have subscripts 1 and -1 mod 4 respectively. □

Theorem 6.2 (Equivalence of 2-adic symbols). *Suppose given two lattices with Jordan decompositions. Then the lattices are isometric if and only if the* 2*-adic symbols of these decompositions are related by a sequence of the sign walk operations in Lemma* 6.1.

Proof. The previous lemma shows that sign walks preserve isometry type. So suppose the lattices are isometric. Refine the Jordan decompositions to fine decompositions, apply Theorem 4.4 to obtain a chain of intermediate fine symbols, and consider the corresponding 2-adic symbols. Lemma 4.3 shows that giver permutation and conversion don't change compartment oddities, so they leave 2-adic symbol unchanged. In the proof of Lemma 5.2 we explained why sign walking between same-scale terms also has no effect. The effects of the remaining sign walk operations are recorded in Lemma 6.1. □

A lattice may have more than one 2-adic symbol, but the only remaining freedom lies in the positions of the signs:

Theorem 6.3. *Suppose two given lattices have 2-adic symbols with the same scales, dimensions, types and signs. Then the lattices are isometric if and only if the symbols are equal, which amounts to having the same compartment oddities.*

Proof. If a 2-adic symbol S of a lattice L admits a sign walk affecting the signs of the terms of scales 2^i, 2^j then we write $\Delta_{i,j}(S)$ for the resulting symbol. No sign walks affect the conditions for $\Delta_{i,j}$ to act on S, since they don't change the type of any term or the oddity mod 4 of any compartment. So we may regard $\Delta_{i,j}$ as acting simultaneously on all 2-adic symbols for L. By its description in terms of negating signs and adjusting compartments' oddities, $\Delta_{i,j}$ may be regarded as an element of order 2 in the group $\{\pm 1\}^T \times (\mathbb{Z}/8)^C$ where T is the number of terms present and C is the number of compartments.

The assertion of the lemma is that if a sequence of sign walks on S restores the original signs, then it also restores the original oddities. We rephrase this in terms of the subgroup A of $\{\pm 1\}^T \times (\mathbb{Z}/8)^C$ generated by the $\Delta_{i,j}$. Namely: projecting A to the $\{\pm 1\}^T$ factor has trivial kernel. This is easy to see because the $\Delta_{i,j}$ are ordered so that they are $\Delta_{i_1,j_1}, \dots, \Delta_{i_n,j_n}$ with $i_1 < j_1 \le i_2 < j_2 \le \dots \le i_n < j_n$. The linear independence of their projections to $\{\pm 1\}^T$ is obvious. □

To get a canonical symbol for a lattice L one starts with any 2-adic symbol S and walks all the minus signs as far left as possible, canceling them when possible. To express this formally, we say two scales *can interact* if their terms are as in Lemma 6.1. (We noted in the previous proof that the ability of two scales to interact is independent of the particular 2-adic symbol representing L.) We define a *signway* as an equivalence class of scales, under the equivalence relation generated by interaction. The language suggests a pathway or highway along which signs can move (or cancel).

One can find the signways without fussing with the conditions in Lemma 6.1. First cut the 2-adic symbol into the "trains" of Conway and Sloane, and then (in rare cases) cut some of the trains into signways. To cut the 2-adic symbol into trains, cut between each pair of consecutive powers of 2 for which both terms have type II. This includes 0-dimensional constituents, which always have type II. To get the signways, inspect each train for "bad" compartments $[q^{\pm 1} r^{\pm 1}]_{0 \text{ or } 4}$ where q, r are consecutive powers of 2. Cut in the middle of each such compartment. To get the three signways in our example,

split here to get two trains

$$\downarrow$$

$$1^2_{\mathrm{II}}\,[2^{-2}4^3]_3 16^1_1\, 32^2_{\mathrm{II}} \qquad 64^{-2}_{\mathrm{II}}\,[128^1 \quad 256^1]_0 512^{-4}_{\mathrm{II}}$$

$$\uparrow$$

and then split the second train here.

Each signway has a term of smallest scale, and by sign walking we may suppose that all minus signs are moved to these terms or canceled with each other. Then we say the symbol is in *canonical form*, which for our example is

$$1_{\mathrm{II}}^{-2}\,[2^2 4^3]_{-1}\,16_1^1\,32_{\mathrm{II}}^2\,64_{\mathrm{II}}^{-2}\,[128^1\,256^{-1}]_4 512_{\mathrm{II}}^4$$

Theorem 6.3 implies:

Corollary 6.4 (Canonical form). *Given lattices L, L' and 2-adic symbols S, S' for them in canonical form, $L \cong L'$ if and only if $S = S'$.* □

Conway and Sloane's discussion of the canonical form is in terms of trains. They asserted that signs can walk up and down the length of a train, so that after walking signs leftward, there is at most one sign per train. But this is not true, as pointed out in [All1]. One cannot walk the minus sign in $[128^1\,256^{-1}]_4$ leftward because there is no way to assign the subscripts in $128_{\pm 3}^{-}\,256_{\pm 1}^{+}$ so that the compartment has oddity 0.

Example 6.5. As an extended demonstration of sign walking, we determine the lattices M with the property that $M \oplus \langle 2,2\rangle \cong L$ where L is from (6.1). Note that $\langle 2,2\rangle = 2_2^2$. Obviously we require

$$M = \underbrace{1_{\mathrm{II}}^{\pm 2}}\,\underbrace{4_?^{\pm 3} 16_?^{\pm 1} 32_{\mathrm{II}}^{\pm 2}}\,\underbrace{64_{\mathrm{II}}^{\pm 2}[128^{\pm 1}}\,\underbrace{256^{\pm 1}]_? 512_{\mathrm{II}}^{\pm 4}}$$

We have marked the signways with underbrackets. The 3rd and 4th of these become the 2nd and 3rd signways of L after summing with 2_2^2. No sign walking is possible between distinct signways. So the isomorphism $M \oplus 2_2^2 \cong L$ shows that the terms in these signways in M can be taken to coincide with the corresponding terms in L. Next, the first two signways of M fuse with the 2_2^2 summand to form the first signway of L. The overall sign of this in L is $-$, so the total number of $-$ signs in the first two signways of M must be odd. By sign walking in the second signway of M, we reduce to

$$M \cong \left(1_{\mathrm{II}}^{-2} 4_t^3 16_u^1 32_{\mathrm{II}}^2 \quad \text{or} \quad 1_{\mathrm{II}}^2 4_t^{-3} 16_u^1 32_{\mathrm{II}}^2\right) \oplus 64_{\mathrm{II}}^{-2}[128^1 256^1]_0 512_{\mathrm{II}}^{-4}$$

where t and u are unknowns. Now we sum with 2_2^2 to get

$$L \cong \left(1_{\mathrm{II}}^{-2}[2^2 4^3]_{2+t} 16_u^1 32_{\mathrm{II}}^2 \quad \text{or} \quad 1_{\mathrm{II}}^2[2^2 4^{-3}]_{2+t} 16_u^1 32_{\mathrm{II}}^2\right) \oplus \cdots$$

Then we sign walk between the first two terms, or between the second and third, to make the signs match those in (6.1). That is,

$$L \cong \left(1_{\mathrm{II}}^2[2^{-2} 4^3]_{6+t} 16_u^1 32_{\mathrm{II}}^2\right) \oplus \cdots$$

Both this and (6.1) represent L, and the signs match, so the subscripts must too. Therefore $6+t=3$ and $u=1$. That is,

$$M \cong 1_{\mathrm{II}}^{\pm 2} 4_5^{\mp 3} 16_1^1 32_{\mathrm{II}}^2 64_{\mathrm{II}}^{-2} [128^1 256^1]_0 512_{\mathrm{II}}^{-4}$$

where one ambiguous sign is $+$ and the other is $-$. The two possibilities are distinct because both are in canonical form. It follows that the isometry group of L has two orbits on summands isomorphic to $\langle 2,2\rangle$.

One can use the ideas of the proof of Theorem 6.3 to give numerical invariants for lattices, if one prefers them to a canonical form. The following invariants come from Theorem 10 of [CS, Ch. 15], which is proven in [Xu]. One records the scales, dimensions and types, the *adjusted oddity* of each compartment, and the overall sign of each signway (the product of the signs of the signway's terms). Here the adjusted oddity of a compartment means its oddity plus 4 for each $-$ sign appearing in its 1st, 3rd, 5th, ... position, with each $-$ sign after that compartment counted as occurring in the "$(k+1)$st" position, where k is the number of terms in the compartment. It is easy to check that sign walking leaves these quantities unchanged.

As an example, the adjusted oddity of the compartment $[2^{-2}4^3]_3$ in

$$1_{\mathrm{II}}^2 [2^{-2}4^3]_3 16_1^1\, 32_{\mathrm{II}}^2\, 64_{\mathrm{II}}^{-2}\, [128^1\, 256^1]_0 512_{\mathrm{II}}^{-4}$$

is $3+4+4+4=7$. The 3 is the ordinary oddity, and the first 4 is because the compartment has sign $-$ in its first position. The last two 4's come from the signs on the terms of scales 64 and 512. For purposes of the adjusted oddity, each of these counts as appearing in the "third" position of the compartment, hence contributes 4 to the adjusted oddity.

These invariants are clumsy because of the definition of adjusted oddity, which has the ugly feature that it depends on signs outside the signway containing the relevant compartment. This goes against the principle that simplified Example 6.5: distinct signways are isolated from each other.

Furthermore, these invariants are really just a complicated way of recording the canonical form while pretending not to. We will show how to construct the unique 2-adic symbol in canonical form having the same invariants as any chosen 2-adic lattice. To do this we first observe that the types of the constituents determine the compartments. The oddities of the compartments are the same mod 4 as the given adjusted oddities. This data controls which sign walks are possible, hence determines the signways. We set the sign of the first term of each signway equal to the given overall sign of that signway, and the other signs to $+$. The signs then allow one to compute the compartment oddities from the adjusted oddities.

7. Equivalences between fine decompositions

In this section we give the deferred proof of Theorem 4.4: two fine symbols represent isometric lattices if and only if they are related by sign walks and giver permutation and conversion. Logically, it belongs anywhere between Theorems 5.1 and 6.2. The next two lemmas are standard; our proofs are adapted from Cassels [Cas, pp. 120–122].

Lemma 7.1. *Suppose L is an integral lattice, that $x, x' \in L$ have the same odd norm, and that their orthogonal complements $x^\perp, x'^\perp$ are either both odd or both even. Then $x^\perp \cong x'^\perp$.*

Proof. First, $(x-x')^2$ is even. If it is twice an odd number then the reflection in $x-x'$ is an isometry of L. This reflection exchanges x and x', so it gives an isometry between $x^\perp$ and $x'^\perp$. This argument applies in particular if $x \cdot x'$ is even. So we may restrict to the case that $x \cdot x'$ is odd and $(x-x')^2$ is divisible by 4. Next, note that $(x+x')^2$ differs from $(x-x')^2$ by $4x \cdot x' \equiv 4 \bmod 8$. So by replacing x' by $-x'$ we may suppose that $(x-x')^2 \equiv 4 \bmod 8$. This replacement is harmless because $\pm x'$ have the same orthogonal complement.

If it happens that $(x-x') \cdot L \subseteq 2\mathbb{Z}_2$ then the reflection in $x-x'$ preserves L and we may argue as before. So suppose some $y \in L$ has odd inner product with $x-x'$. Then the inner product matrix of $x, x-x', y$ is

$$\begin{pmatrix} 1 & 0 & ? \\ 0 & 0 & 1 \\ ? & 1 & ? \end{pmatrix} \quad \bmod 2,$$

which has odd determinant. Therefore these three vectors span a unimodular summand of L, so L has a Jordan decomposition whose unimodular part L_0 contains both x and x'. Note that x's orthogonal complement in L_0 is even just if its orthogonal complement in L is, and similarly for x'. So by discarding the rest of the decomposition we may suppose $L = L_0$, without losing our hypothesis that $x^\perp, x'^\perp$ are both odd or both even. Now, $x^\perp$ is unimodular with $\det(x^\perp) = (\det L)/x^2$ and oddity $o(x^\perp) = o(L) - x^2$, and similarly for x'. Since $x^2 = x'^2$, Theorem 5.1 implies $x^\perp \cong x'^\perp$. □

Lemma 7.2. *Suppose L is an integral lattice and $U, U' \subseteq L$ are isometric even unimodular sublattices. Then $U^\perp \cong U'^\perp$.*

Proof. $U \oplus \langle 1 \rangle$ has an orthogonal basis $x_1, \dots, x_n$ by Lemma 4.1, and we write $x'_1, \dots, x'_n$ for the basis for $U' \oplus \langle 1 \rangle$ corresponding to it under some isometry $U \cong U'$. Apply the previous lemma n times, starting with $L \oplus \langle 1 \rangle$. (In the nth application we need the observation that the orthogonal complements of U, U' in L are both even or both odd. This holds because these orthogonal complements are even or odd according to whether L is even or odd.) □

Lemma 7.3. *Suppose L is an integral lattice and that 1^+_G is a term in some fine symbol for L. Then we may apply a sequence of sign walking and giver permutation and conversion operations to transform any other fine symbol F for L into one possessing a term 1^+_G.*

Proof. We claim first that after some of these operations we may suppose F has a term $1^+_{\cdots}$. Because L is odd, F's terms of scale 1 have the form $1^{\pm}_{R\,\mathrm{or}\,G}$. If F has more than one such term then we can obtain a sign $+$ by sign walking, so suppose it has only one term, of sign $-$. If there are type I terms of scale 4 then again we can use sign walking, so suppose all scale 4 terms have type II. We can do the same thing if there are any terms $2^{\pm 2}_{\mathrm{II}}$. Or terms $2^{\pm 1}_{R\,\mathrm{or}\,G}$, if the compartment consisting of the scale 1 and 2 terms has at least two givers or two receivers. This holds in particular if there is more than one term of scale 2. So we have reduced to the case

$$F = 1^-_{R\,\mathrm{or}\,G}\, 4^{\cdots}_{\mathrm{II}}\, 8^{\cdots}_{\cdots} \cdots \quad \text{or} \quad F = 1^-_{R\,\mathrm{or}\,G}\, 2^{\pm}_{R\,\mathrm{or}\,G}\, 4^{\cdots}_{\mathrm{II}}\, 8^{\cdots}_{\cdots} \cdots$$

where in the latter case one subscript is G and the other is R. (Here and below, the superscript and subscript dots indicate any possibilities for the number of terms at that scale, and their decorations in that position. In particular, there might be no terms of that scale. The dots at the end indicate terms of higher scale than the ones already listed.) By giver permutation we may suppose

$$F = 1^-_{R\,\mathrm{or}\,G}\, 4^{\cdots}_{\mathrm{II}}\, 8^{\cdots}_{\cdots} \cdots \quad \text{or} \quad F = 1^-_G\, 2^{\pm}_R\, 4^{\cdots}_{\mathrm{II}}\, 8^{\cdots}_{\cdots} \cdots$$

None of these cases occur, because these lattices don't represent 1 mod 8, contrary to the hypothesis that some fine decomposition has a term 1^+_G. This non-representation is easy to see because L is $\langle \pm 3\rangle$ or $\langle 5, -2\rangle$ or $\langle 5, 6\rangle$, plus a lattice in which all norms are divisible by 8.

So we may suppose F has a term $1^+_{\cdots}$, and must show that after further operations we may suppose it has a term 1^+_G. We are done unless our term $1^+_{\cdots}$ is 1^+_R. If the compartment C containing it has any givers then we may use giver permutation to complete the proof. So suppose C consists of receivers. If there are 4 receivers then we may convert them to givers, reducing to the previous case. If C has two terms of different scales, neither of which is our 1^+_R term, then we may use sign walking to convert them to givers, again reducing to a known case. Only a few cases remain, none of which actually occur, by a similar argument to the previous paragraph.

Namely, after more sign walking we may take F to be

$$(1^+_R\, 2^\pm_R \text{ or } 1^+_R\, 2^+_R\, 2^\pm_R)4^{\cdots}_{\mathrm{II}}\cdots \quad \text{or} \quad (1^+_R \text{ or } 1^+_R\, 1^\pm_R \text{ or } 1^+_R\, 1^+_R\, 1^\pm_R)2^{\cdots}_{\mathrm{II}}\cdots$$

The first set of possibilities is

$$\big(\langle -1,-2\rangle \text{ or } \langle -1,6\rangle \text{ or } \langle -1,-2,-2\rangle \text{ or } \langle -1,-2,6\rangle\big) \oplus \text{(a lattice with all norms divisible by 8)}$$

none of which represent $1 \bmod 8$. The second set of possibilities is

$$\big(\langle -1\rangle \text{ or } \langle -1,-1\rangle \text{ or } \langle -1,3\rangle \text{ or } \langle -1,-1,-1\rangle \text{ or } \langle -1,-1,3\rangle\big) \oplus \text{(a lattice with all norms divisible by 4)}$$

and only the last two cases represent $1 \bmod 8$. But in these cases every vector x of norm $1 \bmod 8$ projects to $\bar{x} := (1,1,1)$ in $U/2U$, where U is the summand $\langle -1,-1,-1\rangle$ or $\langle -1,-1,3\rangle$. There are no odd-norm vectors orthogonal to x since the orthogonal complement of $\bar{x}$ in $U/2U$ consists entirely of self-orthogonal vectors. So while these lattices admit norm 1 summands, they do not admit fine decompositions with 1^+_G terms. □

Lemma 7.4. *Suppose* $\varepsilon = \pm$. *Then Lemma* 7.3 *holds with* $1^{\varepsilon 2}_{\mathrm{II}}$ *in place of* 1^+_G.

Proof. If F has two terms of scale 1, or a scale 2 term of type I, then we can use sign walking. The only remaining case is $F = 1^{-\varepsilon 2}_{\mathrm{II}}\, 2^{\cdots}_{\mathrm{II}}\, 4^{\cdots}_{\cdots}\cdots$. Write U for the $1^{-\varepsilon 2}_{\mathrm{II}}$ summand and note that any two elements of L with the same image in $L/(2U \oplus U^\perp) = U/2U$ have the same norm mod 4. Direct calculation shows that the norms of the nonzero elements of $U/2U$ are $0,0,2$ or $2,2,2$ mod 4, depending on ε. Now consider the summand $U' \cong 1^{\varepsilon 2}_{\mathrm{II}}$ of L that we assumed to exist. By considering norms mod 4 we see that $U'/2U' \to U/2U$ cannot be injective, so it must have image 0 or $\mathbb{Z}/2$. Since all self-inner products in $U/2U$ vanish, we obtain the absurdity that all inner products in U' are even. □

Proof of Theorem 4.4. The "if" part has already been proven in Lemmas 4.2 and 4.3, so we prove "only if". We assume the result for all lattices of lower dimension. By scaling by a power of 2 we may suppose L is integral and some inner product is odd, so each of F and F' has a nontrivial unimodular term.

First suppose L is odd, so the unimodular terms of F and F' have type I. By rescaling L by an odd number we may suppose F has a term 1^+_G. By Lemma 7.3 we may apply our moves to F' so that it also has a term 1^+_G. The orthogonal complements of the corresponding summands of L are both even (if the unimodular Jordan constituents are 1-dimensional) or both odd (otherwise). By Lemma 7.1 these orthogonal complements are isometric. They come with fine decompositions, given by the remaining terms in F, F'. By induction on dimension these fine decompositions are equivalent by our moves.

If L is even then the same argument applies, using Lemmas 7.4 and 7.2 in place of Lemmas 7.3 and 7.1. □

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References

[All1] D. Allcock, The reflective Lorentzian lattices of rank 3. *Memoirs of the A.M.S.* **1033** (2012). Zbl 1302.11046 MR 3025355

[All2] —— Prenilpotent pairs in the E_{10} root lattice. *Math. Proc. Camb. Phil. Soc.* **164** (2017), 473–483. Zbl 06903453 MR 3784264

[Bar] K. Bartels, *Zur Klassifikation quadratischer Gitter über diskreten Bewertungsringen*. Ph.D. Dissertation, Göttingen 1988. Zbl 0669.10040

[BEF] J. H. Bruinier, S. Ehlen, and E. Freitag, Lattices with many Borcherds products. *Mathematics of Computation* **85** (2016), 1953–1981. Zbl 1404.11042 MR 3471115

[Cas] J. W. S. Cassels, *Rational Quadratic Forms*. Academic Press 1978. Zbl 0395.10029 MR 0522835

[CS] J. H. Conway and N. J. A. Sloane, *Sphere Packings, Lattices and Groups*. 2nd ed., Springer 1993. Zbl 0785.11036 MR 1194619

[Eic] M. Eichler, *Quadratische Formen und Orthogonal Gruppen*. Springer-Verlag, 1952. Zbl 0049.31106 MR 0051875

[HM] G. Höhn and G. Mason, The 290 fixed-point sublattices of the Leech lattice. *Journal of Algebra* **448** (2016), 618–637. Zbl 1360.11078 MR 3438323

[Jon] B. W. Jones, A canonical quadratic form for the ring of 2-adic integers. *Duke Math. J.* **11** (1944), 715–727. Zbl 0060.11006 MR 0012640

[Pal] G. Pall, The arithmetical invariants of quadratic forms. *Bulletin of the A.M.S.* **51** (1945), 185–197. Zbl 0060.11005 MR 0012641

[Tur] I. Turkalj, Totally-reflective genera of integral lattices. Preprint 2016, arXiv:1503.04428.

[Xu] F. Xu, Minimal norm Jordan splittings of quadratic lattices over complete dyadic discrete valuation rings. *Arch. Math.* **81** (2003), 402–415. Zbl 1048.11029 MR 2057061

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Daniel Allcock, Department of Mathematics, University of Texas at Austin, 2515 Speedway, Stop C1200, Austin, Texas 78712-1202, USA

e-mail: allcock@math.utexas.edu

Itamar Gal, Department of Mathematics, University of Texas at Austin,
2515 Speedway, Stop C1200, Austin, Texas 78712-1202, USA

e-mail: igal@math.utexas.edu

Alice Mark, Vanderbilt University, Department of Mathematics,
1326 Stevenson Center, Station B 407807, Nashville, TN 37240, USA

e-mail: alice.h.mark@vanderbilt.edu