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Stabilization distance between surfaces

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Abstract. Define the *1-handle stabilization distance* between two surfaces properly embedded in a fixed 4-dimensional manifold to be the minimal number of 1-handle stabilizations necessary for the surfaces to become ambiently isotopic. For every nonnegative integer m we find a pair of 2-knots in the 4-sphere whose stabilization distance equals m.

Next, using a *generalized stabilization distance* that counts connected sum with arbitrary 2-knots as distance zero, for every nonnegative integer m we exhibit a knot J_m in the 3-sphere with two slice discs in the 4-ball whose generalized stabilization distance equals m. We show this using homology of cyclic covers.

Finally, we use metabelian twisted homology to show that for each m there exists a knot and pair of slice discs with generalized stabilization distance at least m, with the additional property that abelian invariants associated to cyclic covering spaces coincide. This detects different choices of slicing discs corresponding to a fixed metabolising link on a Seifert surface.

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1. Introduction

Given a compact, smooth, oriented 4-manifold W, every second homology class can be represented by some embedded surface [GS, Prop. 1.2.3]. A simple operation called *1-handle stabilization*, illustrated in 3-dimensional space in Figure 1, preserves the homology class represented by a surface while increasing the genus by one. Roughly, a 1-handle stabilization removes $D^2 \times S^0$ from Σ and glues in $S^1 \times D^1$, with some conditions that allow this to occur ambiently in W in a controlled way (see Section 2 for formal definitions). A result of Baykur–Sunukjian [BS] states that any two embedded surfaces in W representing the same second homology class become isotopic after finitely many 1-handle stabilizations.

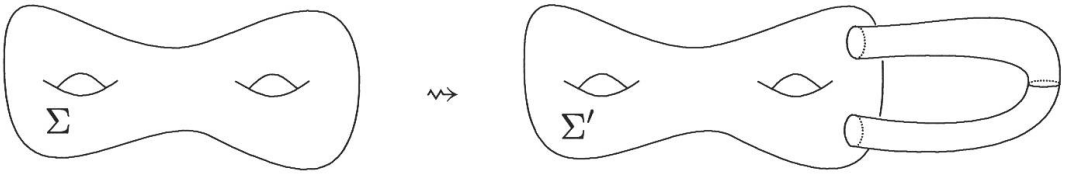


FIGURE 1
An embedded surface Σ (left) is stabilized by addition of a 1-handle, resulting in Σ' (right)

In this paper, we analyze the minimal number of 1-handle additions required to make two surfaces with the same genera isotopic. We call this the *1-handle stabilization distance*, and show that it induces a metric on the collection of ambient isotopy classes of surfaces of a fixed genus representing a given second homology class. There are many invariants capable of distinguishing two surfaces up to ambient isotopy, thereby showing that at least one 1-handle addition is required, but it is more challenging to find more substantial lower bounds on the number of 1-handles needed.

Our first result shows that, even in the simplest possible setting of necessarily null-homologous 2-spheres in S^4, the 1-handle stabilization distance can be arbitrarily large.

Theorem A. *For every nonnegative integer m, there exists a pair of embedded 2-spheres K_1 and K_2 in S^4 with 1-handle stabilization distance m.*

We prove Theorem A by analyzing the effect of 1-handle stabilization on the Alexander module of a surface in S^4. Recall that the first Alexander module $H_1(S^{n+2} \setminus \nu K; \mathbb{Q}[t^{\pm 1}])$ is a classical invariant of an embedded n-sphere K in S^{n+2} that measures the homology of the infinite cyclic cover of the exterior of K, considered as a $\mathbb{Q}[t^{\pm 1}]$-module. In the case of $n = 1$, the order of this $\mathbb{Q}[t^{\pm 1}]$-module is exactly the classical Alexander polynomial $\Delta_K(t)$.

In addition to 1-handle stabilization, one might also wish to allow connected sum with arbitrary knotted 2-spheres, also called *2-knots*. In the context of Theorem A this is uninteresting: any two 2-knots become isotopic with zero 1-handle additions and a single 2-sphere addition to each. However, when considering properly embedded discs in D^4 with fixed boundary we show that the resulting *generalized stabilization distance*, in which 1-handle addition counts as 1 and 2-sphere addition counts as 0, has similarly interesting properties. In particular, the generalized stabilization distance between properly embedded discs in D^4 with fixed boundary can be arbitrarily large. More precisely, a *slice disc* for a 1-knot $J \subset S^3$ is a smoothly properly embedded disc $D^2 \subset D^4$ with boundary the knot J, and we prove the following.

Theorem B. *For every nonnegative integer m, there exists a knot $J \subset S^3$ and a pair of slice discs D_1 and D_2 for J with generalized stabilization distance m.*

To prove Theorem B we again rely on the Alexander module, comparing for $i = 1$ and 2 the kernels of the inclusion-induced maps

$$H_1\big(S^3 \setminus \nu J; \mathbb{Q}[t^{\pm 1}]\big) \to H_1\big(D^4 \setminus \nu D_i; \mathbb{Q}[t^{\pm 1}]\big).$$

Given any embedded surface Σ with boundary J, we then analyze how the kernel of the inclusion induced map

$$H_1\big(S^3 \setminus \nu J; \mathbb{Q}[t^{\pm 1}]\big) \to H_1\big(D^4 \setminus \nu \Sigma; \mathbb{Q}[t^{\pm 1}]\big)$$

can change under 1-handle and 2-sphere addition.

One common way to produce a slice disc for a knot is to surger a spanning surface for the knot along a collection of curves as follows. Given an embedded oriented surface F in S^3 with boundary J, suppose we can find a set of 0-framed curves $\gamma_i \subset F$ that form a half-basis for $H_1(F; \mathbb{Z})$ and which themselves bound disjoint discs Δ_i in D^4. Then the surface

$$F_\Delta := \Big(F \smallsetminus \bigcup_i (\gamma_i \times (0,1))\Big) \cup \Big(\bigcup_i \Delta_i \times \{0,1\}\Big) \subset D^4,$$

is a slice disc for J, after a minor isotopy to smooth corners and make the embedding proper. The methods of Theorem B can often distinguish slice discs which arise from surgering a Seifert surface along two different collections of $\{\gamma_i\}$ curves. However, while fixing the $\{\gamma_i\}$ there can still be multiple choices for the slice discs Δ_i, and Alexander module techniques cannot distinguish the resulting slice discs for J.

For our last main result we detect these second order differences between slice discs, and again show that the distance can be arbitrarily large.

Theorem C. *For every nonnegative integer m, there exists a knot $J \subset S^3$ and a pair of slice discs D_1 and D_2 for J with generalized stabilization distance at least m, such that the kernels*

$$\ker\Big(H_1\big(S^3 \setminus \nu J; \mathbb{Q}[t^{\pm 1}]\big) \to H_1\big(D^4 \setminus \nu D_i; \mathbb{Q}[t^{\pm 1}]\big)\Big)$$

coincide for $i = 1, 2$.

Our primary tool in the proof of Theorem C is *metabelian twisted homology*, or twisted homology coming from maps to *metabelian* groups, i.e., groups G with

$$G^{(2)} := \big[[G,G],[G,G]\big] = 0.$$

These sorts of representations were notably used by Casson–Gordon [CG1, CG2] to give the first examples of algebraically slice knots in S^3 which are not actually slice. The corresponding twisted homology theories have the nice feature of being relatively computable while still being powerful enough to obtain strong conclusions, for example distinguishing mutant knots up to concordance [KL2]. In our case, we take G to be the dihedral group $D_{2n} \cong \mathbb{Z}_2 \ltimes \mathbb{Z}_n$ and construct our representations using maps from the first homology of the double cover of the relevant space to $\mathbb{Z}_n$.

We remark that Theorem B is not a corollary of Theorem C, since the former gives us distance exactly m. Theorem B is also easier to prove, and the method extends straightforwardly to distinguish choices of slice discs for many knots beyond the explicit examples we give, while Theorem C requires more involved arguments and more specialized constructions.

A slightly different analysis of stabilization distance between surfaces was undertaken by [JZ2], who rather than minimizing the number of 1-handle stabilizations necessary to make two surfaces isotopic instead minimized the largest genus of any surface appearing in a sequence of stabilizations and destabilizations connecting the two surfaces.

We also wish to advertise the following problem, which relates to recent work by [JZ1] and [CP]. For a slice knot R, let $n_s(R)$ denote the number of equivalence classes of slice discs for R, where the equivalence relation is generated by connected sum with knotted 2-spheres and ambient isotopy rel. boundary. Note that $n_s(U) = 1$.

Our examples of Theorem B show that for every integer k there is a knot R_k with $n_s(R_k) \geq k$. In fact, the knot $\#^k 9_{46}$ has 2^k natural slice discs obtained by choosing 'left band' or 'right band' slice discs for each $i = 1, \dots, k$; see Figure 3. By considering the kernels of the inclusion induced maps on Alexander modules as we do in the proof of Theorem B, one can see they are all mutually not ambiently isotopic rel. boundary and so $n_s(\#^k 9_{46}) \geq 2^k$.

Problem 1.1. Determine the value of $n_s(R)$ for some nontrivial knot R, or at least whether $n_s(R) < \infty$.

Organization of the paper. In Section 2 we give precise definitions for our notions of stabilization distance. Section 3 constructs a cobordism between surface exteriors corresponding to a stabilization. Our results will follow from analyzing the effects on homology of these cobordisms. Section 4 recalls the notion of generating rank of a module over a commutative PID, records the facts about generating rank that we shall use, and establishes our conventions around twisted

homology. Then Section 5 proves Theorem A, Section 6 proves Theorem B, and Section 7 proves Theorem C.

Conventions. All manifolds, unless otherwise stated, are compact, smooth, and oriented. When N is a properly embedded submanifold of M, we write $X_N := M \smallsetminus \nu(N)$. In our context, we will frequently have a canonical isomorphism $\varepsilon\colon H_1(X_N) \to \mathbb{Z}$ and in this case we let X_N^n denote the corresponding n-fold cyclic cover, for $n \in \mathbb{N} \cup \{\infty\}$. For $n \in \mathbb{N}$, we use $\mathbb{Z}_n$ to denote the finite cyclic group $\mathbb{Z}/n\mathbb{Z}$. Given a surface F, we let $g(F)$ denote its genus.

Immediately before the publication of this paper, the authors learned that original credit for Theorem A belongs to Miyazaki [Miy].

2. Stabilization distances

Fix a compact, oriented, smooth 4-manifold W. The following definition is motivated by that of Juhász and Zemke [JZ2].

Definition 2.1. Let Σ be an oriented surface with boundary, smoothly and properly embedded in W. Let B be an embedding of D^4 into W such that ∂B intersects Σ transversely in a 2-component unlink L and B intersects Σ in two discs Δ_0 and Δ_1, which can be simultaneously isotoped within B to lie in ∂B. Suppose that a 3-dimensional 1-handle $D^2 \times I$ is embedded into the interior of W such that $D^2 \times \{i\} = \Delta_i$ for $i = 0, 1$. Then $\Sigma' := (\Sigma \cap (W \smallsetminus B)) \cup_L (S^1 \times I)$ is a *1-handle stabilization* of Σ. If $S^1 \times I$ can be isotoped into ∂B relative to L, we call the stabilization *trivial.*

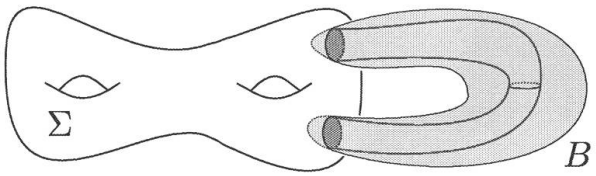


FIGURE 2
A surface Σ with ball B as in Definition 2.1, pre-stabilization

A trivial 1-handle stabilization does not change the fundamental group of the complement of the surface, so frequently there will be no sequence of trivial stabilizations relating two given surfaces. On the other hand, any two homologous surfaces become isotopic after adding finitely many 1-handles [BS].

Definition 2.2. Define the *1-handle stabilization distance* in $\mathbb{N} \cup \{0, \infty\}$ between smoothly and properly embedded surfaces $(F, \partial F) \subset (W, \partial W)$ and $(F', \partial F') \subset (W, \partial W)$ with $\partial F = \partial F'$, homologous in $H_2(W, \partial W; \mathbb{Z})$, to be the minimal $k \in \mathbb{N}$ such that F and F' become ambiently isotopic rel. boundary after each has been stabilized at most k times. We denote this by $d_1(F, F')$. If F and F' are not homologous or have different boundaries then we say that $d_1(F, F') = \infty$.

In particular for any two 2-knots K and J, $d_1(K, J) < \infty$. For distances between slice discs, we obtain stronger results by defining a coarser notion that permits connected sum with locally knotted 2-spheres. By *adding a locally knotted 2-sphere* to a properly embedded surface $(\Sigma, \partial\Sigma) \subset (W, \partial)$ we mean taking a 2-knot S in S^4 and forming the connected sum of pairs

$$(W, \Sigma)\#(S^4, S) = (W, \Sigma\#S).$$

Definition 2.3. Let $(F, \partial F) \subset (W, \partial W)$ and $(F', \partial F') \subset (W, \partial W)$ be smoothly and properly embedded surfaces. If $\partial F = \partial F'$ and $[F] = [F'] \in H_2(W, \partial W; \mathbb{Z})$, we define the *generalized stabilization distance* $d_2(F, F')$ in $\mathbb{N} \cup \{0, \infty\}$ to be the minimal $k \in \mathbb{N}$ such that F and F' become ambiently isotopic rel. boundary after each has been stabilized at most k times and had arbitrarily many locally knotted 2-spheres added. If F and F' are not homologous or have different boundaries then we say that $d_2(F, F') = \infty$.

Note that for any two slice discs D_1, D_2 in D^4 for a fixed knot in S^3, we have that $d_2(D_1, D_2) < \infty$. It is immediate from the definitions that

$$d_2(F, F') \leq d_1(F, F').$$

We also remark that $d_{JZ}(F, F') \leq d_2(F, F')$, where d_{JZ} denotes the Juhász–Zemke stabilization distance [JZ2] between surfaces.

3. Cobordisms corresponding to handle additions

Now we construct cobordisms corresponding to handle additions. The following construction will be used in our proofs of all three main theorems.

Construction 3.1. [A cobordism between surface exteriors.] Let W be a compact, oriented, smooth 4-manifold. Suppose that F_1 is a smoothly and properly embedded surface in W with $\partial F_1 = K \subset \partial W$ and that F_2 has been obtained from F_1 by a 1-handle addition such that $g(F_2) = g(F_1) + 1$. We define an ambient cobordism $T \subset W \times I$ as follows:

$$T := \big(F_1 \times [0, 1/2]\big) \cup \big((D^1 \times D^2) \times \{1/2\}\big) \cup \big(F_2 \times [1/2, 1]\big),$$

where $D^1 \times D^2 \hookrightarrow W$ is an embedding with $\partial D^1 \times D^2 \subset F_1$ and $D^1 \times \partial D^2 \subset F_2$. (That is, $D^1 \times D^2$ is the 3-dimensional 1-handle h in the definition of 1-handle stabilization.) Observe that

$$\partial T = \big(F_1 \times \{0\}\big) \cup_{K\times\{0\}} \big(K \times [0,1]\big) \cup_{K\times\{1\}} F_2 \times \{1\}$$

and so $X_T := (W \times I) \smallsetminus \nu(T)$ is a cobordism rel. X_K from X_{F_1} to X_{F_2}.

Since T is obtained from $F_1 \times [0, 1/2]$ by attaching a single 3-dimensional 1-handle to $F_1 \times \{1/2\}$ (and then flowing upwards), it follows from the rising water principle [GS, Section 6.2] that X_T has a handle decomposition relative to X_{F_1} obtained by attaching a single 5-dimensional 2-handle to $X_{F_1} \times I$. Notice that the attaching sphere of this 2-handle determines an element of $\pi_1(X_{F_1})$ of the form $\gamma = \mu_1 \beta \mu_2^{-1} \beta^{-1}$, where μ_1 and μ_2 are meridians to F_1 near the attaching spheres of h and β is a parallel push-off of the core of h. In particular, γ is null-homologous in $H_1(X_{F_1})$. Taking the dual decomposition, we see that X_T also has a handle decomposition relative to X_{F_2} obtained by attaching a single 5-dimensional 3-handle. By excision, we therefore have that

$$H_k(X_T, X_{F_1}) = \begin{cases} \mathbb{Z} & k = 2 \\ 0 & \text{else} \end{cases} \quad \text{and} \quad H_k(X_T, X_{F_2}) = \begin{cases} \mathbb{Z} & k = 3 \\ 0 & \text{else.} \end{cases}$$

In particular, the inclusion maps $X_{F_i} \to X_T$ induce isomorphisms on first homology. It will be useful for us later on to know that the inclusion induced map $\pi_1(X_{F_1}) \to \pi_1(X_T)$ is surjective, as follows immediately from applying the Seifert-van Kampen theorem to $X_T = (X_{F_1} \times I) \cup (\text{2-handle})$.

We now comment on basepoints for the fundamental group in this context. Let $x_0 \in X_K \subseteq X_T \times \{0\}$, let $\alpha = \{x_0\} \times I \subseteq X_T \times I$, and let $x_1 = \{x_0\} \times 1$. We will always let $\pi_1(X_K) = \pi_1(X_K, x_0)$, $\pi_1(X_{F_1}) = \pi_1(X_{F_1}, x_0)$, $\pi_1(X_T) = \pi_1(X_T, x_0)$, and $\pi_1(X_{F_2}) = \pi_1(X_{F_2}, x_1)$. There are natural inclusion induced maps $\iota\colon \pi_1(X_K, x_0) \to \pi_1(X_T, x_0)$ and $\iota_1\colon \pi_1(X_{F_1}, x_0) \to \pi_1(X_T, x_0)$. Moreover, we use the arc α to define

$$\iota_2\colon \pi_1(X_{F_2}, x_1) \to \pi_1(X_T, x_1) \to \pi_1(X_T, x_0).$$

Later on, we will often omit basepoints from our notation, always using the above arcs and corresponding inclusion maps. This completes Construction 3.1.

Proposition 3.2. *Fix a compact, oriented, smooth 4-manifold W, a (possibly empty) link L in ∂W, a nonnegative number g, and a homology class $A \in H_2(W, \partial W; \mathbb{Z})$ with $\partial A = [L]$. The distance function d_1 defines a metric on the set of ambient isotopy classes rel. boundary of embedded oriented surfaces of genus g in W with boundary L that represent the class $A \in H_2(W, \partial W; \mathbb{Z})$.*

Proof. We use that the distance is finite within the sets considered [BS]. If $d_1(\Sigma, \Sigma') = 0$, then Σ and Σ' are ambiently isotopic. The distance function is flagrantly symmetric.

To see the triangle inequality, suppose F and F' are homologous rel. boundary surfaces which stabilize via k 1-handle additions to a surface S and F' and F'' are homologous rel. boundary surfaces which stabilize via h 1-handle additions to S'. Now consider the sequence of stabilizations and destabilizations from F to S to F' to S' to F'' as a 3-dimensional cobordism T embedded in $W \times I$. We may perturb the embedding of T so that $F\colon W \times I \to I$ restricts to a Morse function on T, where stabilizations correspond to index one critical points, and destabilizations correspond to index two critical points. First we argue that we can rearrange this sequence of stabilizations and destabilizations so that all the stabilizations come first, followed by destabilizations. Our desired result will then follow immediately from letting S'' be the pre-image of a regular value taken after all index one critical points and before all index two critical points, and observing that both F and F'' stabilize via $(k + h)$ 1-handle additions to S''.

In codimension at least two, critical points of an embedded cobordism can be arranged, by ambient isotopy, to appear in order of increasing index [Per], [BP, Theorem 4.1], by the following standard argument, which we include for completeness. Choose a gradient-like embedded vector field subordinate to F [BP, Definition 3.1]. Rearrangement of critical points is possible in general if the ascending manifold of the lower critical point is disjoint from the descending manifold of the higher critical point. Suppose that an index one critical point of T has critical value t_1 higher than critical value t_2 of an index two critical point, and suppose that there are no critical values between t_2 and t_1. The descending manifold of the index 1 critical point of a 3-dimensional cobordism intersects a generic level set $W \times \{t\}$, with $t_2 < t < t_1$ in a 1-dimensional disc. The descending manifold of the index 2 critical point intersects $W \times \{t\}$ also in a 1-dimensional disc. By general position, we can perturb the gradient-like vector field to make the ascending and descending manifolds disjoint, and we may do so simultaneously for all such t. It follows that the critical points can be rearranged by an ambient isotopy, as desired. □

We remark that we do not claim d_2 gives rise to a metric. The next proposition tells us that 2-spheres can be reordered so they come before 1-handle additions.

Proposition 3.3. *Suppose that an embedded surface Σ_2 is obtained from a connected surface Σ_1 by some number m of 1-handle additions, followed by connect summing with a local 2-knot. Then there is an embedded surface Σ' that is obtained from Σ_1 by adding a local 2-knot, and such that Σ_2 is obtained from Σ' by m 1-handle additions.*

Proof. Let Σ_1' denote Σ_1 with the 1-handles attached, so Σ_2 is obtained from Σ_1' by connected sum with a local 2-knot S. The isotopy class of $\Sigma_1' \# S$ is unchanged by where on Σ_1' we take the connected sum, so we can assume that our connected sum takes place far away from the attached 1-handles. But then it is clear that we can attach S first and our 1-handles second. □

4. Generating ranks and twisted homology

4.1. Generating rank of modules over a commutative PID. We recall some facts about generating ranks of finitely generated modules over commutative PIDs.

Let A be a finitely generated module over a commutative PID S. We say that A has *generating rank k over S* if A is generated as an S-module by k elements but not by $k-1$ elements and write $\text{g-rk}_S A = k$. When S is clear from context, we often abbreviate $\text{g-rk}_S A$ by $\text{g-rk}\, A$.

Lemma 4.1. *Let A, B, and C be finitely generated modules over a commutative PID S.*

(1) *If A surjects onto B then $\text{g-rk}_S B \leq \text{g-rk}_S A$.*

(2) *If $B \leq A$ then $\text{g-rk}_S B \leq \text{g-rk}_S A$.*

(3) *Let $0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0$ be a short exact sequence of S-modules. Then $\text{g-rk}_s C \geq \text{g-rk}_S(B) - \text{g-rk}_S(A)$.*

Proof. The first part follows immediately from the definition of generating rank. The second part is easy to check using the classification of finitely generated modules over a commutative PID. The third property follows from taking minimal S-generating sets $\{a_1, \ldots, a_n\}$ and $\{c_1, \ldots, c_m\}$ for A and C respectively, picking $b_i \in g^{-1}(c_i)$ for each $1 \leq i \leq m$, and observing that $\{f(a_1), \ldots, f(a_n), b_1, \ldots, b_m\}$ is an S-generating set for B. □

Remark 4.2. Only (2) uses that S is a PID.

We will also make arguments involving the *order* of a finitely generated module A over a commutative PID S. The classification of finitely generated modules over a PID states that there exist $j, k \in \mathbb{N}$ and elements $s_1, \ldots, s_k \in S$ such that there is a (non-canonical) isomorphism

$$A \cong S^j \oplus TA \cong S^j \oplus \bigoplus_{i=1}^{k} S/\langle s_i \rangle.$$

When $j > 0$ we say that the *order* of A is $|A| = 0$ and when $j = 0$ we say that the order of A is $|A| = \prod_{i=1}^{k} s_i$. This is well-defined up to multiplication by units in S. The key property of order we use is that if $f\colon A \to B$ is a map of S-modules with $\ker(f)$ torsion, then $|\operatorname{Im}(f)| = |A|/|\ker(f)|$.

4.2. Twisted homology. Let X be a CW complex with universal cover $\widetilde{X}$. The cellular chain complex $C_*(\widetilde{X})$ is a chain complex of right $\mathbb{Z}[\pi_1(X)]$-modules. If X is a finite complex then $C_*(\widetilde{X})$ is finitely generated as a $\mathbb{Z}[\pi_1(X)]$-module. Let R be a commutative ring with involution and with unit. Let $\alpha\colon \pi_1(X) \to U_m(R)$ be a unitary representation i.e. $\alpha(g^{-1}) = \overline{\alpha(g)}^T$. This extends to a homomorphism of rings with involution $\mathbb{Z}[\pi_1(X)] \to \mathrm{GL}_m(R)$, and makes R^m into a $(\mathbb{Z}[\pi_1(M)], R)$-bimodule.

Definition 4.3. The kth *twisted homology* of X with respect to α is

$$H_k^\alpha(X; R) := H_k\big(C_*(\widetilde{X}) \otimes_{\mathbb{Z}[\pi_1(X)]} R^m\big).$$

When the ring R is clearly understood, and we are short of space, we shall sometimes omit R from the notation and write $H_k^\alpha(X)$ for $H_k^\alpha(X; R)$.

If X is a finite complex and R is Noetherian then $H_k^\alpha(X; R)$ is finitely generated as an R-module. If $Y \subset X$ is a subcomplex and we choose a path $\gamma\colon I \to X$ from the basepoint then α determines a representation $\pi_1(Y) \to U_m(R)$ and we write $H_k^\alpha(Y; R)$ for the resulting twisted homology. The inclusion induced map $H_k^\alpha(Y; R) \to H_k^\alpha(X; R)$ depends on the choice of γ, but nonetheless we omit γ from the notation.

Remark 4.4. Given X and $\alpha\colon \pi_1(X) \to U_m(R)$ as above, let $X^\alpha \to X$ be the cover corresponding to $\ker(\alpha)$. Then $\mathbb{Z}[\pi_1(X)]$ acts on $C_*(X^\alpha)$ and it follows immediately from our definitions that

$$H_k^\alpha(X; R) \cong H_k\big(C_*(X^\alpha) \otimes_{\mathbb{Z}[\pi_1(X)]} R^m\big).$$

It is sometimes more convenient to compute with this smaller covering space.

4.3. Rational Alexander modules. For any knot or slice disc L, let $\mathcal{A}(L)$ denote the Alexander module of L with integral coefficients and let $\mathcal{A}_{\mathbb{Q}}(L)$ denote the Alexander module of L with rational coefficients. That is, let X_L be the exterior of L and as usual let $\varepsilon\colon \pi_1(X_L) \to \mathbb{Z}$ denote the abelianization map. Then $\mathcal{A}(L) := H_1(X_L, \mathbb{Z}[t^{\pm 1}])$ and $\mathcal{A}_{\mathbb{Q}}(L) := H_1(X_L; \mathbb{Q}[t^{\pm}])$, where for $R = \mathbb{Z}, \mathbb{Q}$ the ring $R[t^{\pm 1}]$ has a $\mathbb{Z}[\pi_1(X_L)]$-structure determined by ε. We remark that $\mathbb{Q}$ is flat as a $\mathbb{Z}$-module, and so $\mathcal{A}_{\mathbb{Q}}(L) \cong \mathcal{A}(L) \otimes_{\mathbb{Z}} \mathbb{Q}$.

5. Pairs of 2-knots with arbitrary 1-handle distance

In this section, we prove that for every nonnegative integer m, there exists a pair of 2-knots K and J in the 4-sphere with 1-handle stabilization distance m, which is an immediate consequence of the following proposition.

Proposition 5.1. *For each $m \in \mathbb{N}$, there exists a knotted 2-sphere K in S^4 such that the minimal number of 1-handle stabilizations needed to make K an unknotted surface is exactly m.*

Proof of Theorem A. Let $m \in \mathbb{N}$, let K be as in Proposition 5.1, and let J be an unknotted 2-sphere. Since every stabilization of an unknotted 2-sphere is an unknotted surface, we obtain immediately that $d_1(K, J) = m$. □

The next proposition is the key algebraic input into the proof of Proposition 5.1.

Proposition 5.2. *Let $F_1 \subset S^4$ be a smoothly embedded oriented surface and suppose that F_2 is obtained from F_1 by a 1-handle stabilization. Then there is a polynomial $p \in \mathbb{Q}[t^{\pm 1}]$ and a short exact sequence*

$$0 \to \mathbb{Q}[t^{\pm 1}]/\langle p \rangle \to H_1\big(S^4 \smallsetminus \nu F_1; \mathbb{Q}[t^{\pm 1}]\big) \to H_1\big(S^4 \smallsetminus \nu F_2; \mathbb{Q}[t^{\pm 1}]\big) \to 0.$$

Proof. We consider the relative cobordism X_T between X_{F_1} and X_{F_2} from Construction 3.1, with $W = S^4$. We will consider the infinite cyclic cover $\widetilde{X}_T$. Recall that X_T is obtained from $X_{F_1} \times I$ by attaching a single 5-dimensional 2-handle along $\gamma \times \{1\}$ for $\gamma = \mu_1 \beta \mu_2^{-1} \beta^{-1}$, where μ_1 and μ_2 are meridians of F_1 in S^4 near the attaching spheres of the 1-handle and β is a parallel push-off of the core of this 1-handle. Since $H_1(F_1; \mathbb{Z}) \cong \mathbb{Z}$, and the attaching sphere of the 2-handle is null homologous, the abelianization homomorphism $\pi_1(X_{F_1}) \to \mathbb{Z}$ extends to a homomorphism $\pi_1(X_T) \to \mathbb{Z}$. From now on in this proof we consider homology with $\mathbb{Q}[t^{\pm 1}]$-coefficients induced by this homomorphism. We also note that the handle decomposition lifts to a relative handle decomposition of $\widetilde{X}_T$ with one orbit of 2-handles under the deck transformation action of $\mathbb{Z}$.

Using this relative handle decomposition we obtain that $H_k(X_T, X_{F_1}; \mathbb{Q}[t^{\pm 1}]) = 0$ for $k \neq 2$ and $H_2(X_T, X_{F_1}; \mathbb{Q}[t^{\pm 1}]) \cong \mathbb{Q}[t^{\pm 1}]$. Since dually X_T is obtained from $X_{F_2} \times I$ by attaching a single 5-dimensional 3-handle, we have that $H_k(X_T, X_{F_2}\, \mathbb{Q}[t^{\pm 1}]) = 0$ for $k \neq 3$. Now consider the long exact sequence of the pair (X_T, X_{F_1}) with $\mathbb{Q}[t^{\pm 1}]$-coefficients.

$$\cdots \to H_2(X_T) \to H_2(X_T, X_{F_1}) \to H_1(X_{F_1}) \to H_1(X_T) \to H_1(X_T, X_{F_1}).$$

Since $H_1(X_T, X_{F_1}) = 0$ and $H_2(X_T, X_{F_1}) \cong \mathbb{Q}[t^{\pm 1}]$, and since $\mathbb{Q}[t^{\pm 1}]$ is a PID, this yields a short exact sequence

$$0 \to \mathbb{Q}[t^{\pm 1}]/\langle p \rangle \to H_1(X_{F_1}) \to H_1(X_T) \to 0$$

for some $p \in \mathbb{Q}[t^{\pm 1}]$. Now the long exact sequence of the pair (X_T, X_{F_2}) yields

$$0 = H_2(X_T, X_{F_2}) \to H_1(X_{F_2}) \to H_1(X_T) \to H_1(X_T, X_{F_2}) = 0,$$

from which it follows that the inclusion induced map $H_1(X_{F_2}) \to H_1(X_T)$ is an isomorphism, and so we obtain the desired short exact sequence

$$0 \to \mathbb{Q}[t^{\pm 1}]/\langle p \rangle \to H_1(X_{F_1}) \to H_1(X_{F_2}) \to 0. \qquad \square$$

For the reader's convenience, we now describe two common constructions of slice discs.

Construction 5.3. Given a subset $Y \subseteq S^3$ and $J \subseteq I$ that is either an interval $[a,b]$ or a point $\{a\}$, write Y_J for $Y \times J \subseteq S^3 \times I$. We think of D^4 as $D^4 \cong S^3_{[0,1]}/S^3_1$.

The banding construction. Let K be a knot with disjointly embedded bands $\beta_1, \dots, \beta_n$ in S^3 such that the result of banding K via $\{\beta_i\}_{i=1}^n$ is the $(n+1)$-component unlink U_{n+1}, which could be capped off via $(n+1)$ discs in S^3. Then, up to smoothing corners,

$$D := K_{[0,1/3]} \cup \left(\cup_{i=1}^n \beta_i\right)_{1/3} \cup (U_{n+1})_{[1/3,2/3]} \cup \left(\cup_{i=1}^{n+1} D^2\right)_{2/3}$$

is a ribbon disc for K.

The surgery construction. Let K be a knot with a genus g Seifert surface F and a collection of g disjoint curves $\alpha_1, \dots, \alpha_g \subset F$ which are 0-framed by F and which generate a $\mathbb{Z}^g$ summand of $H_1(F)$. Suppose also that the link $\cup_{i=1}^g \alpha_i \subset S^3$ is an unlink. Then, up to smoothing corners,

$$D = K_{[0,1/3]} \cup \left(F \smallsetminus \nu(\cup_{i=1}^g \alpha_i)\right)_{1/3} \cup \cup_{i=1}^g (\alpha_i^+ \sqcup \alpha_i^-)_{[1/3,2/3]} \cup \cup_{i=1}^n (D^2 \sqcup D^2)_{2/3}$$

is a ribbon disc for K. We note that this construction is easily adapted to build a slice disc for K under the weaker assumption that $\cup_{i=1}^g \alpha_i$ is merely strongly slice.

Example 5.4 (The knot 9_{46} and its two standard slice discs.). Let $R := 9_{46}$, and let D_j for $j = 1, 2$ be the slice discs indicated by the left and right bands, respectively, of the left part of Figure 3. Observe that R has a genus 1 Seifert surface F (illustrated on the right of Figure 3), and for $j = 1, 2$ let D'_j be the slice disc obtained by surgery of F along α_j. Referring back to Construction 5.3 for our explicit description of D_j and D'_j, we can recognize these as isotopic discs in D^4, since

$$R_{[1/6,1/3]} \cup (\beta_j)_{1/3} \cup (U_2^j)_{[1/3,2/3]} \subset D_j$$

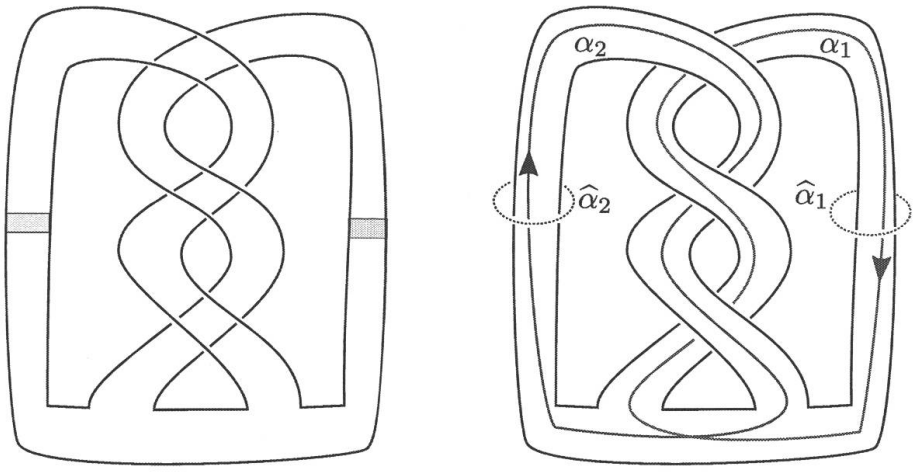


FIGURE 3
The knot $R = 9_{46}$ has slice discs D_1 (left band) and D_2 (right band)

and

$$R_{[1/6,1/3]} \cup \big(F \smallsetminus \nu(\alpha_j)\big)_{1/3} \cup \big(\alpha_j^+ \sqcup \alpha_j^-\big)_{[1/3,2/3]} \subset D_j'$$

are isotopic rel. boundary as subsets of $S^3 \times [1/6, 2/3]$.

The oriented curves α_1, α_2 represent a basis for $H_1(F)$ with respect to which the Seifert pairing is given by

$$A = \begin{bmatrix} 0 & 2 \\ 1 & 0 \end{bmatrix}.$$

The Alexander module is therefore presented by

$$tA - A^T = \begin{bmatrix} 0 & 2t-1 \\ t-2 & 0 \end{bmatrix},$$

and hence is isomorphic to $\mathbb{Z}[t^{\pm 1}]/\langle t-2\rangle \oplus \mathbb{Z}[t^{\pm 1}]/\langle 2t-1\rangle$, where $\widehat{\alpha}_1$ and $\widehat{\alpha}_2$ represent the generators of each summand.

Moreover, the inclusion induced maps $\iota_j \colon \mathcal{A}_{\mathbb{Q}}(R) \to \mathcal{A}_{\mathbb{Q}}(D_j)$ are given by projection onto summands:

$$\begin{aligned} \mathcal{A}_{\mathbb{Q}}(R) \cong \mathbb{Q}[t^{\pm 1}]/\langle 2t-1\rangle \oplus \mathbb{Q}[t^{\pm 1}]/\langle t-2\rangle &\xrightarrow{\iota_1} \mathbb{Q}[t^{\pm 1}]/\langle 2t-1\rangle \cong \mathcal{A}_{\mathbb{Q}}(D_1) \\ (x,y) &\mapsto x \\ \mathcal{A}_{\mathbb{Q}}(R) \cong \mathbb{Q}[t^{\pm 1}]/\langle 2t-1\rangle \oplus \mathbb{Q}[t^{\pm 1}]/\langle t-2\rangle &\xrightarrow{\iota_2} \mathbb{Q}[t^{\pm 1}]/\langle t-2\rangle \cong \mathcal{A}_{\mathbb{Q}}(D_2) \\ (x,y) &\mapsto y. \end{aligned}$$

Note that $\ker(\iota_1) \cap \ker(\iota_2) = \{0\} \subseteq \mathcal{A}_{\mathbb{Q}}(R)$.

A detailed computation with these slice discs can be found in [CP, Section 5.1]. To see that the induced maps are as claimed, we argue by the rising water principle [GS, Section 6.2]. There is a handle decomposition of X_{D_i} relative to X_R consisting of one 2-handle attached along $\widehat{\alpha}_i$ (corresponding to the band), followed by two 3-handles corresponding to the maxima, and a 4-handle. Only the 2-handle affects first homology, by killing the class represented by $\widehat{\alpha}_i$.

Proof of Proposition 5.1. Let $D := D_2 \subset D^4$ be the "right band" slice disc for the 9_{46} knot shown via a blue band on the left of Figure 3. Let K_0 be the 2-knot obtained from doubling this disc, that is $K_0 = D \cup_{9_{46}} D \subset D^4 \cup D^4 = S^4$. Let $K := \#_{i=1}^m K_0$.

First we use Proposition 5.2 to show that if K stabilizes to an unknotted surface by n 1-handle additions then $n \geq m$. We know that

$$H_1\big(S^3 \smallsetminus \nu(9_{46}); \mathbb{Q}[t^{\pm 1}]\big) \cong \mathbb{Q}[t^{\pm 1}]/\langle 2t-1\rangle \oplus \mathbb{Q}[t^{\pm 1}]/\langle t-2\rangle$$

where the inclusion induced map to $H_1(D^4 \smallsetminus \nu(D); \mathbb{Q}[t^{\pm 1}]) \cong \mathbb{Q}[t^{\pm 1}]/\langle t-2\rangle$ is given by projection onto the second factor. By using the Mayer–Vietoris sequence corresponding to the decomposition

$$S^4 \smallsetminus \nu K_0 = \big(D^4 \smallsetminus \nu(D)\big) \cup_{S^3 \smallsetminus \nu(9_{46})} \big(D^4 \smallsetminus \nu(D)\big),$$

we can compute that

$$H_1\big(S^4 \smallsetminus \nu K_0; \mathbb{Q}[t^{\pm 1}]\big) = \mathbb{Q}[t^{\pm 1}]/(t-2).$$

Since Alexander modules are additive under connected sum of 2-knots we therefore have that

$$H_1(S^4 \smallsetminus \nu K; \mathbb{Q}[t^{\pm 1}]) = \bigoplus_{i=1}^{m} \big(\mathbb{Q}[t^{\pm 1}]/(t-2)\big).$$

We therefore need to show that one requires at least m stabilizations to trivialize the Alexander module of K. Note that the generating rank of $H_1(S^4 \smallsetminus \nu K; \mathbb{Q}[t^{\pm 1}])$ is m. We claim that the result of stabilizing an embedded surface whose Alexander module has generating rank k is an embedded surface with generating rank at least $k-1$. To see the claim, we use Proposition 5.2 and the fact that if a $\mathbb{Q}[t^{\pm 1}]$-module M has generating rank k and a submodule N has generating rank 1, then the quotient M/N has generating rank at least $k-1$, by Lemma 4.1 (3). By the claim and the fact that the generating rank of $H_1(S^4 \smallsetminus \nu K; \mathbb{Q}[t^{\pm 1}])$ is m, it follows by induction that $d_1(K, J) \geq m$.

It remains to show that we can make K unknotted via m 1-handle attachments. Recall that the slice disc D is constructed by a band move "cutting" one of the bands of the obvious Seifert surface Σ for 9_{46} in Figure 3, and then capping

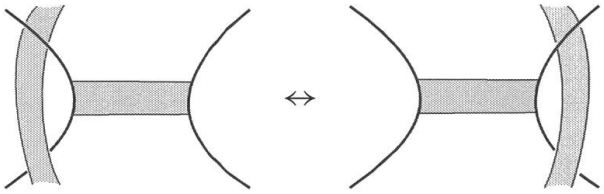


FIGURE 4
A 'band-swim' move preserves the isotopy class
of a surface presented by a banded knot diagram

off the resulting 2-component unlink with disjoint discs. A single stabilization, tubing these two discs together, results in an embedded genus one surface. This surface could also be obtained by capping off the 2-component unlink with an annulus instead of two discs, and hence is isotopic to the result of pushing the aforementioned Seifert surface into D^4. We assert that $D \cup \Sigma \subset S^4$ is an unknotted genus one surface, and prove this by direct manipulation of handle diagrams for the embedding of the surface in D^4, using the banded knot diagram moves of Swenton [Swe].[1]

The data of an unlink and bands attached to it with the property that the result of performing the corresponding band moves is also an unlink provides instructions for embedding a surface in S^4: the unlink's components correspond to 0-handles, the bands to 1-handles, and the unlink obtained by banding can be capped off with 2-handles in an essentially unique way, in the sense that any two choices of discs in S^3 capping off the unlink yield isotopic surfaces in S^4. This uses the main result of [Liv], that any two sets of embedded discs in S^3 are isotopic rel. boundary in D^4. We remark that isotopy of banded knot diagrams in S^3 together with cancellation/creation of band-unknot pairs, sliding of bands across each other, and the 'band-swim move' illustrated in Figure 4 preserve the isotopy class of the presented surface (see Swenton [Swe] for more details).

The banded diagram on the far left of Figure 5 gives $D \cup \Sigma$. The top two bands correspond to the Seifert surface, and the green band is the band of the disc D.

The center left of Figure 5 gives the 'dual' band description corresponding to turning our handle diagram upside down. The center right figure is obtained by an isotopy of the banded diagram in S^3, and we perform a 'band-swim' move of the green band through the red band to obtain the diagram on the far right of Figure 5.

[1] The reader who is familiar with doubly slice knots may instead observe that $D \cup \Sigma$ is a stabilization of the unknotted 2-knot obtained by gluing the 'left band' and 'right band' discs together, and hence is itself unknotted. We give the longer argument here to be self-contained.

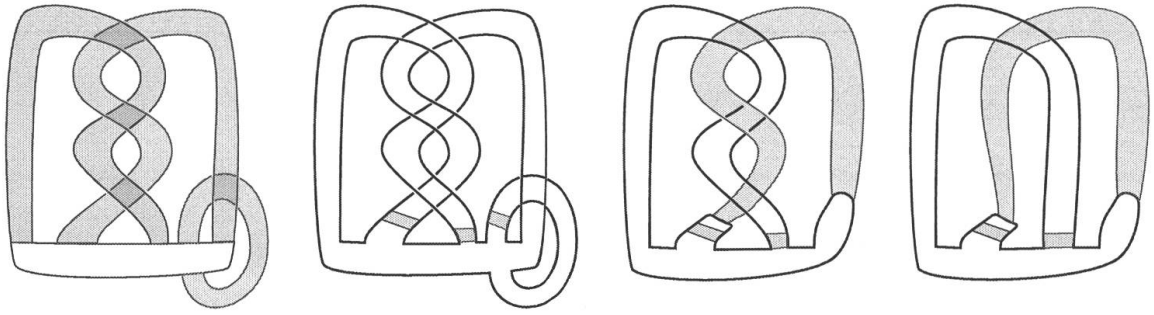


FIGURE 5
Simplifying a banded knot diagram for $D \cup \Sigma$

Now obtain the diagram on the left of Figure 6 by an isotopy of the diagram in S^3, before sliding the green band across the red band to obtain the central diagram. We can then cancel the right-hand unknot with the red band, corresponding to canceling a pair of 0- and 1-handles, in order to obtain the standard diagram for an unknotted torus seen on the right of Figure 6. □

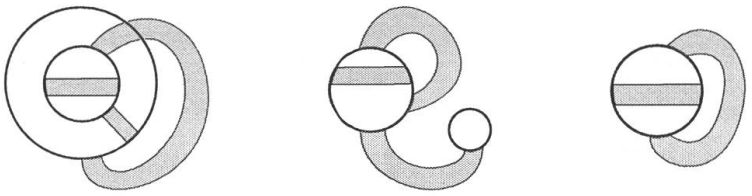


FIGURE 6
Further simplifications of the banded knot diagram for $D \cup \Sigma$, resulting in the standard diagram for an unknotted torus (right)

6. Pairs of slice discs with large generalized stabilization distance

In this section we prove Theorem B. We use the classical Alexander module to show that for every nonnegative integer m there is a knot K with slice discs D and D' such that $d_2(D, D')$ equals m. To do this, we investigate the kernel of the induced map on fundamental groups from the knot exterior to the slice disc exteriors by using the homology of cyclic covering spaces.

First, we note that connected sum with a knotted 2-sphere has no effect on the kernel of the map on fundamental groups.

Proposition 6.1. *Suppose that F_2 has been obtained from F_1 by connected sum with a knotted 2-sphere S. Then*

$$\ker\big(i_1 \colon \pi_1(X_K) \to \pi_1(X_{F_1})\big) = \ker\big(i_2 \colon \pi_1(X_K) \to \pi_1(X_{F_2})\big).$$

Proof. Let $X_S := S^4 \setminus \nu S$ be the exterior of S in S^4. Construct X_{F_2} from X_{F_1} and X_S by identifying thickened meridians $S^1 \times D^2 \subset \partial X_{F_1}$ and $S^1 \times D^2 \subset \partial X_S$ in the boundaries and smoothing corners. By the Seifert-van Kampen theorem we have that

$$\pi_1(X_{F_2}) \cong \pi_1(X_{F_1}) *_{\mathbb{Z}} \pi_1(X_S).$$

So $\pi_1(X_{F_1})$ is isomorphic to a subgroup of $\pi_1(X_{F_2})$ in such a way that the inclusion-induced maps factor as

$$\pi_1(X_{F_1}) \hookrightarrow \pi_1(X_{F_1}) *_{\mathbb{Z}} \pi_1(X_S) \xrightarrow{\cong} \pi_1(X_{F_2}).$$

It follows that $\ker(i_1) = \ker(i_2)$. □

The following proposition is central to the rest of the paper, and so we state it in some generality. In particular, in later sections we will want to apply this result with twisted coefficients, so in the name of efficiency we state and prove the full version here.

Proposition 6.2. *Let F_1 and F_2 be properly embedded surfaces in D^4 with $\partial F_j = K$, where F_2 has been obtained from F_1 by g 1-handle additions such that $g(F_2) = g(F_1) + g$. Let $T \subseteq D^4 \times I$ be the 3-manifold built as in Construction* 3.1. *Suppose that $\phi\colon \pi_1(X_K) \to \mathrm{GL}_m(R)$ extends over $\pi_1(X_T)$ to a map $\Phi\colon \pi_1(X_T) \to \mathrm{GL}_m(R)$. For $j = 1, 2$ define*

$$P_j := \ker\big(H_1^{\phi}(X_K; R) \to H_1^{\Phi}(X_{F_j}; R)\big).$$

Then $P_1 \subseteq P_2$ and, assuming in addition that R is a PID, P_2 is generated as an R-module by $P_1 \cup \{x_i\}_{i=1}^{gm}$ for some choice of $x_i \in P_2$.

Proof. The case of general g follows immediately from repeated application of the $g = 1$ case, which we now prove.

Recall that X_T is obtained from $X_{F_1} \times I$ by attaching a single 5-dimensional 2-handle along $\gamma \times \{1\}$ for γ a simple closed curve representing $[\gamma] = \mu_1 \beta \mu_2^{-1} \beta^{-1}$ in $\pi_1(X_{F_1})$, where μ_1 and μ_2 are meridians of F_1 in D^4 near the attaching spheres of the 1-handle, and β is a parallel push-off of the core of this 1-handle.

There is a CW pair $(X_T^{CW}, X_{F_1}) \simeq (X_T, X_{F_1})$ where X_T^{CW} is a CW complex obtained by attaching a single 2-cell to X_{F_1} along γ. The universal cover $\widetilde{X}_T^{CW} \to X_T^{CW}$ induces a pull-back covering $\widetilde{X}_{F_1} \to X_{F_1}$, with relative cellular chain complex

$$C_*\big(\widetilde{X}_T^{CW}, \widetilde{X}_{F_1}\big) \simeq C_*\big(\widetilde{X}_T, \widetilde{X}_{F_1}\big)$$

with $C_2(\widetilde{X}_T^{CW}, \widetilde{X}_{F_1}) \cong \mathbb{Z}[\pi_1(X_T)]$ and $C_k(\widetilde{X}_T^{CW}, \widetilde{X}_{F_1}) = 0$ for $k \neq 2$. By tensoring with R^m we have that

$$C_k^{\Phi}\left(X_T^{CW}, X_{F_1}; R\right) \cong C_k\left(\widetilde{X}_T^{CW}, \widetilde{X}_{F_1}\right) \otimes_{\mathbb{Z}[\pi_1(X_T)]} R^m$$

is isomorphic to R^m for $k = 2$ and is zero otherwise. Since $C_*^{\Phi}(X_T, X_{F_1}; R) \simeq C_*^{\Phi}(X_T^{CW}, X_{F_1}; R)$, we therefore obtain that $H_k^{\Phi}(X_T, X_{F_1}; R) = 0$ for $k \neq 2$ and $H_2^{\Phi}(X_T, X_{F_1}; R) \cong R^m$.

Since dually X_T is obtained from $X_{F_2} \times I$ by attaching a single 5-dimensional 3-handle, we have that $H_k^{\Phi}(X_T, X_{F_2}; R) = 0$ for $k \neq 3$. For $j = 1, 2$ the long exact sequence in twisted homology with R-coefficients corresponding to the triple (X_T, X_{F_j}, X_K) is

(1)

$$\cdots \to H_3^{\Phi}(X_T, X_{F_j}) \to H_2^{\Phi}(X_{F_j}, X_K) \xrightarrow{g_j} H_2^{\Phi}(X_T, X_K) \xrightarrow{h_j} H_2^{\Phi}(X_T, X_{F_j}) \to \ldots$$

and so we see that g_2 is surjective.

Now consider the following diagram, which is commutative since all maps are induced by various inclusions and natural long exact sequences. The horizontal sequences come from long exact sequences of various pairs and all homology is appropriately twisted with coefficients in R.

$$\begin{array}{ccccccc}
H_2^{\Phi}(X_{F_1}) & \longrightarrow & H_2^{\Phi}(X_{F_1}, X_K) & & & & H_1^{\Phi}(X_{F_1}) \\
\downarrow & & \downarrow g_1 & \searrow^{\partial_1} & & \nearrow^{j_1} & \downarrow \\
H_2^{\Phi}(X_T) & \longrightarrow & H_2^{\Phi}(X_T, X_K) & \xrightarrow{\partial_T} & H_1^{\phi}(X_K) & \xrightarrow{j_T} & H_1^{\Phi}(X_T) \\
\uparrow & & g_2 \uparrow & \nearrow_{\partial_2} & & \searrow_{j_2} & \uparrow \\
H_2^{\Phi}(X_{F_2}) & \longrightarrow & H_2^{\Phi}(X_{F_2}, X_K) & & & & H_1^{\Phi}(X_{F_2})
\end{array}$$

Since g_2 is surjective, we have that $P_2 = \ker(j_2) = \mathrm{Im}(\partial_2) = \mathrm{Im}(\partial_T)$. Also,

$$P_1 = \ker(j_1) = \mathrm{Im}(\partial_1) = \mathrm{Im}(\partial_T \circ g_1) \subseteq \mathrm{Im}(\partial_T) = P_2.$$

So we have established the first conclusion of this proposition.

To establish the second conclusion, we recall from above that $H_2^{\Phi}(X_T, X_{F_1}; R) \cong R^m$ has R-generating rank m. Considering the long exact sequence of Equation (1), we see that

$$\begin{aligned}
\mathrm{coker}(g_1) &= H_2^{\Phi}(X_T, X_K)/\mathrm{Im}(g_1) = H_2^{\Phi}(X_T, X_K)/\ker(h_1) \\
&\cong \mathrm{Im}(h_1) \subseteq H_2^{\Phi}(X_T, X_{F_1})
\end{aligned}$$

and so $\mathrm{coker}(g_1)$ has generating rank no more than m as an R-module, by Lemma 4.1 (2). We can therefore let $\{a_i\}_{i=1}^m$ be elements of $H_2(X_T, X_K)$ which represent generators of $\mathrm{coker}(g_1)$. Hence together with $\mathrm{Im}(g_1)$ the $\{a_i\}_{i=1}^m$

generate $H_2(X_T, X_K)$ as an R-module. Therefore $\partial_T(\mathrm{Im}(g_1) \cup \{a_i\}_{i=1}^m)$ generates $\mathrm{Im}\,\partial_T = P_2$. It follows that

$$\begin{aligned} P_1 \cup \{\partial_T(a_i)\}_{i=1}^m &= \mathrm{Im}(\partial_1) \cup \{\partial_T(a_i)\}_{i=1}^m \\ &= \mathrm{Im}(\partial_T \circ g_1) \cup \{\partial_T(a_i)\}_{i=1}^m \\ &= \partial_T\big(\mathrm{Im}(g_1) \cup \{a_i\}_{i=1}^m\big) \end{aligned}$$

generates $\mathrm{Im}(\partial_T) = P_2$ as an R-module, and so we can let $x_i = \partial_T(a_i)$ for $i = 1, \ldots, m$. □

Proposition 6.3. *Let Δ_1 and Δ_2 be slice discs for a knot K. Let $P_j := \ker(\mathcal{A}_{\mathbb{Q}}(K) \to \mathcal{A}_{\mathbb{Q}}(\Delta_j))$ for $j = 1, 2$. Suppose that $\text{g-rk}(P_1) = \text{g-rk}(P_2) = n$ and that $\text{g-rk}(P_1 \cap P_2) = k$. Then $d_2(\Delta_1, \Delta_2) \geq n - k$.*

Proof. Suppose that F is a genus g surface to which both Δ_1 and Δ_2 stabilize by g 1-handle additions and some number of 2-knot additions. We will show that $g \geq n - k$. By Proposition 3.3, for $j = 1, 2$ there exist a disc Δ_j' obtained from Δ_j by connected sum with some number of knotted 2-spheres such that F is obtained from Δ_j' by g 1-handle additions. It follows from Proposition 6.1 that for $j = 1, 2$ we have

$$P_j' := \ker\big(\mathcal{A}_{\mathbb{Q}}(K) \to \mathcal{A}_{\mathbb{Q}}(\Delta_i')\big) = P_j.$$

Let $P := \ker(\mathcal{A}_{\mathbb{Q}}(K) \to \mathcal{A}_{\mathbb{Q}}(F))$. By Proposition 6.2, we see that both P_1' and P_2' are submodules of P. We now argue that the generating rank of P, considered as a $\mathbb{Q}[t^{\pm 1}]$-module, is at least $2n - k$. To see this we show that $\mathrm{Im}(P_1' \oplus P_2' \to P)$ has generating rank at least $2n - k$ and apply Lemma 4.1 (2). Let $i_1 \colon P_1' \to P$ and $i_2 \colon P_2' \to P$ be the inclusion maps. Both P_1' and P_2' are submodules of P, so

$$\begin{aligned} \ker(i_1 \oplus -i_2 \colon P_1' \oplus P_2' \to P) &= \big\{(p_1, p_2) \in P_1' \oplus P_2' \mid i_1(p_1) = i_2(p_2) \in P\big\} \\ &\cong P_1' \cap P_2'. \end{aligned}$$

We obtain a short exact sequence

$$0 \to P_1' \cap P_2' \to P_1' \oplus P_2' \to \mathrm{Im}(i_1 \oplus -i_2) \to 0,$$

and conclude by Lemma 4.1 (3) that $\text{g-rk}(\mathrm{Im}(i_1 \oplus -i_2)) \geq 2n - k$. Therefore by Lemma 4.1 (2), $\text{g-rk}(P) \geq 2n - k$. Note that this uses that $\mathbb{Q}[t^{\pm 1}]$ is a PID.

However, Proposition 6.2 applied with $m = 1$ also tells us that there exist some $x_1, \ldots, x_g$ in P such that P is generated by $P_1' \cup \{x_1, \ldots, x_g\}$. Therefore the generating rank of P is at most $n + g$, and so we have $n + g \geq \text{g-rk}(P) \geq 2n - k$, from which it follows as desired that $g \geq n - k$. □

The next proposition completes the proof of Theorem B.

Proposition 6.4. *Let K_0 be the knot 9_{46} and let $K = \#_{i=1}^n K_0$. Let $\Delta_1 = \natural_{i=1}^n D_1$ and let $\Delta_2 := \natural_{i=1}^n D_2$ be the 'left band only' and 'right band only' slice discs. Then*

$$d_2(\Delta_1, \Delta_2) = n.$$

Proof. First, note that we can obtain both Δ_1 and Δ_2 from surgery on a genus n Seifert surface for K and so $d_2(\Delta_1, \Delta_2) \leq n$.

There is an identification

$$\mathcal{A}_{\mathbb{Q}}(K) \cong \bigoplus_{i=1}^n \mathcal{A}_{\mathbb{Q}}(K_0) \cong \bigoplus_{i=1}^n \big(\mathbb{Q}[t^{\pm 1}]/\langle 2t-1\rangle \oplus \mathbb{Q}[t^{\pm 1}]/\langle t-2\rangle\big)$$

such that

$$P_1 := \ker\big(\mathcal{A}_{\mathbb{Q}}(K) \to \mathcal{A}_{\mathbb{Q}}(\Delta_1)\big) = \bigoplus_{i=1}^n \mathbb{Q}[t^{\pm 1}]/\langle t-2\rangle$$

and

$$P_2 := \ker\big(\mathcal{A}_{\mathbb{Q}}(K) \to \mathcal{A}_{\mathbb{Q}}(\Delta_2)\big) = \bigoplus_{i=1}^n \mathbb{Q}[t^{\pm 1}]/\langle 2t-1\rangle.$$

In particular, $P_1 \cap P_2 = \{0\}$. Now, g-rk(P_1) = g-rk$(P_2) = n$, and g-rk$(P_1 \cap P_2) = 0$. It follows from Proposition 6.3 that $d_2(\Delta_1, \Delta_2) \geq n$ as required. □

7. Secondary lower bounds using metabelian twisted homology

We now construct subtler examples of pairs of slice discs with high stabilization distance.

7.1. Satellite knots and satellite slice discs. Our examples come from the satellite construction. Let R and J be knots and let $\eta \subset S^3 \smallsetminus R$ be an unknotted simple closed curve in the complement of R. Recall that $S^3 \smallsetminus \nu(\eta) \cup X_J \cong S^3$, where the meridian of η is identified with the longitude of J, and vice versa. The image of $R \subset S^3 \smallsetminus \nu(\eta)$ under this homeomorphism is by definition the satellite knot $R_\eta(J)$.

It is a well known fact that if R and J are slice knots and η is any unknot in the complement of R, then the satellite knot $R_\eta(J)$ is also slice. It will be useful to have an explicit construction of a slice disc Δ_D for $R_\eta(J)$ coming from a choice of slice discs Δ_0 for R and D for J, together with compatible degree 1 maps $f\colon X_{R_\eta(J)} \to X_R$ and $g\colon X_{\Delta_D} \to X_{\Delta_0}$.

Construction 7.1 (Satellite slice discs and degree 1 maps). Let R be a knot with slice disc Δ_0 and let η be an unknotted curve in $S^3 \smallsetminus \nu(R)$. Identify $D^4 \supset \Delta_0$ as $D^2 \times D^2$ in such a way that when we consider $\partial(D^2 \times D^2) = (S^1 \times D^2) \cup (D^2 \times S^1)$ we have $D^2 \times S^1 = \nu(\eta)$ and so $R = \partial\Delta_0 \subseteq S^1 \times D^2$.

Now let J be a knot with slice disc D. We obtain a slice disc denoted Δ_D for $R_\eta(J)$ by considering

$$\Delta_0 \subseteq D \times D^2 = \nu(D) \subset D^4.$$

Note that $X_{\Delta_D} = X_{\Delta_0} \cup_{S^1 \times D^2} X_D$, where $S^1 \times D^2$ is identified with $\nu(\eta) \subseteq X_R \subset \partial X_{\Delta_0}$ and with $S^1 \times D \subset \partial X_D$, and that this identification is evidently compatible with the decomposition $X_{R_\eta(J)} = (X_R \smallsetminus \nu(\eta)) \cup_{T^2} X_J$.

For every knot J there is a standard degree 1 map $f_0 \colon X_J \to X_U$ which sends μ_J to μ_U and λ_J to λ_U, and for any slice disc D there is a similar degree one map $g_0 \colon X_D \to X_E$, where E denotes the standard slice disc for the unknot. For the sake of completeness, we give this construction, emphasizing that one can choose g_0 to be an extension of f_0.

Parametrize

$$\nu(\partial X_J) = \partial X_J \times [0, \delta] = \{(p, s, t) \in S^1 \times ([0, 2\pi]/\sim) \times [0, \delta]\},$$

where $\{(p, 0, 0)\} = \lambda_J$ and $\{(1, s, 0)\} = \mu_J$. Now let $F \subset X_J$ be a (truncated) Seifert surface for J with tubular neighborhood $\nu(F) = F \times [0, \varepsilon]$. We can assume that

$$\nu(F) \cap \nu(\partial X_J) = \{(p, s, t) \in S^1 \times [0, \varepsilon] \times [0, \delta]\},$$

as illustrated in Figure 7.

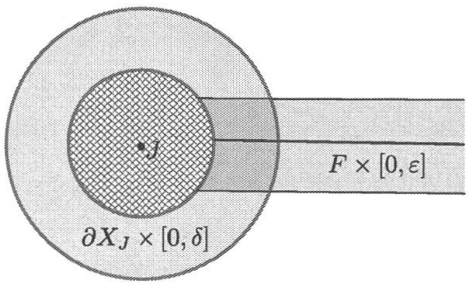


FIGURE 7

A cross section of X_J near its boundary. Note that the cross-hatched region represents $\nu(J)$ and is therefore not part of X_J.

We write $X_U = S \times D$ for $S = ([0,\varepsilon]/\sim) \cong S^1$ and $D = (S^1 \times [0,\delta])/(S^1 \times \delta) \cong D^2$. Define f_0 on $\nu(\partial X_J)$ by

$$f_0(p,s,t) = \begin{cases} (s,(p,t)) & \text{if } 0 \le s \le \varepsilon \\ (\varepsilon,(p,t)) & \text{if } \varepsilon < s, \end{cases}$$

and then extend over the rest of $\nu(F) = F \times [0,\varepsilon]$ by $f_0(y,s) = (s,(0,\delta))$. Finally, for any x in neither $\nu(F)$ nor $\nu(\partial X_K)$, we define $f_0(x) = (\varepsilon,(0,\delta))$.

The construction of g_0 is very similar, only with a compact orientable 3-manifold G with boundary $\partial G = F \cup_J D$ playing the role of the Seifert surface: we extend f_0 as defined above on X_J over $X_J \times I$, then over the rest of $\nu(\partial X_D)$, then over $\nu(G) \cong G \times I$ and then send the entirety of $X_D \setminus (\nu(\partial X_D) \cup \nu(G))$ to a single point in X_E.

Here are the details, which closely parallel the construction of f_0, though with extra care taken to ensure that $g_0|_{X_J} = f_0$:

First parametrize a neighborhood of the slice disc D as $D^2 \times D^2$, naturally a manifold with corners, such that $S^1 \times D^2$ is a tubular neighborhood of J and $S^1 \times S^1 = \partial X_J$. Consider a collar on this part of ∂X_D as follows. We think of X_D as a manifold with corners, with ∂X_J the corner set, dividing ∂X_D as $X_J \cup_{\partial X_J} D^2 \times S^1$. Then we consider a collar on the $D^2 \times S^1$ part of the boundary that restricts on X_J to a collar for ∂X_J in X_J. Parametrize this collar as

$$\nu\left(D^2 \times S^1\right) = D^2 \times S^1 \times [0,\delta] = \left\{(p,s,t) \in D^2 \times ([0,2\pi]/\sim) \times [0,\delta]\right\},$$

where $\{(p,0,0)\}$ is a push-off of the slice disc with boundary λ_J and $\{(1,s,0)\} = \mu_J$.

Now let $G \subset X_D$ be a (truncated) 3-manifold with $\partial G = F \cup \{(p,0,0)\}$, with tubular neighborhood $\nu(G) = G \times [0,\varepsilon]$. We note that the existence of such a 3-manifold follows from a standard obstruction theoretic argument, see, e.g., [Lic, Lemma 8.14]. We can assume this restricts to the tubular neighborhood of F used above in the definition of f_0, and that

$$\nu(G) \cap \nu(D^2 \times S^1) = \left\{(p,s,t) \in D^2 \times [0,\varepsilon] \times [0,\delta]\right\}.$$

We write $X_E = S \times B$ for $S = ([0,\varepsilon]/\sim) \cong S^1$ and $B = (D^2 \times [0,\delta])/(D^2 \times \delta) \cong D^3$. Note that we have a natural inclusion $D \subset B$ corresponding to $X_U = S \times D \subset S \times B = X_E$. Define g_0 on $\nu(D^2 \times S^1)$ by

$$g_0(p,s,t) = \begin{cases} (s,(p,t)) & \text{if } 0 \le s \le \varepsilon \\ (\varepsilon,(p,t)) & \text{if } \varepsilon < s, \end{cases}$$

and then extend over the rest of $\nu(G) = G \times [0,\varepsilon]$ by $g_0(y,s) = (s,(0,\delta))$. Finally, for any x in neither $\nu(G)$ nor $\nu(D^2 \times S^1)$, we define $g_0(x) = (\varepsilon,(0,\delta))$.

By using the above decompositions $X_{R_\eta(J)} = (X_R \smallsetminus \nu(\eta)) \cup_{T^2} X_J$ and $X_{\Delta_D} = X_{\Delta_0} \cup_{S^1\times D^2} X_D$, we obtain compatible degree 1 maps

$$f = \mathrm{Id} \cup f_0 \colon X_{R_\eta(J)} \to X_R \quad \text{and} \quad g = \mathrm{Id} \cup g_0 \colon X_{\Delta_D} \to X_{\Delta_0}.$$

This completes Construction 7.1.

Recall that for a connected space X equipped with a surjective map $\varepsilon\colon \pi_1(X) \to \mathbb{Z}$, we let $\mathcal{A}(X)$ denote the induced $\mathbb{Z}[t^{\pm 1}]$-twisted first homology, and for a knot or disc L we often let $\mathcal{A}(L)$ denote $\mathcal{A}(X_L)$.

Proposition 7.2. *Let R, Δ_0, η, J, and D be as above. Suppose that the linking number of η and R in S^3 is 0. Letting f and g be the degree 1 maps discussed above, the following diagram commutes, where the horizontal maps are the usual inclusion induced maps:*

$$\begin{array}{ccc} \mathcal{A}\big(R_\eta(J)\big) & \longrightarrow & \mathcal{A}(\Delta_D) \\ \big\downarrow{\scriptstyle f_*} & & \big\downarrow{\scriptstyle g_*} \\ \mathcal{A}(R) & \longrightarrow & \mathcal{A}(\Delta_0). \end{array}$$

Moreover, f_* and g_* are isomorphisms and so

$$\ker\big(\mathcal{A}\big(R_\eta(J)\big) \to \mathcal{A}(\Delta_D)\big) = f_*^{-1}\Big(\ker\big(\mathcal{A}(R) \to \mathcal{A}(\Delta_0)\big)\Big) \cong \ker\big(\mathcal{A}(R) \to \mathcal{A}(\Delta_0)\big)$$

is independent of the choice of slice disc D for J.

Proof. The fact that the diagram commutes follows immediately from the compatibility of f and g as defined in Construction 7.1. Since the linking number of R and η is 0, the fact that f_* is an isomorphism is a standard fact (one can also imitate the proof of Proposition 7.8 in a simpler setting). Briefly, one compares the Mayer–Vietoris sequences for $X_{R_\eta(J)} = X_{R\cup\eta} \cup_{S^1\times S^1} X_J$ and $X_R = X_{R_\eta(U)} = X_{R\cup\eta} \cup_{S^1\times S^1} X_U$. The fact that the winding number of η is zero implies that the induced representations $\pi_1(X_J) \to \mathbb{Z}$ and $\pi_1(X_U) \to \mathbb{Z}$ are trivial, so $H_1(X_J; \mathbb{Z}[t^{\pm 1}]) \cong H_1(X_U; \mathbb{Z}[t^{\pm 1}]) \cong \mathbb{Z}[t^{\pm 1}]$.

To see that g_* induces an isomorphism consider the following diagram, where the rows are the Mayer–Vietoris sequences in $\mathbb{Z}[t^{\pm 1}]$-coefficients corresponding to the decompositions $X_{\Delta_D} = X_{\Delta_0} \cup_{S^1\times D^2} X_D$ and $X_{\Delta_0} = X_{\Delta_0} \cup_{S^1\times D^2} X_E$. We have replaced the H_0 terms with zeroes, since the maps from $H_0(S^1\times D^2; \mathbb{Z}[t^{\pm 1}])$ are injective.

$$\begin{array}{ccccccc} H_1\big(S^1\times D^2; \mathbb{Z}[t^{\pm 1}]\big) & \to & H_1\big(X_{\Delta_0}; \mathbb{Z}[t^{\pm 1}]\big) \oplus H_1\big(X_D; \mathbb{Z}[t^{\pm 1}]\big) & \to & H_1\big(X_{\Delta_D}; \mathbb{Z}[t^{\pm 1}]\big) & \to & 0 \\ \big\downarrow & & \big\downarrow{\scriptstyle \mathrm{Id}\oplus(g_0)_*} & & \big\downarrow{\scriptstyle g_*} & & \\ H_1\big(S^1\times D^2; \mathbb{Z}[t^{\pm 1}]\big) & \to & H_1\big(X_{\Delta_0}; \mathbb{Z}[t^{\pm 1}]\big) \oplus H_1\big(X_E; \mathbb{Z}[t^{\pm 1}]\big) & \to & H_1\big(X_{\Delta_0}; \mathbb{Z}[t^{\pm 1}]\big) & \to & 0 \end{array}$$

Since the linking number of η and R is 0, the cores of the copies of $S^1 \times D^2$ along which the spaces are glued, when thought of as fundamental group elements, map trivially to $\mathbb{Z}$ via the appropriate version of ε. Therefore $H_1(S^1 \times D^2; \mathbb{Z}[t^{\pm 1}]) \cong H_1(S^1 \times D^2; \mathbb{Z}) \otimes \mathbb{Z}[t^{\pm 1}] \cong \mathbb{Z}[t^{\pm 1}]$. Similarly, since $S^1 \times D^2 \to X_D$ and $S^1 \times D^2 \to X_E$ are $\mathbb{Z}$-homology equivalences, the maps $\pi_1(X_D) \to \mathbb{Z}$ and $\pi_1(X_E) \to \mathbb{Z}$ are likewise trivial, and so the maps $H_1(S^1 \times D^2; \mathbb{Z}[t^{\pm 1}]) \to H_1(X_D; \mathbb{Z}[t^{\pm 1}])$ and $H_1(S^1 \times D^2; \mathbb{Z}[t^{\pm 1}]) \to H_1(X_E; \mathbb{Z}[t^{\pm 1}])$ are isomorphisms. It follows that the diagram above reduces to the diagram:

$$\begin{array}{ccc} H_1(X_{\Delta_0}; \mathbb{Z}[t^{\pm 1}]) & \xrightarrow{\cong} & H_1(X_{\Delta_D}; \mathbb{Z}[t^{\pm 1}]) = \mathcal{A}(\Delta_D) \\ \downarrow \mathrm{Id} & & \downarrow g_* \\ H_1(X_{\Delta_0}; \mathbb{Z}[t^{\pm 1}]) & \xrightarrow{\cong} & H_1(X_{\Delta_0}; \mathbb{Z}[t^{\pm 1}]) = \mathcal{A}(\Delta_0). \end{array}$$

Therefore the right hand vertical map is an isomorphism induced by g, as required. □

Example 7.3. Let R be the slice knot 6_1, with unknotted curve $\eta \in S^3 \smallsetminus \nu(R)$ as shown on the left of Figure 8. We will be interested in the satellite knot $R_\eta(J)$, depicted on the right of Figure 8, for certain choices of J. Note that η does not intersect F and so $R_\eta(J)$ has a genus 1 Seifert surface F_J as shown on the right of Figure 8. The illustrated homologically essential 0-framed curve on F_J (that, in a mild abuse of notation, we also call γ) is isotopic to the knot J when thought as a curve in S^3.

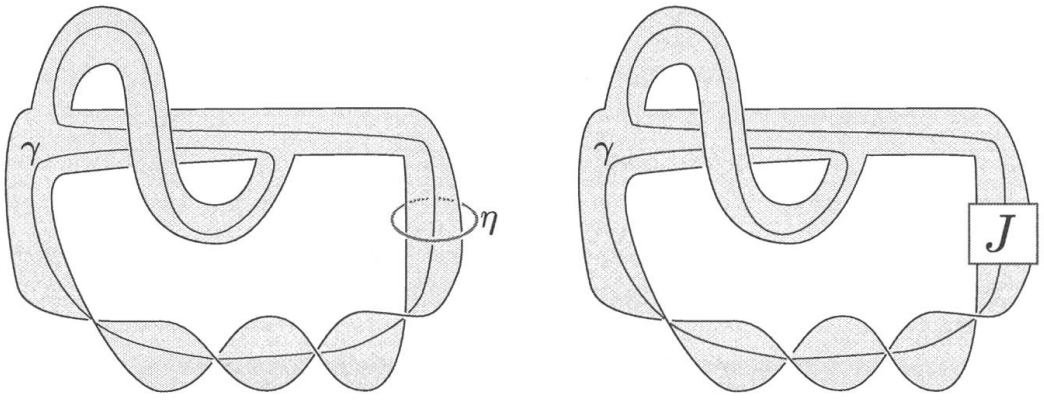


FIGURE 8
The knot $R = 6_1$ with a genus 1 Seifert surface F, a 0-framed curve γ on F, and an infection curve η (left) and the satellite knot $R_\eta(J)$ (right)

Let Δ_0 denote the standard slice disc for R, obtained by surgering F along γ. Given a slice disc D for J, in Construction 7.1 we built a slice disc Δ_D for $R_\eta(J)$. In this context, one can interpret this construction as follows. Push the interior of F_J into the interior of D^4, then remove a small neighborhood of γ in F_J. This creates two new boundary components, which may be capped off with parallel copies of D to yield Δ_D. We note that a single 1-handle attachment to Δ_D that connects the two parallel copies of D returns the (pushed in) Seifert surface F_J, and so if D and D' are two different slice discs for J we always have that $d_2(\Delta_D, \Delta_{D'}) \leq 1$, even if $d_2(D, D')$ is large.

As in Example 5.4, we can pick a basis for the first homology of the Seifert surface F for which the Seifert matrix is given by

$$A = \begin{bmatrix} 1 & 1 \\ 0 & -2 \end{bmatrix}$$

and manipulate $tA - A^T$ to see that $\mathcal{A}(R) \cong \mathbb{Z}[t^{\pm 1}]/\langle (2t-1)(t-2) \rangle$. We have that $\mathcal{A}(\Delta_0) \cong \mathbb{Z}[t^{\pm 1}]/\langle 2t-1 \rangle$, and that the kernel of the inclusion induced map $\mathcal{A}(R) \to \mathcal{A}(\Delta_0)$ is exactly $(t-2)\mathcal{A}(R)$. Details can be found in, e.g., [CP, Section 5.2]. Additionally, by substituting $t = -1$ into the above computations we discover the homology of the 2-fold branched covers: $H_1(\Sigma_2(R)) \cong \mathbb{Z}_9$ and $\ker(H_1(\Sigma_2(R); \mathbb{Z}) \to H_1(\Sigma_2(D^4, \Delta_0); \mathbb{Z})) = 3\,\mathbb{Z}_9$.

7.2. Metabelian twisted homology. We will use twisted homology coming from metabelian representations that factor through the dihedral group $D_{2n} \cong \mathbb{Z}_2 \ltimes \mathbb{Z}_n$. As noted in the introduction, these representations originate in the work of Casson–Gordon [CG1, CG2]. Our perspective on these representations is particularly indebted to the work of [HKL], as well as [KL1, Let, Fri].

Construction 7.4. Consider a knot K with preferred meridian μ_0, an abelianization map $\varepsilon\colon \pi_1(X_K) \to \mathbb{Z}$, and a map $\psi\colon H_1(X_K^2) \to \mathbb{Z}_n$ for some prime n, where X_K^2 is the 2-fold cyclic cover of X_K. Assume that the map ψ factors as

$$\psi\colon H_1(X_K^2) \to H_1\big(\mathrm{E}_{t2}(K)\big) \xrightarrow{\chi} \mathbb{Z}_n,$$

where the first map is induced by the inclusion $X_K^2 \subset \mathrm{E}_{t2}(K)$, so that ψ is determined by χ. Define

$$\phi_\chi\colon \pi_1(X_K) \to \mathbb{Z}_2 \ltimes \mathbb{Z}_n \text{ by } \phi_\chi(\gamma) = \big([\varepsilon(\gamma)], \psi(\mu_0^{-\varepsilon(\gamma)}\gamma)\big),$$

noting that $\mu_0^{-\varepsilon(\gamma)}\gamma \in \ker(\pi_1(X_K) \to \mathbb{Z}_2)$ and so represents an element in $\pi_1(X_K^2)$.

Letting $\xi_n = e^{2\pi i/n}$, we have a standard map

$$\alpha\colon \mathbb{Z}_2 \ltimes \mathbb{Z}_n \to \mathrm{GL}_2\big(\mathbb{Z}[\xi_n]\big)$$
$$(a,b) \mapsto \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}^a \begin{bmatrix} \xi_n^b & 0 \\ 0 & \xi_n^{-b} \end{bmatrix}.$$

In particular, we obtain a representation $\alpha_\chi = \alpha \circ \phi_\chi$ of $\pi_1(X_K)$ into $\mathrm{GL}_2(\mathbb{Z}[\xi_n])$. We will be interested in the corresponding twisted homology $H_*^{\alpha_\chi}(X_K, \mathbb{Z}[\xi_n])$, especially when $\mathbb{Z}[\xi_n]$ is a PID, e.g., when $n = 3$ and $\mathbb{Z}[\xi_3]$ is the ring of *Eisenstein integers*. For a connected space X together with a map $\phi\colon \pi_1(X) \to \mathbb{Z}_2 \ltimes \mathbb{Z}_n$, we will sometimes let $H_*^{\phi}(X; \mathbb{Z}[\xi_n])$ be shorthand for $H_*^{\alpha\circ\phi}(X; \mathbb{Z}[\xi_n])$. When the coefficients are clearly understood and we are short of space, we shall abbreviate this still further to $H_*^{\phi}(X)$.

Remark 7.5. We will often have two compact connected spaces $X \subset Y$ and a map $\alpha_\psi = \alpha \circ \phi_\psi\colon \pi_1(Y) \to \mathrm{GL}_2(\mathbb{Z}[\xi_n])$ arising as above from $\varepsilon\colon \pi_1(Y) \to \mathbb{Z}$ and $\psi\colon Y^2 \to \mathbb{Z}_n$. We wish to consider the inclusion induced maps

$$i_k\colon H_k^{\alpha_\psi \circ i_*}\big(X, \mathbb{Z}[\xi_n]\big) \to H_k^{\alpha_\psi}\big(Y, \mathbb{Z}[\xi_n]\big).$$

To understand this map when $k = 0$, pick a CW structure on X with a single 0-cell x and 1-cells $g_1, \ldots, g_m$ and extend it to a CW structure on Y by first adding 1-cells $g_{m+1}, \ldots, g_{m+m'}$. Of course, there may be many additional n-cells for $n \geq 2$, but these will not impact H_0 computations. The relevant twisted cellular chain complexes are

$$C_0^{\alpha_\psi \circ i_*}(X) \cong C_0^{\alpha_\psi}(Y) \cong \mathbb{Z}[\xi_n]^2, \quad C_1^{\alpha_\psi \circ i_*}(X) \cong \mathbb{Z}[\xi_n]^{2m}, \quad \text{and}$$
$$C_1^{\alpha_\psi}(Y) \cong \mathbb{Z}[\xi_n]^{2(m+m')}$$

with differential maps given by the matrices

$$d_1^X = \big[\, [\alpha_\psi(g_1) - \mathrm{Id}] \;\; [\alpha_\psi(g_2) - \mathrm{Id}] \;\; \ldots \;\; [\alpha_\psi(g_m) - \mathrm{Id}] \,\big]$$
$$d_1^Y = \big[\, [\alpha_\psi(g_1) - \mathrm{Id}] \;\; [\alpha_\psi(g_2) - \mathrm{Id}] \;\; \ldots \;\; [\alpha_\psi(g_m) - \mathrm{Id}] \;\; \ldots \;\; [\alpha_\psi(g_{m+m'}) - \mathrm{Id}] \,\big].$$

It follows that the map i_0 is always a surjection, and is an isomorphism if and only if

$$\mathrm{Span}\big\{\mathrm{Im}[\alpha_\psi(g_i) - \mathrm{Id}]\big\}_{i=1}^{m} = \mathrm{Span}\big\{\mathrm{Im}[\alpha_\psi(g_i) - \mathrm{Id}]\big\}_{i=1}^{m+m'}.$$

In order to ensure that i_0 is an isomorphism, it therefore suffices to check that the two maps $\phi_\psi \circ i_*$ and ϕ_ψ have the same image in $\mathbb{Z}_2 \ltimes \mathbb{Z}_n$. In the rest of this section, whenever we claim that i_0 is an isomorphism it will be because these two images agree, though in the interest of brevity we will often leave that verification to the reader.

We will need a computation of the twisted homology of a knot complement with respect to certain abelian representations into $\mathrm{GL}_2(\mathbb{Z}[\xi_n])$. It will be convenient to have the following notation.

Notation 7.6. Let X be a connected space equipped with a surjection $\varepsilon\colon \pi_1(X) \twoheadrightarrow \mathbb{Z}$, and let ξ be a root of unity. Define $\mathcal{A}_\xi(X) := \mathcal{A}(X) \otimes_{\mathbb{Z}[t^{\pm 1}]} \mathbb{Z}[\xi]$, where $\mathbb{Z}[\xi]$ has the $\mathbb{Z}[t^{\pm 1}]$-module structure induced by $t \cdot a := \xi a$.

Also, for any $\mathbb{Z}[\xi]$-module M, let $\overline{M}$ denote the module with conjugate $\mathbb{Z}[\xi]$-structure and let $M^{1\oplus\bar{1}} := M \oplus \overline{M}$.

Lemma 7.7. *Let X be a connected space with a surjection $\varepsilon\colon \pi_1(X) \twoheadrightarrow \mathbb{Z}$, and define $\phi\colon \pi_1(X) \to \mathrm{GL}_2(\mathbb{Z}[\xi_n])$ by*

$$\gamma \mapsto \begin{bmatrix} \xi_n^{\varepsilon(\gamma)} & 0 \\ 0 & \xi_n^{-\varepsilon(\gamma)} \end{bmatrix}.$$

Then $H_1^\phi(X; \mathbb{Z}[\xi_n]) \cong \mathcal{A}_\xi(X) \oplus \mathcal{A}_{\bar{\xi}}(X) \cong \mathcal{A}_\xi(X)^{1\oplus\bar{1}}$.

Proof. First, note that $H_1^\phi(X; \mathbb{Z}[\xi_n]) \cong H_1^\theta(X; \mathbb{Z}[\xi_n])^{1\oplus\bar{1}}$, where $\theta\colon \pi_1(X) \to \mathbb{Z}[\xi_n]^\times$ is given by $\theta(\gamma) = \xi_n^{\varepsilon(\gamma)}$. So it suffices to show that $H_1^\theta(X; \mathbb{Z}[\xi_n]) \cong \mathcal{A}_\xi(X)$.

Let $X^\infty \to X$ be the ε-induced $\mathbb{Z}$-cover of X. Note that $\theta(\gamma) = 0$ if and only if $\varepsilon(\gamma) \equiv 0 \mod n$, and so the θ-induced cover of X is the n-fold cyclic cover X^n. We can compute $H_1^\theta(X; \mathbb{Z}[\xi_n])$ as

$$H_1^\theta\big(X; \mathbb{Z}[\xi_n]\big) = H_1\big(C_*(X^n) \otimes_{\mathbb{Z}[\mathbb{Z}_n]} \mathbb{Z}[\xi_n]\big) = H_1\big(C_*(X^\infty) \otimes_{\mathbb{Z}[t^{\pm 1}]} \mathbb{Z}[\xi_n]\big).$$

The Künneth spectral sequence [Wei, Theorem 5.6.4, p. 143] tells us that since $C_*(X_\infty)$ is a bounded below complex of flat (in fact free) $\mathbb{Z}[t^{\pm 1}]$-modules, there is a boundedly converging upper right quadrant spectral sequence:

$$E^2_{p,q} = \mathrm{Tor}_p^{\mathbb{Z}[t^{\pm 1}]}\big(H_q(X^\infty), \mathbb{Z}[\xi_n]\big) \Rightarrow H_{p+q}\big(C_*(X^\infty) \otimes_{\mathbb{Z}[t^{\pm 1}]} \mathbb{Z}[\xi_n]\big).$$

The only $E^2_{p,q}$ which could potentially contribute to $H_1(C_*(X^\infty) \otimes_{\mathbb{Z}[t^{\pm 1}]} \mathbb{Z}[\xi_n])$ are $(p,q) \in \{(1,0),(0,1)\}$. The only relevant differential could be $d^2_{2,0}\colon E^2_{2,0} \to E^2_{0,1}$. However.

$$\begin{aligned} E^2_{2,0} &= \mathrm{Tor}_2^{\mathbb{Z}[t^{\pm 1}]}\big(H_0(X^\infty), \mathbb{Z}[\xi_n]\big) = \mathrm{Tor}_2^{\mathbb{Z}[t^{\pm 1}]}\big(\mathbb{Z}[t^{\pm 1}]/\langle t-1\rangle, \mathbb{Z}[\xi_n]\big) \\ &= \mathrm{Tor}_2^{\mathbb{Z}[t^{\pm 1}]}\big(\mathbb{Z}, \mathbb{Z}[\xi_n]\big) = 0, \end{aligned}$$

since as a $\mathbb{Z}[t^{\pm 1}]$-module $\mathbb{Z}$ has a length 1 projective resolution. Therefore the spectral sequence collapses on the 1-line at the E^2 page, and it suffices to compute $E^2_{0,1}$ and $E^2_{1,0}$. We have that

$$\begin{aligned} E^2_{1,0} &= \operatorname{Tor}_1^{\mathbb{Z}[t^{\pm 1}]}\big(H_0(X^\infty), \mathbb{Z}[\xi_n]\big) \\ &= \operatorname{Tor}_1^{\mathbb{Z}[t^{\pm 1}]}\big(\mathbb{Z}[t^{\pm 1}]/\langle t-1\rangle, \mathbb{Z}[\xi_n]\big) \\ &\cong \big\{x \in \mathbb{Z}[\xi_n] \mid (t-1)\cdot x = 0\big\} \\ &\cong \big\{x \in \mathbb{Z}[\xi_n] \mid (\xi_n - 1)x = 0\big\} = 0. \end{aligned}$$

Finally, since

$$E^2_{0,1} = \operatorname{Tor}_0^{\mathbb{Z}[t^{\pm 1}]}\big(H_1(X^\infty), \mathbb{Z}[\xi_n]\big) \cong H_1(X^\infty) \otimes_{\mathbb{Z}[t^{\pm 1}]} \mathbb{Z}[\xi_n] = \mathcal{A}_\xi(X)$$

we obtain our desired result. □

Recall that given a slice knot R with slice disc Δ_0, a slice knot J with slice disc D, and an unknot η in the complement of R, in Construction 7.1 we built degree one maps $f\colon X_{R_\eta(J)} \to X_R$ and $g\colon X_{\Delta_D} \to X_{\Delta_0}$. The following proposition analyzes the f- and g-induced maps on certain twisted first homology modules under some additional conditions.

Proposition 7.8. *Let R be a slice knot with slice disc Δ_0 and J be a slice knot with slice disc D. Let η be an unknot in the complement of R which generates $\mathcal{A}(R)$. Suppose that n is prime and $\chi\colon H_1(E_2(R)) \to \mathbb{Z}_n$ is a nontrivial map such that ϕ_χ extends to $\Phi\colon \pi_1(X_{\Delta_0}) \to \mathbb{Z}_2 \ltimes \mathbb{Z}_n$. There are identifications*

$$\begin{aligned} H_1^{\phi_\chi \circ f_*}\big(X_{R_\eta(J)}, \mathbb{Z}[\xi_n]\big) &\cong H_1^{\phi_\chi}\big(X_R, \mathbb{Z}[\xi_n]\big) \oplus \mathcal{A}_{\xi_n}(J)^{1\oplus \bar{1}} \\ H_1^{\Phi \circ g_*}\big(X_{\Delta_D}, \mathbb{Z}[\xi_n]\big) &\cong H_1^{\Phi}\big(X_{\Delta_0}, \mathbb{Z}[\xi_n]\big) \oplus \mathcal{A}_{\xi_n}(D)^{1\oplus \bar{1}}. \end{aligned}$$

Moreover, these are natural with respect to inclusion maps; in particular

$$P := \ker\Big(H_1^{\phi_\chi \circ f_*}\big(X_{R_\eta(J)}, \mathbb{Z}[\xi_n]\big) \to H_1^{\Phi \circ g_*}\big(X_{\Delta_D}, \mathbb{Z}[\xi_n]\big)\Big)$$

splits as the direct sum of the corresponding kernels $P_R \oplus P_J^{1\oplus \bar{1}}$, where

$$\begin{aligned} P_R &:= \ker\Big(H_1^{\phi_\chi}\big(X_R, \mathbb{Z}[\xi_n]\big) \to H_1^{\Phi}\big(X_{\Delta_0}, \mathbb{Z}[\xi_n]\big)\Big) \\ P_J^{1\oplus \bar{1}} &:= \ker\big(\mathcal{A}_{\xi_n}(J)^{1\oplus \bar{1}} \to \mathcal{A}_{\xi_n}(D)^{1\oplus \bar{1}}\big) = \ker\big(\mathcal{A}_{\xi_n}(J) \to \mathcal{A}_{\xi_n}(D)\big)^{1\oplus \bar{1}}. \end{aligned}$$

The proof of Proposition 7.8, while somewhat long and notation heavy, essentially follows from careful consideration of the relationship between four Mayer–Vietoris long exact sequences. These sequences are related by the maps induced from the following commutative diagram, where we remind the reader that horizontal maps are inclusions and vertical maps are defined as in Construction 7.1:

$$\begin{array}{ccc} X_{R_\eta(J)} = X_R \smallsetminus \nu(\eta) \cup X_J & \xrightarrow{i_\eta \cup i_J} & X_{\Delta_0} \cup X_D = X_{\Delta_D} \\ {\scriptstyle f=\mathrm{Id}\cup f_0}\big\downarrow & & \big\downarrow{\scriptstyle \mathrm{Id}\cup g_0=g} \\ X_R = X_R \smallsetminus \nu(\eta) \cup X_U & \xrightarrow[i_\eta \cup i_U]{} & X_{\Delta_0} \cup X_E = X_{\Delta_0}. \end{array}$$

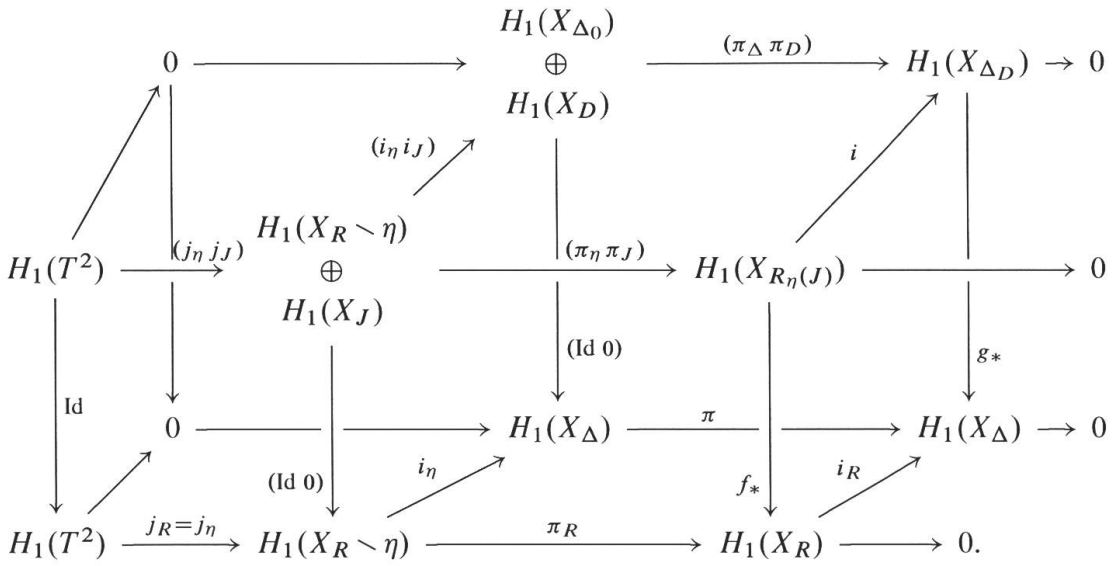
Proof. We abbreviate $X_R \smallsetminus \nu(\eta)$ by $X_R \smallsetminus \eta$ and let $\xi = \xi_n = e^{2\pi i/n}$.

Since $\eta \in \pi_1(X_R)^{(1)}$, when we restrict $(\alpha \circ \phi_\chi) \circ f_*$ to $\pi_1(X_J)$ we see that every element of $\pi_1(X_J)$ is sent to a matrix of the form

$$\begin{bmatrix} \xi^b & 0 \\ 0 & \xi^{-b} \end{bmatrix}$$

for some $b \in \mathbb{Z}_n$. In particular, this restriction factors through $H_1(X_J; \mathbb{Z}) \cong \mathbb{Z}$. The fact that η generates $\mathcal{A}(R)$ implies that the lifts of η to X_R^2 generate $TH_1(X_R^2)$, since $TH_1(X_R^2) \cong \mathcal{A}(R)/\langle t^2 - 1 \rangle$ [Fri, Lemma 2.2]. However, the longitudes of η are identified with the meridians of J in $X_{R_\eta(J)}$, and so since χ is a nontrivial (hence surjective) character, the map $\pi_1(X_J) \to \mathbb{Z}_n$ given by $\gamma \mapsto b(\gamma) \in \mathbb{Z}_n$ is surjective. Henceforth, unless otherwise specified, all homology in this proof is taken to be twisted with $\mathbb{Z}[\xi]$-coefficients induced by (restrictions of) the maps ϕ_χ and Φ, composed with f_* or g_* as appropriate.

We are in the setting of Lemma 7.7 and therefore $H_1(X_J) \cong \mathcal{A}_\xi(J)^{1\oplus\bar{1}}$ and $H_1(X_D) \cong \mathcal{A}_\xi(D)^{1\oplus\bar{1}}$. The decompositions outlined in Construction 7.1 are related by inclusion and degree one maps in such a way that, when we take homology with twisted $\mathbb{Z}[\xi]$-coefficients, we obtain a commutative diagram. Note that the twisted homology $H_1(X_U) = H_1(X_E) = H_1(S^1 \times D^2) = 0$, by Lemma 7.7, since each of these spaces have trivial Alexander module. Also, the maps $H_0(T^2) \to H_0(X_*)$ for $* = U, J$ and $H_0(S^1 \times D^2) \to H_0(X_*)$ for $* = E, D$ are isomorphisms, as follows from an analysis as in Remark 7.5. All horizontal sequences are exact, since they arise from Mayer–Vietoris sequences. We have simplified the following diagram using these observations:



For reasons of concision, in the above diagram we use $(f_1\ f_2)$ to variously refer to any of the maps

$$\begin{bmatrix} f_1 \\ f_2 \end{bmatrix}, \quad [f_1\ f_2], \quad \text{or} \quad \begin{bmatrix} f_1 & 0 \\ 0 & f_2 \end{bmatrix}$$

as appropriate.

We immediately obtain that

$$[\pi_\Delta\ \pi_D]\colon H_1(X_{\Delta_0}) \oplus H_1(X_D) \to H_1(X_{\Delta_D})$$

is an isomorphism, which is the second identification of the proposition. We also see that

$$H_1(X_R) = \operatorname{Im}(\pi_R) \cong H_1(X_R \smallsetminus \eta)/\ker(\pi_R) = H_1(X_R \smallsetminus \eta)/\operatorname{Im}(j_R)$$

and similarly that

$$H_1(X_{R_\eta(J)}) = \operatorname{Im}\big([\pi_\eta\ \pi_J]\big) \cong \big(H_1(X_R \smallsetminus \eta)\ \oplus\ H_1(X_J)\big)/\operatorname{Im}\begin{bmatrix} j_R \\ j_J \end{bmatrix}.$$

We can directly compute that

$$H_1(T^2) = H_1\big(C_*(\widetilde{T^2}) \otimes_{\mathbb{Z}[\pi_1(T^2)]} \mathbb{Z}[\xi]^2\big) \cong \big(\mathbb{Z}[\xi]/(\xi - 1)\big)^{1 \oplus \bar{1}}$$

is generated as a $\mathbb{Z}[\xi]$-module by $\alpha \otimes [0,1]$ and $\alpha \otimes [1,0]$, where α is the curve on T^2 identified with μ_η in $X_R \smallsetminus \eta$ and λ_J in X_J. Since $[\lambda_J] = 0 \in H_1(X_J^\infty)$, we see that

$$j_J\big(\alpha \otimes [0,1]\big) = j_J\big(\alpha \otimes [1,0]\big) = 0 \text{ in } H_1(X_J)$$

and hence that $j_J = 0$.

It follows that the map induced by $[\pi_\eta\ \pi_J]$ from $H_1(X_R \smallsetminus \eta)/\operatorname{Im}(j_\eta) \oplus H_1(X_J)$ to $H_1(X_{R_\eta(J)})$ is an isomorphism, and that our desired isomorphism is given by the composition[2]

$$\Phi\colon H_1(X_R) \oplus H_1(X_J) \xrightarrow{\begin{bmatrix} \pi_R^{-1} & 0 \\ 0 & \mathrm{Id} \end{bmatrix}} H_1(X_R \smallsetminus \eta)/\operatorname{Im}(j_\eta) \oplus H_1(X_J) \xrightarrow{[\pi_\eta\ \pi_J]} H_1(X_{R_\eta(J)}). \tag{2}$$

It remains to show that $\Phi^{-1}(\ker(i)) = \ker(i_R) \oplus \ker(i_J)$, which will follow from some diagram chasing,

Claim 7.9. $\Phi^{-1}\big(\ker(i)\big) \subseteq \ker(i_R) \oplus \ker(i_J)$.

Let $x \in \ker(i)$. Since $(\pi_\eta \oplus \pi_J)$ is onto, there exists $a \in H_1(X_R \smallsetminus \eta)$ and $b \in H_1(X_J)$ such that $(\pi_\eta \oplus \pi_J)(a,b) = x$. Moreover, $(\pi_R(a), b) = \Phi^{-1}(x)$, so it suffices to show that

$$i_R\big(\pi_R(a)\big) = 0 \in H_1(X_{\Delta_0}) \text{ and } i_J(b) = 0 \in H_1(X_D).$$

Observe that by the commutativity of our large diagram,

$$\pi_R(a) = \big(\pi_R \circ [\mathrm{Id}\ 0]\big)(a,b) = \big(f_* \circ [\pi_\eta\ \pi_J]\big)(a,b) = f_*(x).$$

Therefore

$$(i_R \circ \pi_R)(a) = (i_R \circ f_*)(x) = (g_* \circ i)(x) = g_*(0) = 0.$$

In order to show that $i_J(b) = 0$, observe that

$$\left([\pi_\Delta\ \pi_D] \circ \begin{bmatrix} i_\eta & 0 \\ 0 & i_J \end{bmatrix}\right)(a,b) = \big(i \circ [\pi_\eta\ \pi_J]\big)(a,b) = i(x) = 0.$$

But $[\pi_\Delta\ \pi_D]$ is an isomorphism, and so it follows that

$$\begin{bmatrix} i_\eta & 0 \\ 0 & i_J \end{bmatrix}(a,b) = \big(i_\eta(a), i_J(b)\big) = 0.$$

So $i_J(b) = 0$ as desired. This completes the proof of the claim that $\Phi^{-1}(\ker(i)) \subseteq \ker(i_R) \oplus \ker(i_J)$.

[2] The labels of the maps in Equation (2) are mild abuses of notation. In particular, $\pi_R\colon H_1(X_R \smallsetminus \eta) \to H_1(X_R)$ is not itself an isomorphism and hence does not have an inverse until we mod out by $\operatorname{Im}(j_\eta)$, and $[\pi_\eta\ \pi_J]$ actually has domain $H_1(X_R \smallsetminus \eta) \oplus H_1(X_J)$, though it of course induces a well-defined map on $H_1(X_R \smallsetminus \eta)/\operatorname{Im}(j_\eta) \oplus H_1(X_J)$. Nevertheless, we hope the reader finds the reminder of how these maps are induced sufficiently helpful so as to outweigh the indignity of slightly misleading labels.

Claim 7.10. $\Phi^{-1}\big(\ker(i)\big) \supseteq \ker(i_R) \oplus \ker(i_J)$.

It suffices to show that both $\ker(i_R)$ and $\ker(i_J)$ are contained in $\Phi^{-1}(\ker(i))$. Observe that if $b \in \ker(i_J)$ then

$$i\big(\Phi(b)\big) = i\big(\pi_J(b)\big) = \pi_D\big(i_J(b)\big) = \pi_D(0) = 0,$$

so $b \in \Phi^{-1}(\ker(i))$. Now let $\alpha \in \ker(i_R)$ to show that $\Phi(\alpha) \in \ker(i)$. Let $a \in H_1(X_R \smallsetminus \eta)$ be such that $\pi_R(a) = \alpha$, and observe that $\Phi(\alpha) = \pi_\eta(a)$. We have that

$$(\pi \circ i_\eta)(a) = (i_R \circ \pi_R)(a) = i_R(\alpha) = 0.$$

Since π is an isomorphism, this implies that $i_\eta(a) = 0$ and hence that

$$i\big(\Phi(\alpha)\big) = i\big(\pi_\eta(a)\big) = \pi_\Delta\big(i_\eta(a)\big) = \pi_\Delta(0) = 0,$$

as desired. This completes the proof of the claim that $\Phi^{-1}(\ker(i)) \supseteq \ker(i_R) \oplus \ker(i_J)$.

The last two claims combine to show that $\Phi^{-1}(\ker(i)) = \ker(i_R) \oplus \ker(i_J)$, which completes the proof of Proposition 7.8. □

Note that given a properly embedded disc D in D^4 and a knotted 2-sphere S in S^4, we can decompose $X_{D\#S} = X_D \cup_{S^1 \times D^2} X_S$. It follows that the double cover is decomposed analogously; gluing in the branch set and applying a straightforward Mayer–Vietoris argument tells us that

$$H_1\big(\Sigma_2(D^4, D\#S)\big) \cong H_1\big(\Sigma_2(D^4, D)\big) \oplus H_1\big(\Sigma_2(S^4, S)\big).$$

Given $\chi\colon H_1\big(\Sigma_2(K)\big) \to \mathbb{Z}_n$ that extends to $\chi_D\colon H_1\big(\Sigma_2(D^4, D)\big) \to \mathbb{Z}_n$, define

$$\chi_{D\#S}\colon H_1\big(\Sigma_2(D^4, D\#S)\big) \cong H_1\big(\Sigma_2(D^4, D)\big) \oplus H_1\big(\Sigma_2(S^4, S)\big) \xrightarrow{\chi_D \oplus 0} \mathbb{Z}_n.$$

We can now show an analogue of Proposition 6.1 in the context of twisted homology.

Proposition 7.11. *Let D be a properly embedded disc in D^4 with boundary K, and let S be a knotted 2-sphere in S^4. Let $\chi\colon H_1(\Sigma_2(K)) \to \mathbb{Z}_n$ be a map that extends to $\chi_D\colon H_1(\Sigma_2(D^4, D)) \to \mathbb{Z}_n$, and let $\chi_{D\#S}$ be as above. Then*

$$\ker\Big(H_1^{\phi_\chi}(X_K) \to H_1^{\phi_{\chi_D}}(X_D)\Big) = \ker\Big(H_1^{\phi_\chi}(X_K) \to H_1^{\phi_{\chi_{D\#S}}}(X_{D\#S})\Big).$$

Proof. For a submanifold $Y \subset X_{D\#S}$ we can restrict $\phi_{\chi_{D\#S}}$ to $\pi_1(Y)$ and, by a mild abuse of notation we let $H_*^{\phi_{\chi_{D\#S}}}(Y)$ denote the resulting twisted homology with $\mathbb{Z}[\xi_n]$-coefficients.

We shall use the decomposition $X_{D\#S} = X_D \cup_{S^1\times D^2} X_S$. First we compute the homology of $S^1\times D^2$ and X_S. Letting t denote the generator of $\pi_1(S^2\times D^2) \cong \mathbb{Z}$, we can pick a cell structure for (a space homotopy equivalent to) $S^1 \times D^2$ consisting of a single 0-cell and a single 1-cell and use this to compute

$$\begin{aligned} H_1^{\phi_{\chi_{D\#S}}}(S^1 \times D^2) &\cong \ker(\phi_{\chi_{D\#S}}(t) - \mathrm{Id}) \\ &= \ker\left(\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} \xi^b & 0 \\ 0 & \xi^{-b} \end{bmatrix} - \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} : \mathbb{Z}[\xi_n]^2 \to \mathbb{Z}[\xi_n]^2\right), \\ &\qquad\qquad \text{for some } b \in \mathbb{Z} \\ &= \ker\left(\begin{bmatrix} -1 & \xi^{-b} \\ \xi^b & -1 \end{bmatrix}\right) \cong \{(x, y) \in \mathbb{Z}[\xi_n]^2 \mid \xi^b x = y\} \cong \mathbb{Z}[\xi_n]. \end{aligned}$$

Claim 7.12. We have that

$$H_1^{\phi_{\chi_{D\#S}}}(X_S) \cong \mathbb{Z}[\xi_n] \oplus \left(\mathcal{A}(S) \otimes_{\mathbb{Z}[t^{\pm 1}]} \mathbb{Z}[\xi_n]^2\right),$$

where on the right we have the action of $\mathbb{Z}[t^{\pm 1}]$ on $\mathbb{Z}[\xi_n]^2$ given by $t \cdot [x, y] = [y, x]$.

To see this, use the Künneth spectral sequence [Wei, Theorem 5.6.4] as in the proof of Lemma 7.7. Since $H_0(X^\infty) \cong \mathbb{Z}$, we obtain

$$\begin{aligned} E^2_{0,1} &= \mathcal{A}(S) \otimes_{\mathbb{Z}[t^{\pm 1}]} \mathbb{Z}[\xi_n]^2 \\ E^2_{1,0} &= \mathrm{Tor}_1^{\mathbb{Z}[t^{\pm 1}]}\left(H_0(X_S^\infty), \mathbb{Z}[\xi_n]^2\right) \cong H_1^{\phi_{\chi_{D\#S}}}(S^1) \cong \mathbb{Z}[\xi_n] \\ E^2_{2,0} &= \mathrm{Tor}_2^{\mathbb{Z}[t^{\pm 1}]}\left(H_0(X_S^\infty), \mathbb{Z}[\xi_n]^2\right) \cong H_2^{\phi_{\chi_{D\#S}}}(S^1) = 0. \end{aligned}$$

Since $E^2_{2,0} = 0$ it follows that $E^2_{0,1} \cong E^\infty_{0,1}$. We also have $E^2_{1,0} \cong E^\infty_{1,0}$. The spectral sequence therefore gives rise to a short exact sequence of $\mathbb{Z}[\xi_n]$-modules

$$0 \to \mathcal{A}(S) \otimes_{\mathbb{Z}[t^{\pm 1}]} \mathbb{Z}[\xi_n]^2 \to H_1^{\phi_{\chi_{D\#S}}}(X_S) \to \mathbb{Z}[\xi_n] \to 0,$$

which splits since the last module is free. This completes the proof of the claim.

Moreover, comparing the spectral sequences for $S^1 \times D^2$ and X_S using naturality, it follows that the map $\mathbb{Z}[\xi_n] \cong H_1^{\phi_{\chi_{D\#S}}}(S^1 \times D^2) \to H_1^{\phi_{\chi_{D\#S}}}(X_S)$ is injective and maps onto $\mathbb{Z}[\xi_n]$.

Since the restriction of

$$\phi_{\chi_{D\#S}} : \pi_1(X_{D\#S}) \to \mathbb{Z}_2 \ltimes \mathbb{Z}_n$$

to $\pi_1(X_S)$ is the map $\gamma \mapsto ([\varepsilon_S(\gamma)], 0)$ we have that

$$H_0^{\phi_{\chi_{D\#S}}}(S^1 \times D^2) \to H_0^{\phi_{\chi_{D\#S}}}(X_S)$$

is an isomorphism, see Remark 7.5. The Mayer–Vietoris sequence for $X_{D\#S} = X_D \cup_{S^1\times D^2} X_S$ with $\mathbb{Z}[\xi_n]$-coefficients therefore gives us that

$$H_1^{\phi_{\chi_{D\#S}}}(X_{D\#S}) \cong H_1^{\phi_{\chi_D}}(X_D) \oplus \big(\mathcal{A}(S) \otimes_{\mathbb{Z}[t^{\pm 1}]} \mathbb{Z}[\xi_n]^2\big),$$

since $H_1^{\phi_{\chi_{D\#S}}}(S^1 \times D^2) \cong \mathbb{Z}[\xi_n]$ maps onto the $\mathbb{Z}[\xi_n]$-summand of $H_1^{\phi_{\chi_{D\#S}}}(X_S)$.

Since $X_K \subset X_D$, the inclusion induced map $H_1^{\phi_\chi}(X_K) \to H_1^{\phi_{\chi_{D\#S}}}(X_{D\#S})$ factors as

$$H_1^{\phi_\chi}(X_K) \to H_1^{\phi_{\chi_D}}(X_D) \to H_1^{\phi_{\chi_D}}(X_D) \oplus \big(\mathcal{A}(S) \otimes_{\mathbb{Z}[t^{\pm 1}]} \mathbb{Z}[\xi_n]^2\big) \cong H_1^{\phi_{\chi_{D\#S}}}(X_{D\#S}).$$

We saw that the central map is a split injection, the inclusion of the $H_1^{\phi_{\chi_D}}(X_D)$ direct summand. It follows that

$$\ker\big(H_1^{\phi_\chi}(X_K) \to H_1^{\phi_{\chi_{D\#S}}}(X_{D\#S})\big) \cong \ker\big(H_1^{\phi_\chi}(X_K) \to H_1^{\phi_{\chi_D}}(X_D)\big)$$

as desired. □

7.3. Construction of examples and proof of Theorem C. Recall from Notation 7.6 that for a space X and a root of unity ξ, we define

$$A_\xi(X) := \mathcal{A}(X) \otimes_{\mathbb{Z}[t^{\pm 1}]} \mathbb{Z}[\xi].$$

Now let J_0 be a ribbon knot with preferred ribbon disc D_0 such that

$$\mathcal{A}_{\xi_3}(J_0)/\ker\big(\mathcal{A}_{\xi_3}(J_0) \to \mathcal{A}_{\xi_3}(D_0)\big)$$

is nonzero. The knot $J := J_0\# - J_0$ has two preferred slice (in fact ribbon) discs: D_1 consists of $D_0 \natural - D_0$ and D_2 is the standard ribbon disc for any knot of the form $K\# - K$ obtained by spinning. Note that $\mathcal{A}(J) \cong \mathcal{A}(J_0) \oplus \mathcal{A}(J_0)$, $\mathcal{A}(D_1) \cong \mathcal{A}(D_0) \oplus \mathcal{A}(D_0)$, and by the next lemma $\mathcal{A}(D_2) \cong \mathcal{A}(J_0)$.

Lemma 7.13. *The spun slice disc satisfies* $\mathcal{A}(D_2) \cong \mathcal{A}(J_0)$.

Proof. Let $J_0^\dagger$ be a tangle $D^1 \subseteq D^3$ arising from removing a trivial ball-arc pair (D^3, D^1) from (S^3, J_0). Note that

$$\mathcal{A}(J_0^\dagger) = H_1(D^3 \setminus \nu J_0^\dagger) \cong \mathcal{A}(J_0)$$

and

$$D^4 \setminus \nu D_2 \cong (D^3 \setminus \nu J_0^\dagger) \times I \simeq D^3 \setminus \nu J_0^\dagger.$$

It follows that $\mathcal{A}(D_2) \cong \mathcal{A}(J_0)$ as claimed. □

Moreover, the map $i_1\colon \mathcal{A}(J) \to \mathcal{A}(D_1)$ is given by $(x, y) \mapsto (i_0(x), i_0(y))$ and the map $i_2\colon \mathcal{A}(J) \to \mathcal{A}(D_2)$ is given by $(x, y) \mapsto x + y$.

Example 7.14. One example of such a knot is $J_0 = 6_1$. As noted in Example 7.3, $\mathcal{A}(J_0) = \mathbb{Z}[t^{\pm 1}]/\langle (2t-1)(t-2)\rangle$, $\mathcal{A}(D_0) = \mathbb{Z}[t^{\pm 1}]/\langle t-2\rangle$ and the map $i_0\colon \mathcal{A}(J_0) \to \mathcal{A}(D_0)$ is given by multiplication by $2t-1$. In particular, we have that

$$\begin{aligned}\mathcal{A}_{\xi_3}(J_0)/\ker\big(\mathcal{A}_{\xi_3}(J_0) \to \mathcal{A}_{\xi_3}(D_0)\big) &\cong \mathbb{Z}[\xi_3]/\langle (2\xi_3-1)(\xi_3-2), \xi_3-2\rangle \\ &\cong \mathbb{Z}_7[x]/\langle x-2\rangle \neq 0.\end{aligned}$$

Here the $\mathbb{Z}_7$ comes from $\xi_3^2 + \xi_3 + 1 = 0$, combined with $\xi_3 - 2 = 0$.

Now we prove the following more explicit version of Theorem C.

Theorem 7.15. *Let (R, η, Δ_0) be as in Example 7.3 and let J_0 be a ribbon knot with preferred ribbon disc D_0 such that $\mathcal{A}_{\xi_3}(J_0)/\ker\big(\mathcal{A}_{\xi_3}(J_0) \to \mathcal{A}_{\xi_3}(D_0)\big)$ is nonzero. Let $J = J_0\# -J_0$, D_1, and D_2 be defined as above. Then for any $g \geq 0$, the knot $K := \#_{i=1}^{4g} R_\eta(J)$ has ribbon discs Δ_1, the boundary connected sum of $4g$ copies of Δ_{D_1}, and Δ_2, the boundary connected sum of $4g$ copies of Δ_{D_2}, such that*

$$\ker\big(\mathcal{A}_{\mathbb{Q}}(K) \to \mathcal{A}_{\mathbb{Q}}(\Delta_1)\big) \cong \ker\big(\mathcal{A}_{\mathbb{Q}}(K) \to \mathcal{A}_{\mathbb{Q}}(\Delta_2)\big).$$

and yet

$$d_2(\Delta_1, \Delta_2) \geq g.$$

As discussed in Example 7.3, since both Δ_{D_1} and Δ_{D_2} are obtained from surgery on a genus 1 Seifert surface for $R_\eta(J)$, we know that $d_2(\Delta_{D_1}, \Delta_{D_2}) \leq 1$. It follows that $d_2(\Delta_1, \Delta_2) \leq 4g$, though we are not able to determine $d_2(\Delta_1, \Delta_2)$ precisely.

Remark 7.16. The proof that $d_2(\Delta_1, \Delta_2) \geq g$ is somewhat long and involved, so for the reader's convenience we outline the key points in advance:

We suppose that F is a genus $h \leq g$ surface to which both Δ_1 and Δ_2 stabilize by addition of h 1-handles and some number of local 2-knots, in order to show $h = g$.

For $j = 1, 2$ let Δ_j' be a disc obtained from Δ_j by 2-knot addition which stabilizes to F via h 1-handle additions. Let $T = T_1 \cup -T_2$ denote the standard cobordism built as in Construction 3.1, so X_T is a cobordism from $X_{\Delta_1'}$ through X_F to $X_{\Delta_2'}$. Our first main argument proving Claim 7.17 below shows that there exists a highly nontrivial character on $H_1(\Sigma_2(K))$ giving rise to a representation $\pi_1(X_K) \to \mathbb{Z}_2 \ltimes \mathbb{Z}_3$ that extends over X_T to a map Φ with certain nice properties.

Just as in the proof of Theorem B, we compare $\ker(H_1^\Phi(X_K) \to H_1^\Phi(X_{\Delta_1}))$ and $\ker(H_1^\Phi(X_K) \to H_1^\Phi(X_{\Delta_2}))$. Essentially by Proposition 7.11 and the careful construction of Φ, we are able to work with $\ker(H_1^\Phi(X_K) \xrightarrow{\iota_1} H_1^\Phi(X_{\Delta_1'}))$ and $\ker(H_1^\Phi(X_K) \xrightarrow{\iota_2} H_1^\Phi(X_{\Delta_2'}))$ instead. By the construction of our examples, work before the statement of Theorem 7.15, and Proposition 7.8, we can show that $\ker(\iota_2)/(\ker(\iota_1) \cap \ker(\iota_2))$ has generating rank x at least $2g$. We then use Proposition 6.2 to show that $\ker(\iota_F)$ both contains $\ker(\iota_2)$ and is generated by $\ker(\iota_1)$ together with some other $2h$ elements. It follows that $\ker(\iota_2)/(\ker(\iota_1) \cap \ker(\iota_2))$ has generating rank x no more than $2h$, and hence $2g \leq x \leq 2h$ so $g \leq h$. We assumed $h \leq g$ so $g = h$ as desired.

Proof of Theorem 7.15. Fix $g \in \mathbb{N}$, and let K, Δ_1, and Δ_2 be as above. Define $N = 4g$, $\xi := \xi_3$, and recall that for any knot or slice disc L we have $\mathcal{A}_\xi(L) := \mathcal{A}(L) \otimes_{\mathbb{Z}[t^{\pm 1}]} \mathbb{Z}[\xi]$. By Proposition 7.2 we have identifications

$$\mathcal{A}(K) \cong \bigoplus_{i=1}^{N} \mathcal{A}\big(R_\eta(J)\big) \cong \bigoplus_{i=1}^{N} \mathcal{A}(R)$$

and

$$\mathcal{A}(\Delta_j) \cong \bigoplus_{i=1}^{N} \mathcal{A}(\Delta_{D_j}) \cong \bigoplus_{i=1}^{N} \mathcal{A}(\Delta_0) \quad \text{for} \quad j = 1, 2$$

in such a way that $\ker(\mathcal{A}(K) \to \mathcal{A}(\Delta_1))$ and $\ker(\mathcal{A}(K) \to \mathcal{A}(\Delta_2))$ are both identified with a sum $\bigoplus_{i=1}^{N} \ker(\mathcal{A}(R) \to \mathcal{A}(\Delta_0))$, and in particular are equal. Since $\mathcal{A}_\mathbb{Q}(L) \cong \mathcal{A}(L) \otimes \mathbb{Q}$ for any knot or slice disc L, our first conclusion follows.

Now suppose that F is a genus $h \leq g$ surface to which both Δ_1 and Δ_2 stabilize by addition of h 1-handles and some number of local 2-knots. We shall show under these assumptions that $h \geq g$. As in the proof of Proposition 6.3, for $j = 1, 2$ there exist discs Δ_j' obtained from Δ_j by connected sum with local 2-knots such that F is obtained from Δ_j' by h 1-handle additions. For $j = 1, 2$ we write $\Delta_j' = \Delta_j \# S_j$ for some local 2-knot S_j.

Note that $f\colon X_{R_\eta(J)} \to X_R$ lifts to give a degree one map $X^2_{R_\eta(J)} \to X^2_R$, which extends to give $\overline{f}\colon \Sigma_2(R_\eta(J)) \to \Sigma_2(R)$. Moreover, Proposition 7.2 implies that $\overline{f}$ induces an isomorphism on first homology. So we obtain an isomorphism

$$\mathfrak{f}\colon H_1\big(\Sigma_2(K)\big) \cong \oplus_{i=1}^{N} H_1\Big(\Sigma_2\big(R_\eta(J)\big)\Big) \xrightarrow{\oplus_{i=1}^{N} \overline{f}_*} \oplus_{i=1}^{N} H_1\big(\Sigma_2(R)\big) \cong H_1\big(\Sigma_2(R_N)\big)$$

where we let R_N denote the connected sum of N copies of R.

Let T_1 and T_2 be appropriate unions of the simple cobordisms built in Construction 3.1, such that X_{T_1} is a cobordism from $X_{\Delta_1'}$ to X_F rel. X_K and X_{T_2} is a cobordism from $X_{\Delta_2'}$ to X_F rel. X_K. We let $X_T := X_{T_1} \cup_{X_F} -X_{T_2}$.

Claim 7.17. There exists a map

$$\chi = (\chi_i)_{i=1}^N \colon \oplus_{i=1}^N H_1\big(\Sigma_2(R)\big) \to \mathbb{Z}_3$$

with at least $2g$ of the χ_i nonzero such that $\phi_{\chi\circ\mathfrak{f}} \colon \pi_1(X_K) \to \mathbb{Z}_2 \ltimes \mathbb{Z}_3$ extends over $\pi_1(X_T)$ to a map $\Phi \colon \pi_1(X_T) \to \mathbb{Z}_2 \ltimes \mathbb{Z}_3$ and for $j = 1, 2$ the composition

$$\pi_1(X_{S_j}) \to \pi_1(X_{\Delta_j}) *_{\mathbb{Z}} \pi_1(X_{S_j}) \cong \pi_1(X_{\Delta_j'}) \to \pi_1(X_T) \xrightarrow{\Phi} \mathbb{Z}_2 \ltimes \mathbb{Z}_3$$

is given by $\gamma \mapsto ([\varepsilon(\gamma)], 0)$.

We will always construct our extensions in stages, first extending over

$$Y = X_{\Delta_1'} \cup (X_K \times I) \cup X_{\Delta_2'}$$

and then extending over the rest of X_T.

Note that $H_1(\Sigma_2(R)) \cong \mathbb{Z}_9$ and that it follows from Proposition 7.2 that

$$\begin{aligned} (3)\quad \ker\Big(H_1\big(\Sigma_2(K)\big) \to H_1\big(\Sigma_2(D^4, \Delta_j)\big)\Big) \\ \cong \ker\Big(\bigoplus_{i=1}^N H_1\big(\Sigma_2(R)\big) \to \bigoplus_{i=1}^N H_1\big(\Sigma_2(D^4, \Delta_0)\big)\Big) \\ (4)\quad \cong \bigoplus_{i=1}^N 3\,\mathbb{Z}_9\,. \end{aligned}$$

It follows that for $j = 1, 2$ and for any character $\chi \colon H_1(\Sigma_2(R_N)) \to \mathbb{Z}_3$ we have that $\chi \circ \mathfrak{f}$ extends to a map χ_j on $H_1(\Sigma_2(D^4, \Delta_j))$, up to a priori extending its range to $\mathbb{Z}_{3^a}$ for some $a \geq 1$. However, since our slice discs Δ_j are in fact ribbon discs, the inclusion induced map $\pi_1(X_K) \to \pi_1(X_{\Delta_j})$ is surjective for $j = 1, 2$. So we can take $a = 1$.

Note that any map $\chi \circ \mathfrak{f} \colon H_1(\Sigma_2(K)) \to \mathbb{Z}_3$ induces $\overline{\chi \circ \mathfrak{f}} \colon H_1(X_K^2) \to \mathbb{Z}_3$ by precomposition with the natural inclusion induced map $H_1(X_K^2) \to H_1(\Sigma_2(K))$. Since inclusion induces isomorphisms of $H_1(X_K)$ with $H_1(X_T)$, in order to show that a given $\phi_{\chi\circ\mathfrak{f}}$ extends over $\pi_1(X_T)$ it suffices to extend the corresponding $\overline{\chi \circ \mathfrak{f}}$ first over $\pi_1(X^2_{\Delta_1'} \cup (X_K^2 \times I) \cup X^2_{\Delta_2'})$ and then over $\pi_1(X_T^2)$.

Now, consider the Mayer–Vietoris sequence for $X^2_{\Delta_1'} \cup (X_K^2 \times I) \cup X^2_{\Delta_2'}$, which we note is diffeomorphic to $X^2_{\Delta_1'} \cup_{X_K^2} X^2_{\Delta_2'}$:

$$H_1(X_K^2) \xrightarrow{i_1' \oplus i_2'} H_1(X^2_{\Delta_1'}) \oplus H_1(X^2_{\Delta_2'}) \xrightarrow{j_1 \oplus j_2} H_1(X^2_{\Delta_1'} \cup_{X_K^2} X^2_{\Delta_2'}) \to 0.$$

For $j = 1, 2$ we have that $H_1(X^2_{\Delta'_j}) \cong H_1(X^2_{\Delta_j}) \oplus H_1(\Sigma_2(S^4, S_j))$ in such a way that $i'_j \colon H_1(X^2_K) \to H_1(X^2_{\Delta'_j})$ is given by $i_j \oplus 0$, where $i_j \colon H_1(X^2_K) \to H_1(X^2_{\Delta_j})$ is the inclusion-induced map. We therefore obtain, recalling that the map $H_1(X^2_K) \to H_1(X^2_{\Delta_j})$ is surjective since Δ_j is a ribbon disc, that

$$H_1\big(X^2_{\Delta'_1} \cup_{X^2_K} X^2_{\Delta'_2}\big) \cong H_1(X^2_{\Delta_1}) \oplus H_1\big(\Sigma_2(S^4, S_1)\big) \oplus H_1\big(\Sigma_2(S^4, S_2)\big).$$

Therefore any $\overline{\chi \circ \mathfrak{f}}$ can be extended over

$$X^2_{\Delta'_1} \cup \big(X^2_K \times I\big) \cup X^2_{\Delta'_2} = \big(X^2_{\Delta_1} \cup X^2_{S_1}\big) \cup \big(X^2_K \times I\big) \cup \big(X^2_{\Delta_2} \cup X^2_{S_2}\big) \subset \partial X^2_T$$

so that the extension is trivial on the $H_1(\Sigma_2(S^4, S_1)) \oplus H_1(\Sigma_2(S^4, S_2))$-summand. Moreover, such a map extends over $H_1(X^2_T)$ if and only if it vanishes on

$$H := \ker\Big(H_1\big(X^2_{\Delta'_1} \cup \big(X^2_K \times I\big) \cup X^2_{\Delta'_2}\big) \to H_1(X^2_T)\Big).$$

Note that our maps $\overline{\chi \circ \mathfrak{f}}$ have been chosen to vanish on $H_1(\Sigma_2(S^4, S_1)) \oplus H_1(\Sigma_2(S^4, S_2))$, and hence vanish on H if and only if they vanish on

$$H \cap H_1\big(X^2_{\Delta_1}\big) = \ker\Big(H_1(X^2_{\Delta_1}) \to H_1\big(X^2_T\big)\Big).$$

Moreover, $\ker\Big(H_1(X^2_{\Delta_1}) \to H_1(X^2_T)\Big)$ is isomorphic to a quotient of $\ker(H_1(X^2_K) \to H_1(X^2_T))$.

For a space X with surjection $\varepsilon \colon H_1(X) \to \mathbb{Z}$, we consider the map

$$e = e_X \colon \pi_1(X) \to \mathrm{GL}_2(\mathbb{Z})$$
$$\gamma \mapsto \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}^{\varepsilon(\gamma)}.$$

Note that the e_X maps for $X = X_K, X_{\Delta'_j}, X_F, X_T$ are compatible, since inclusion $X_K \hookrightarrow X_*$ induces an isomorphism on first homology. The proof of Proposition 6.2 implies that

$$\begin{aligned}\ker\big(H_1^e(X_K) \to H_1^e(X_{T_1})\big) &\cong \ker\big(H_1^e(X_K) \to H_1^e(X_F)\big) \\ &\cong \ker\big(H_1^e(X_K) \to H_1^e(X_{T_2})\big).\end{aligned}$$

Proposition 6.2 also tells us that this kernel is generated by $\ker(H_1^e(X_K) \to H_1^e(X_{\Delta'_1}))$ along with some $2h$ elements $\{x_k\}_{k=1}^{2h} \subseteq H_1^e(X_K)$.

By the topologists' Shapiro lemma [DK, p. 100], there is a canonical identification $H_1^e(X) \cong H_1(X^2)$ for all X, and so

$$\begin{aligned}\ker\big(H_1(X_K^2) \to H_1(X_{T_1}^2)\big) &\cong \ker\big(H_1(X_K^2) \to H_1(X_F^2)\big)\\ &\cong \ker\big(H_1(X_K^2) \to H_1(X_{T_2}^2)\big)\end{aligned}$$

and this kernel is generated by $\ker(H_1(X_K^2) \to H_1(X_{\Delta_1'}^2))$ along with some $2h$ elements $\{x_k\}_{k=1}^{2h} \subseteq H_1(X_K^2)$.

Therefore, since every map $H_1(X_K^2) \to \mathbb{Z}_3$ extends over $H_1(X_{\Delta_1'}^2 \cup_{X_K^2} X_{\Delta_2'}^2)$ in our prescribed fashion, in order to ensure that $\overline{\chi \circ \mathfrak{f}}$ extends over $H_1(X_T^2)$ it is enough to have $(\overline{\chi \circ \mathfrak{f}})(x_k) = 0$ for all $k = 1, \ldots, 2h$. It follows from Equation (3) that $\mathrm{Hom}(H_1(\Sigma_2(K)), \mathbb{Z}_3) \cong \mathbb{Z}_3^N$. Using our assumption that $h \leq g$, we have

$$N - 2h = (4g) - 2h \geq (4g) - 2g \geq 2g.$$

A linear algebraic argument as in the proof of [KiL, Theorem 6.1] shows that if A is an abelian group with $\mathrm{Hom}(A, \mathbb{F}) \cong \mathbb{F}^N$ then, given any m elements $a_1, \ldots, a_m \in A$ there exists a character $\chi = (\chi_i)_{i=1}^N \in \mathrm{Hom}(A, \mathbb{F})$ such that $\chi(a_j) = 0$ for all $j = 1, \ldots, m$ and such that at least $N - m$ of the χ_i maps are nonzero. It therefore follows that there exists some $\chi = (\chi_i)_{i=1}^N$ such that $\chi \circ \mathfrak{f}$ vanishes on $\{x_1, \ldots, x_{2h}\}$ and at least $N - 2h \geq 2g$ of the χ_i are nonzero. This completes the proof of Claim 7.17.

Let $\chi = (\chi_i)_{i=1}^N$ be such a map. By reordering the summands, without loss of generality we may assume that $\chi_1, \ldots, \chi_m$ are nonzero for some $m \geq 2g$ and that $\chi_{m+1}, \ldots, \chi_N$ are zero. Let $\phi := \phi_{\chi \circ \mathfrak{f}}$ and let $\Phi\colon \pi_1(X_T) \to \mathbb{Z}_2 \ltimes \mathbb{Z}_3$ be the corresponding extension of ϕ over $\pi_1(X_T)$.

Observe that X_K is the union of N copies of $X_{R_\eta(J)}$, glued along $(N-1)$ copies of $S^1 \times I$, and that, for $j = 1, 2$, $X_{\Delta_j'}$ is the union of N copies of $X_{\Delta_{D_j}}$, glued along $(N-1)$ copies of $S^1 \times I \times I$, along with a single copy of X_{S_j} glued along $S^1 \times D^2$ away from all the other identifications. These decompositions are compatible.

Let ϕ_i denote the restriction of ϕ to the fundamental group of the ith copy of $X_{R_\eta(J)}$ and respectively let Φ_i denote the restriction of Φ to the ith copy of $\pi_1(X_{\Delta_{D_j}})$. Recall that there are some choices of basepoints and paths implicit here – see the note at the end of Construction 3.1. It is then straightforward to argue that our maps are related by the following commutative diagram, where unlabeled arrows are induced by inclusion and Φ_{χ_i} denotes the unique extension of ϕ_{χ_i} to $\pi_1(X_{\Delta_0})$:

$$\begin{array}{ccccc}
 & & \overset{\phi_i}{\frown} & & \\
\pi_1(X_{R_\eta(J)}) & \xrightarrow[f_*]{} & \pi_1(X_R) & \xrightarrow[\phi_{\chi_i}]{} & \mathbb{Z}_2 \ltimes \mathbb{Z}_3 \\
\downarrow & & \downarrow & & \downarrow = \\
\pi_1(X_{\Delta_{D_j}}) & \xrightarrow{g_*} & \pi_1(X_{\Delta_0}) & \xrightarrow{\Phi_{\chi_i}} & \mathbb{Z}_2 \ltimes \mathbb{Z}_3 .
\end{array}$$

$$\underset{\Phi_i}{\smile}$$

For $1 \leq i \leq m$, the map χ_i is nontrivial and so Proposition 7.8 implies that

$$H_1^{\phi_i}(X_{R_\eta(J)}) \cong H_1^{\phi_{\chi_i}}(X_R) \oplus \mathcal{A}_\xi(J)^{1\oplus\bar{1}}$$

and

$$H_1^{\Phi_i}(X_{\Delta_{D_j}}) \cong H_1^{\Phi_{\chi_i}}(X_{\Delta_0}) \oplus \mathcal{A}_\xi(D_j)^{1\oplus\bar{1}}$$

in such a way that $\ker(H_1^{\phi_i}(X_{R_\eta(J)}) \to H_1^{\Phi_i}(X_{\Delta_{D_j}}))$ is identified with

$$\ker\big(H_1^{\phi_{\chi_i}}(X_R) \to H_1^{\Phi_{\chi_i}}(X_{\Delta_0})\big) \oplus \ker\big(\mathcal{A}_\xi(J) \to \mathcal{A}_\xi(D_j)\big)^{1\oplus\bar{1}}.$$

Now consider a portion of the Mayer–Vietoris sequences in twisted homology for $X_K = \cup_{i=1}^N X_{R_\eta(J)}$ and $X_{\Delta_j} = \cup_{i=1}^N X_{\Delta_{D_j}}$ for $j = 1, 2$:

$$\begin{array}{ccccc}
\oplus_{i=1}^{N-1} H_1^{\phi_i}(S^1 \times I) & \xrightarrow{u} & \oplus_{i=1}^{N} H_1^{\phi_i}(X_{R_\eta(J)}) & \xrightarrow{v} & H_1^{\phi}(X_K) \\
\downarrow \mathrm{Id} & & \downarrow \oplus_{i=1}^n \iota_j^i & & \downarrow \iota_j \\
\oplus_{i=1}^{N-1} H_1^{\Phi_i}(S^1 \times I \times I) & \xrightarrow{U_j} & \oplus_{i=1}^{N} H_1^{\Phi_i}(X_{\Delta_{D_j}}) & \xrightarrow{V_j} & H_1^{\Phi}(X_{\Delta_j}).
\end{array}$$

In the above diagram, by a mild abuse of notation we refer to the restriction of ϕ_i to $\pi_1(S^1 \times I)$ as just ϕ_i, and similarly for $\Phi_i|_{\pi_1(S^1\times I\times I)}$.

We wish to show that $\ker(\iota_2)/(\ker(\iota_1) \cap \ker(\iota_2))$ has generating rank at least $2g$. In order to do this, we focus on a submodule Q of $\oplus_{i=1}^N H_1^{\phi_i}(X_{R_\eta(J)})$ and analyze how $v(Q)$ intersects $\ker(\iota_1)$ and $\ker(\iota_2)$.

Claim 7.18. The module $Q := \oplus_{i=1}^m \mathcal{A}_\xi(J)^{1\oplus\bar{1}} \subset \oplus_{i=1}^N H_1^{\phi_i}(X_{R_\eta(J)})$ is carried isomorphically by v to a subgroup of $H_1^{\phi}(X_K)$ such that for $q \in Q$ we have that $v(q) \in \ker(\iota_j)$ if and only if $q \in \ker\big(\oplus_{i=1}^N \iota_j^i\big)$.

First, use Proposition 6.2 to decompose

$$\bigoplus_{i=1}^{N} H_1^{\phi_i}(X_{R_\eta(J)}) \cong \bigoplus_{i=1}^{m} \big(H_1^{\phi_{\chi_i}}(X_R) \oplus \mathcal{A}_\xi(J)^{1\oplus\bar{1}}\big) \oplus \bigoplus_{i=m+1}^{N} H_1^{\phi_i}(X_{R_\eta(J)}).$$

We can then observe that since

$$(S^1 \times I)_i \subset (X_R)_i \cap (X_R)_{i+1} \subset (X_{R_\eta(J)})_i \cap (X_{R_\eta(J)})_{i+1}$$

we have

$$\ker(v) = \mathrm{Im}(u) \subseteq \bigoplus_{i=1}^{m} H_1^{\phi_{\chi_i}}(X_R) \oplus \bigoplus_{i=m+1}^{N} H_1^{\phi_i}(X_{R_\eta(J)}).$$

Similarly, we have that

$$\ker(V_j) = \mathrm{Im}(U_j) \subseteq \bigoplus_{i=1}^{m} H_1^{\Phi_{\chi_i}}(X_{\Delta_0}) \oplus \bigoplus_{i=m+1}^{N} H_1^{\Phi_i}(X_{\Delta_{D_j}}).$$

That is, $\ker(v)$ and $\ker(V_j)$ respectively intersect the $\mathcal{A}_\xi(J)^{1\oplus\bar{1}}$ and $\mathcal{A}_\xi(D_j)^{1\oplus\bar{1}}$ summands trivially.

In order to show that $\iota_j^i(x) = 0$ if and only if $\iota_j(v(x)) = 0$, suppose that x is an element of the ith copy of $\mathcal{A}_\xi(J)^{1\oplus\bar{1}}$ for some $1 \le i \le m$. One direction follows immediately from the commutativity of our diagram: if $\iota_j^i(x) = 0$, then $\iota_j(v(x)) = V_j(\iota_j^i(x)) = V_j(0) = 0$. So suppose now that $\iota_j(v(x)) = 0$. It follows that $\iota_j^i(x) \in \ker(V_j) = \mathrm{Im}(U_j)$, and so there exists $y \in \oplus_{i=1}^{n-1} H_1(S^1)$ such that $U_j(y) = \iota_j^i(x)$. Observe that $\iota_j^i(x - u(y)) = \iota_j^i(x) - U_j(y) = 0$, so $x - u(y) \in \ker(\iota_j^i)$. However, since

$$\iota_j^i(x) \in \bigoplus_{i=1}^{m} \mathcal{A}_\xi(D_j)^{1\oplus\bar{1}}$$

and

$$\iota_j^i\big(u(y)\big) = U_j(y) \in \mathrm{Im}(U_j) \subseteq \bigoplus_{i=1}^{m} H_1^{\Phi_{\chi_i}}(X_{\Delta_0}) \oplus \bigoplus_{i=m+1}^{N} H_1^{\Phi_i}(X_{\Delta_{D_j}})$$

we must have $\iota_j^i(x) = 0 = U_j(y)$, as desired. This completes the proof of Claim 7.18.

For $j = 1, 2$ we have by Claim 7.18 that

$$P_j := v(Q) \cap \ker(\iota_j) \cong Q \cap v^{-1}\big(\ker(\iota_j)\big) = Q \cap \bigoplus_{i=1}^{m} \ker(\iota_j^i). \tag{5}$$

We now argue that the subset $P_2/(\ker(\iota_1) \cap P_2)$ of $\ker(\iota_2)/(\ker(\iota_1) \cap \ker(\iota_2))$ has generating rank at least $2g$, noting that by Lemma 4.1 (2) this implies as desired that $\ker(\iota_2)/(\ker(\iota_1) \cap \ker(\iota_2))$ has generating rank at least $2g$.

By the splitting of the kernel from Proposition 7.8 we have that

$$(6) \qquad Q \cap \bigoplus_{i=1}^{m} \ker(\iota_j^i) = \bigoplus_{i=1}^{m} \mathcal{A}_\xi(J)^{1\oplus\bar{1}} \cap \bigoplus_{i=1}^{m} \ker(\iota_j^i) = \bigoplus_{i=1}^{m} \ker\left(\mathcal{A}_\xi(J)^{1\oplus\bar{1}} \to \mathcal{A}_\xi(D_j)^{1\oplus\bar{1}}\right).$$

From our computations of the maps $\mathcal{A}_\xi(J) \to \mathcal{A}_\xi(D_j)$ before the statement of Theorem 7.15, we also have

$$(7) \quad \ker\left(\mathcal{A}_\xi(J)^{1\oplus\bar{1}} \to \mathcal{A}_\xi(D_j)^{1\oplus\bar{1}}\right) = \begin{cases} \ker\left(\iota_0^\xi \colon \mathcal{A}_\xi(J_0) \to \mathcal{A}_\xi(D_0)\right)^{1\oplus\bar{1}} & j = 1 \\ \{(x,-x) \mid x \in \mathcal{A}_\xi(J_0)\} & j = 2. \end{cases}$$

Observe that by Claim 7.18 together with Equations (5) and (7) we have

$$\begin{aligned} P_2/\left(\ker(\iota_1) \cap P_2\right) &= P_2/\left(\ker(\iota_1) \cap v(Q) \cap \ker(\iota_2)\right) \\ &= P_2/\left(P_2 \cap P_1\right) \\ &\cong \bigoplus_{i=1}^{m} \{(x,-x) \mid x \in \mathcal{A}_\xi(J_0)\} / \bigoplus_{i=1}^{m} \{(x,-x) \mid x \in \ker(\iota_0^\xi)\} \\ &\cong \bigoplus_{i=1}^{m} \mathcal{A}_\xi(J_0)/\ker(\iota_0^\xi). \end{aligned}$$

Since $\mathcal{A}_\xi(J_0)/\ker(\iota_0^\xi)$ is nonzero, the classification theorem of finitely generated modules over commutative PIDs implies that the generating rank of $P_2/\left(\ker(\iota_1) \cap P_2\right)$ is $m \geq n = 2g$.

Now we finish the proof that $h \geq g$ by showing that the generating rank of $\ker(\iota_2)/(\ker(\iota_1) \cap \ker(\iota_2))$ is no more than $2h$. Let $P_F := \ker(H_1^\phi(X_K) \to H_1^\Phi(X_F))$. By Proposition 6.2 applied to Δ_1' and F, we have that P_F is generated as a $\mathbb{Z}[\xi]$-module by $\ker(H_1^\phi(X_K) \to H_1^\Phi(X_{\Delta_1'}))$ together with some $2h$ elements $x_1, \ldots, x_{2h}$. Here we use that the ring of Eisenstein integers $\mathbb{Z}[\xi]$ is a Euclidean domain and is therefore a PID. However, by Proposition 7.11 we have that

$$\ker\left(H_1^\phi(X_K) \to H_1^\Phi(X_{\Delta_1'})\right) = \ker\left(H_1^\phi(X_K) \to H_1^\Phi(X_{\Delta_1})\right) = \ker(\iota_1).$$

So for any submodule P of P_F, the quotient module $P/(P \cap \ker(\iota_1))$ is isomorphic to a submodule of $P_F/\ker(\iota_1)$ and hence, by Lemma 4.1 (2), has generating rank at most $2h$. But Proposition 6.2 applied to Δ_2' and F together with the fact that by Proposition 7.11

$$\ker\left(H_1^\phi(X_K) \to H_1^\Phi(X_{\Delta_2'})\right) = \ker\left(H_1^\phi(X_K) \to H_1^\Phi(X_{\Delta_2})\right) = \ker(\iota_2)$$

implies that $\ker(\iota_2)$ is contained in P_F. We can therefore conclude as desired that

$$2h \geq \text{g-rk}\Big(\ker(\iota_2)/\big(\ker(\iota_2) \cap \ker(\iota_1)\big)\Big) \geq 2g. \qquad \square$$

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