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Objekttyp: **Article**

Zeitschrift: **L'Enseignement Mathématique**

Band (Jahr): **64 (2018)**

Heft 3-4

PDF erstellt am: **01.05.2024**

Persistenter Link: https://doi.org/10.5169/seals-842104

L'Enseignement Mathématique (2) 64 (2018), 477–496 DOI 10.4171/LEM/64-3/4-12

Moser's shadow problem

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Abstract. Moser's shadow problem asks to estimate the shadow function $\mathfrak{s}_b(n)$, which is the largest number such that for each bounded convex polyhedron P with n vertices in 3-space there is some direction $\mathbf{v}$ (depending on P) such that, when illuminated by parallel light rays from infinity in direction $\mathbf{v}$, the polyhedron casts a shadow having at least $\mathfrak{s}_b(n)$ vertices. A general version of the problem allows unbounded polyhedra as well, and has associated shadow function $\mathfrak{s}_u(n)$. This paper presents correct order of magnitude asymptotic bounds on these functions. The bounded shadow problem has answer $\mathfrak{s}_b(n) = \Theta\big(\log(n)/(\log(\log(n))\big)$. The unbounded shadow problem is shown to have the different asymptotic growth rate $\mathfrak{s}_u(n) = \Theta\big(1\big)$. Results on the bounded shadow problem follow from 1989 work of Chazelle, Edelsbrunner and Guibas on the (bounded) silhouette span number $\mathfrak{s}_b^*(n)$, defined analogously but with arbitrary light sources. We complete the picture by showing that the unbounded silhouette span number $\mathfrak{s}_u^*(n)$ grows as $\Theta\big(\log(n)/(\log(\log(n))\big)$.

Mathematics Subject Classification (2010). Primary: 52B10; Secondary: 51N15, 65D18, 68U05, 90C05.

Keywords. Moser's shadow problem, silhouette span problem, 3-dimensional polytopes and polyhedra, shadows and silhouettes.

1. Introduction

This paper gives complete answers to several variants of a problem raised in 1966 by Leo Moser [Mo] in an influential list of problems in discrete and combinatorial geometry, later reprinted in 1991 [Mos]. Problem 35 of Moser's list is as follows.[1]

[1] We have changed the original notation f to $\mathfrak{s}$ in stating Problems 1 and 2.

Problem 1. Estimate the largest $\mathfrak{s} = \mathfrak{s}(n)$ such that every convex polyhedron of n vertices has an orthogonal projection onto the plane with $\mathfrak{s}(n)$ vertices on the 'outside'.

A nearly equivalent problem was formulated in a 1968 paper of G. C. Shephard [She2, Problem VIII].

Problem 2. Find a function $\mathfrak{s}(v)$ such that every convex polyhedron with v vertices possesses a projection which is an n-gon with $n \geq \mathfrak{s}(v)$.

This problem has been called *Moser's shadow problem* ([CEG, p. 140], [CFG, Problem B10]), because such projections can be viewed as the shadow of the polyhedron cast by parallel light rays coming from a light source "at infinity".

The problem can be formulated in two variants, depending on whether or not unbounded polyhedra are allowed. Shephard's version of the problem [She1, She2] definitely restricts to bounded polyhedra since he treats polyhedra that are the convex hull of a finite set of points. Moser's original problem statement does not explicitly indicate whether polyhedra are required to be bounded, though he probably had bounded polyhedra in mind. In any case the unbounded version of the problem is of interest because polyhedra defined as intersections of half-spaces naturally arise in linear programming, and certain linear programming algorithms have an interpretation in terms of shadows.

In this paper we consider both the bounded and unbounded case. To distinguish the bounded case from the general (unbounded) case we let $\mathfrak{s}_b(n)$ denote the minimal value over bounded polyhedra (i.e., 3-polytopes) having n vertices, and $\mathfrak{s}_u(n)$ denote the minimal value allowing unbounded polyhedra with n vertices as well (counting only bounded vertices). We call *Moser's shadow problem* the problem of determining the growth rate of $\mathfrak{s}_b(n)$. We also formulate in analogy *Moser's unbounded shadow problem*, which concerns the growth rate of $\mathfrak{s}_u(n)$.

A related problem, the *silhouette span problem*, was formulated by Chazelle, Edelsbrunner and Guibas in 1989 [CEG]. It is a variant of the shadow problem that allows more freedom in the location of the light source from which the shadow is cast. It considers shadows cast by point light sources at finite distance from the polytope. The corresponding *bounded silhouette span number* $\mathfrak{s}_b^*(n)$ is defined analogously as the shadow number, maximizing over all finite locations of the light source. It is also possible to define the *unbounded silhouette span number*, $\mathfrak{s}_u^*(n)$. Its formal definition is a little subtle, and is given in Section 2.

Chazelle, Edelsbrunner and Guibas [CEG, Theorem 4] determined the exact asymptotics of the bounded silhouette span function $\mathfrak{s}_b^*(n)$.

Theorem 1 (Chazelle–Edelsbrunner–Guibas). *The bounded n-vertex silhouette span number $\mathfrak{s}_b^*(n)$ for 3-dimensional convex polytopes satisfies*

$$\mathfrak{s}_b^*(n) = \Theta\left(\frac{\log(n)}{\log\left(\log(n)\right)}\right).$$

In this paper, our object is to determine the asymptotic growth rates of the other three functions $\mathfrak{s}_b(n)$, $\mathfrak{s}_u(n)$ and $\mathfrak{s}_u^*(n)$, as $n \to \infty$. In particular, the original Moser shadow problem corresponds to $\mathfrak{s}_b(n)$.

Our first result puts on record a complete solution to Moser's shadow problem in the bounded polyhedron case.

Theorem 2. *The bounded n-vertex shadow number $\mathfrak{s}_b(n)$ for 3-dimensional convex polytopes satisfies*

$$\mathfrak{s}_b(n) = \Theta\left(\frac{\log(n)}{\log\left(\log(n)\right)}\right).$$

As we shall explain below, this result should be attributed to Chazelle, Edelsbrunner and Guibas, in the sense that all the ingredients for a proof are present in their 1989 paper [CEG]. However, although they mentioned the shadow problem, they did not point out that their results implied a solution. In Section 3 we provide the missing steps for the proof of Theorem 2.

The remainder of the paper is devoted to the unbounded polyhedron versions of the shadow and silhouette span problems. In Section 4 we prove that the unbounded shadow function $\mathfrak{s}_u(n)$ is eventually constant.

Theorem 3. *The unbounded n-vertex shadow number $\mathfrak{s}_u(n)$ for 3-dimensional convex polyhedra satisfies*

$$\mathfrak{s}_u(n) = \Theta(1).$$

In fact $\mathfrak{s}_u(n) = 3$ for all $n \geq 3$ (and $\mathfrak{s}_u(1) = 1$ and $\mathfrak{s}_u(2) = 2$).

Finally, in Section 5 we treat the unbounded version of the silhouette span problem. There is a subtlety in generalizing the definition of silhouette span to unbounded polyhedra. Certain edges visible in the shadow may not correspond to an edge of the unbounded polyhedron itself. Our definition, which in the bounded polyhedron case is equivalent to that used in [CEG, Sect. 5.3], allows as potentially visible edges corresponding to the recession directions of the unbounded polyhedron. See Section 2. We obtain the following result, which shows the order of magnitude of the silhouette span number does not decrease when one allows unbounded polyhedra.

Theorem 4. *The unbounded n-vertex silhouette span number $\mathfrak{s}_u^*(n)$ for 3-dimensional convex polyhedra satisfies*

$$\mathfrak{s}_u^*(n) = \Theta\left(\frac{\log(n)}{\log\big(\log(n)\big)}\right).$$

This result is proved by reduction to the bounded silhouette span case. Notice that our results show that the shadow and silhouette span problems have different growth rates in the unbounded case (in contrast with the bounded case, where both coincide).

1.1. Related work. After Moser's original formulation in 1966, the problem was restated several times [CFG, Mo, Mos, She2]. The problem book of Croft, Falconer and Guy [CFG, Problem B10] reports that Moser conjectured $\mathfrak{s}_b(n) = \mathcal{O}\big(\log(n)\big)$ and it sketches the construction of a polytope whose shadow number is of this order of magnitude. Shephard [She2, Problem VIII] did not conjecture a value for $\mathfrak{s}_b(n)$. However, in the dual formulation terms of sections [She2, Problem VI], he proposed a lower bound for the silhouette span problem of the form n^α for some constant $0 < \alpha < 1$.

The 1989 paper of Chazelle, Edelsbrunner and Guibas [CEG] treated a diverse set of problems concerning the combinatorial and computational complexity of diverse stabbing problems in dimensions two and three, among which the silhouette span problem. Their approach to the silhouette span problem (in the bounded case) exploited the polarity operation, and was shown to be equivalent to the *cross-section span problem*: finding the maximal number of facets of the polar polytope which can be intersected with a plane. This problem is actually another of the problems in Shephard's list [She2, Problem VI]. Both problems are solved and shown to be of order $\Theta\big(\log(n)/\log\big(\log(n)\big)\big)$.

The fact that $\mathfrak{s}_b^*(n) \geq \mathfrak{s}_b(n)$, yielding an upper bound for $\mathfrak{s}_b(n)$, was noted in [CEG, pp. 174–175]. As we remarked above, [CEG] also contains ingredients sufficing to prove a lower bound for $\mathfrak{s}_b(n)$. Indeed, under polarity the shadow problem can be seen to correspond to maximizing the number of facets that can be intersected with a plane that goes through the origin. Although in [CEG] the authors only claim results for the silhouette span problem and the cross-section span problem, their lower bound proof for cross-section span only uses planes through the origin [CEG, Lemma 5.1], and hence is also valid for Moser's shadow problem. Thus Theorem 2 follows from the results in [CEG]. However, the relevant bound in Lemma 5.1 is stated for an unnamed function $c_d^*(n)$ and their paper did not remark on its consequences for the shadow problem, which has been considered open until now.

Glisse et al. [GLMP] studied the expected shadow number of a random 3-polytope obtained by a Poisson point process on the sphere and showed it to be of order $\Theta(\sqrt{n})$.

1.2. Higher-dimensional generalized shadow problems. Shadow problems can be generalized to higher dimensions by considering k-dimensional shadows/silhouettes of d-dimensional polytopes.

The special higher-dimensional case of 2-dimensional projections of d-dimensional polyhedra has been studied in connection with linear programming algorithms. The *shadow vertex simplex algorithm* is a parametric version of the simplex algorithm in linear programming introduced by Gass and Saaty [GS] in 1955. The analysis of this algorithm leads to the study of 2-dimensional shadows of d-dimensional polyhedra. A variant of the algorithm was studied in detail by Borgwardt [Bor1, Bor2, Bor3, Bor4]. Later Spielman and Teng [ST] and Kelman and Spielman [KS] studied the shadow vertex simplex algorithm in connection with average-case analysis of linear programming problems.

Several different types of higher-dimensional shadow problems can be considered:

(1) *Worst case problems* concern the problem of *maximizing* shadow numbers for a fixed number of vertices. The worst case behavior of the shadow vertex method is related to polyhedra having large shadows, For dimension $d = 3$ it is easily seen that for all $n \geq 4$ there are polyhedra having all vertices visible in a shadow: one may take a suitable oblique cone over a base that is an $(n-1)$-gon. Amenta and Ziegler [AZ] and Gärtner, Helbling, Ota and Takahashi [GHOT] (see also [GJM]) present constructions of bad examples of 2-dimensional shadows in all higher dimensions d.

(2) *Average case problems* concern the average size of k-dimensional shadows taken with respect to some measure on the set of directions. Such problems for 2-dimensional shadows arose from the average case analysis of the shadow vertex algorithm. In the 1980's Borgwardt [Bor1, Bor2, Bor3, Bor4] developed a polynomial time average case analysis of the variant of the simplex method for linear programming that uses the shadow vertex pivot rule. The shadow vertex simplex algorithm later provided the fundamental example used in Spielman and Teng's [ST] theory of smoothed analysis of algorithms. Their analysis requires obtaining some control on the (average) size of shadows, as a function of the numbers of variables and constraints in the linear program. Further developments of smoothed analysis are given in Deshpande and Spielman [DS], Kelner and Spielman [KS], Vershynin [Var], and Dadush and Huiberts [DH].

(3) *Minimax case problems* for 2-dimensional shadows in dimensions $d \geq 4$ generalize the shadow problem treated in this paper. Tóth [Tot] has studied line stabbing numbers of convex subdivisions in all dimensions, extending the analysis of Chazelle et al. [CEG]. His lower bounds induce lower bounds for 2-dimensional shadow numbers of d-polyhedra, however his examples for upper bounds are not face-to-face, and hence do not arise from convex polytopes.

The general minimax problem for k-dimensional shadows is:

Problem 3. Estimate the growth rate of the maximal number $\mathfrak{s}_b(n,d,k)$ (resp. $\mathfrak{s}_b^*(n,d,k)$) such that every d-polytope with n vertices has a k-dimensional shadow (resp. silhouette) with $\mathfrak{s}_b(n,d,k)$ (resp. $\mathfrak{s}_b^*(n,d,k)$) vertices. Do the same for maximizing over all d-polyhedra $\mathfrak{s}_u(n,d,k)$ (resp. $\mathfrak{s}_u^*(n,d,k)$.)

To our knowledge all these minimax problems are open in dimensions $d \geq 4$; and so are the analogue silhouette span questions.

2. Definitions

We follow the terminology for convex polytopes in Ziegler [Zie, pp. 4–5], and define a *polyhedron* in $\mathbb{R}^d$ to be a finite intersection of closed half-spaces, which may be unbounded, and a *polytope* in $\mathbb{R}^d$ to be the convex hull of a finite set of points; that is, a bounded polyhedron. Faces of dimensions 0, 1 and $d-1$ of a d-dimensional polyhedron are called *vertices*, *edges*, and *facets*, respectively. We say that a polyhedron is *pointed* if it does not contain a full line. This paper exclusively considers the 3-dimensional case $\mathbb{R}^3$.

A *shadow* of a (possibly unbounded) polyhedron P in $\mathbb{R}^3$ is the image of P under an affine projection $\pi_V : \mathbb{R}^3 \to V$ onto a two-dimensional affine flat V. The *shadow number* $\mathfrak{s}(P)$ of P is the maximum number of vertices on the boundary of one of its shadows. In this definition we may restrict π_V to be orthogonal projections onto a linear subspace V perpendicular to a given unit vector $\mathbf{v} \in \mathbb{S}^2$, which we define to be the *shadow* in direction $\mathbf{v}$. Alternatively, the shadow number $\mathfrak{s}(P)$ of P can also be interpreted as the maximal number of 1-dimensional faces of the "cylinder" resulting from the Minkowski sum $P + \mathbb{R}\mathbf{v}$, varying over all directions $\mathbf{v}$.

The *n-vertex bounded shadow number* $\mathfrak{s}_b(n)$ and the *n-vertex unbounded shadow number* $\mathfrak{s}_u(n)$ are given by

$$\mathfrak{s}_b(n) := \min\{\mathfrak{s}(P) : \ P \text{ is a bounded 3-polyhedron with } n \text{ vertices}\},$$
$$\mathfrak{s}_u(n) := \min\{\mathfrak{s}(P) : \ P \text{ is a 3-polyhedron with } n \text{ (bounded) vertices}\}.$$

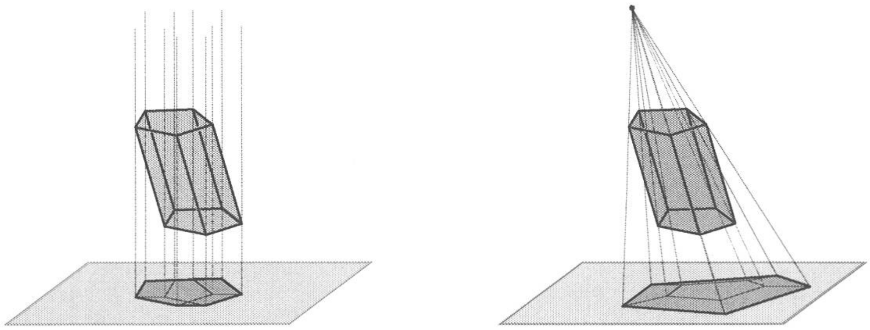

FIGURE 2.1
A shadow and a silhouette of a polytope

The definition of silhouette span of a bounded polyhedron P given in [CEG, Section 5.3, p. 174], is an intrinsic definition as a subset of the boundary of P. Here we use an alternative definition, equivalent as far as the bounded silhouette span is concerned, that parallels the "cylinder" definition of shadow numbers and is better suited for unbounded polyhedra.

For a (possibly unbounded) polyhedron $P \subset \mathbb{R}^3$ and a point $p \in \mathbb{R}^3$ outside P, let

$$C_p(P) = \overline{\{p + \lambda \mathbf{v} \ : \ \mathbf{v} \in P - p, \lambda \geq 0\}}$$

be the closure of the cone with apex p spanned by P. A *silhouette* of P with respect to p is a section of $C_p(P)$ with a transversal plane (for example, a plane separating p from P). The *size* of a silhouette is its number of vertices (in bijection with the rays of the cone), and the *silhouette span* $\mathfrak{s}^*(P)$ is the size of the largest silhouette of P.

In [CEG], Chazelle et al. define the silhouette of a bounded polytope P with respect to a point p outside P as the collection of faces F of P that allow a supporting plane H of P such that p lies in H and F is in the relative interior of $P \cap H$; and measure its size as its number of vertices. To avoid confusion, we may call this the *pre-silhouette* of P with respect to p (such complexes are sometimes referred to as the *shadow-boundary* of P from p, see for example [She3]). When p is not coplanar with any facet of P, the pre-silhouette is a collection of edges and vertices in the boundary of P (but otherwise it might also contains facets). In this case, central projection from p maps the pre-silhouette bijectively to the boundary of the silhouette. Note that if there is a 2-dimensional face in the pre-silhouette, at most two of its vertices can be in the pre-silhouette (by the relative interior condition), and these remain in the pre-silhouette even after a small perturbation of the point p. Hence, silhouettes of maximal size can always be attained from points p that do not lie in any plane

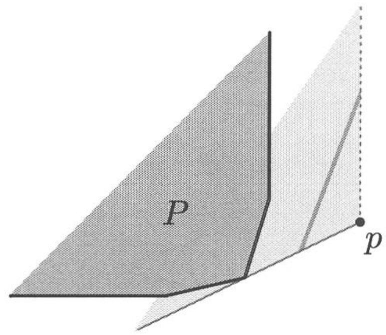

FIGURE 2.2
A 2-dimensional unbounded polyhedron P as seen from a point p. The cone with apex p spanned by P is not closed, one boundary edge (dotted lines) is missing. Any transversal section of the closure of this cone gives a silhouette, one example is the highlighted segment.

supporting a facet of P, and both definitions give exactly the same silhouette spans.

However, this definition of pre-silhouettes is not well adapted for unbounded polyhedra. If P is unbounded, we wish to consider also as part of the silhouette those faces of the recession cone that are visible from p at infinity. Indeed, silhouettes can be interpreted by projecting onto a canvas that separates P from a viewer placed at p. Unbounded facets are seen as half-open polytopes, in which part of the boundary may be missing, as it corresponds to limit directions at infinity. An example with missing boundary is sketched in Figure 2.2 for the planar case.

Our definition includes this extra boundary (this is why the closure is needed in the definition of $C_p(P)$). In order to reformulate the definition of pre-silhouettes to this set-up, one should consider also some extra unbounded edges of P in the directions of the recession cone. To each such unbounded edge it adds a "vertex at infinity". The *silhouette size* of P viewed from p would now count the additional "vertices at infinity" included this way.

We now define the bounded silhouette span function as a min-max quantity. The *n-vertex bounded silhouette span number* $\mathfrak{s}_b^*(n)$ and the *n-vertex unbounded silhouette span number* $\mathfrak{s}_u^*(n)$ are given by

$$\mathfrak{s}_b^*(n) := \min\{\mathfrak{s}^*(P) : \ P \text{ is a bounded 3-polyhedron with } n \text{ vertices}\},$$
$$\mathfrak{s}_u^*(n) := \min\{\mathfrak{s}^*(P) : \ P \text{ is a is a 3-polyhedron with } n \text{ (bounded) vertices}\}.$$

These four functions satisfy the following inequalities,

$$\begin{array}{ccc} \mathfrak{s}_b^*(n) & \geq & \mathfrak{s}_u^*(n) \\ \text{IV} & & \text{IV} \\ \mathfrak{s}_b(n) & \geq & \mathfrak{s}_u(n). \end{array}$$

The two horizontal inequalities hold because the unbounded numbers minimize over a larger set than the bounded numbers, for both the shadow problem and the silhouette span problem. The vertical inequality between silhouette span numbers and shadow numbers holds because silhouettes from light sources that are sufficiently far away in the direction of a parallel projection have at least as many vertices as shadows obtained by that parallel projection (see [CEG, pp. 174–175] for the bounded case; a similar argument holds for unbounded shadows and silhouettes).

3. Moser's (bounded) shadow problem

As discussed in Section 2, the shadow number is bounded from above by the silhouette span number, and hence the upper bound $\mathfrak{s}_b(n) \leq \mathfrak{s}_b^*(n) = \mathcal{O}\big(\log(n)/\log\big(\log(n)\big)\big)$ follows from the upper bound for the silhouette span problem implied by Theorem 1 (originally from [CEG, Lemma 5.15]):

Corollary 3.1. *The bounded n-vertex shadow number $\mathfrak{s}_b(n)$ for 3-dimensional polytopes satisfies*

$$\mathfrak{s}_b(n) = \mathcal{O}\left(\frac{\log(n)}{\log\big(\log(n)\big)}\right).$$

For the proof in [CEG, Lemma 5.15], Chazelle et al. construct a polytope with n vertices whose silhouette from each point of view has size at most $\mathcal{O}\big(\log(n)/\log\big(\log(n)\big)\big)$. Since shadows can be regarded as a special kind of silhouettes, this is also an upper bound for the shadow number. However the construction in [CEG, Section 5.2] is very involved, requiring some quite technical steps. Constructing upper bound examples for the shadow number problem is actually simpler. For completeness, in Appendix A we present an alternative direct construction that establishes this upper bound.

To prove Theorem 2, it suffices hence to provide a matching lower bound for the shadow number. To this end, we use a lower bound result for *minimal line span* proved by Chazelle et al. [CEG, Lemma 3.2]. Although they used it to prove a lower bound for the silhouette span number, it actually serves to prove a stronger result, a lower bound for the shadow number.

Proposition 3.2. *The bounded n-vertex shadow number $\mathfrak{s}_b(n)$ for 3-dimensional polytopes satisfies*

$$\mathfrak{s}_b(n) = \Omega\left(\frac{\log(n)}{\log\left(\log(n)\right)}\right).$$

Proof. Let P be a bounded polytope in $\mathbb{R}^3$ with n vertices. The intersection of the normal fan of P with the unit 2-sphere $\mathbb{S}^2$ is a spherical polyhedral subdivision $\mathcal{D}$ of $\mathbb{S}^d$ into n regions (see [Zie, Section 7]).

Now, consider the central (gnomonic) projection $\gamma : \mathbb{S}^2_- \longrightarrow H$ that maps the open lower hemisphere $\mathbb{S}^2_- := \mathbb{S}^2 \cap \{(x_1, x_2, x_3) : x_3 < 0\}$ bijectively to the plane $H := \{(x_1, x_2, x_3) : x_3 = -1\}$, by mapping $\mathbf{v} \in \mathbb{S}^2_-$ to the unique intersection point of the line through 0 and $\mathbf{v}$ with H.

By rotating P if needed, we may assume that the lower hemisphere $\mathbb{S}^2_-$ intersects at least $\lceil n/2 \rceil$ regions of $\mathcal{D}$. In this case, the central projection of the lower hemisphere induces a polyhedral subdivision of the plane into has at least $\lceil n/2 \rceil$ regions. By [CEG, Lemma 3.2], there is a line ℓ that stabs $\Omega\left(\log(n)/\log\left(\log(n)\right)\right)$ cells of this subdivision. Let $H_{\mathbf{v}}$ be the linear plane that contains its preimage $\gamma^{-1}(\ell)$. By construction, $H_{\mathbf{v}}$ intersects the interior of $\Omega\left(\log(n)/\log\left(\log(n)\right)\right)$ cells of $\mathcal{D}$.

Let $\mathbf{v}$ be a normal vector to $H_{\mathbf{v}}$, and $\pi_{\mathbf{v}}$ the orthogonal projection along $\mathbf{v}$. Then regions of $\mathcal{D}$ whose interior is intersected by $H_{\mathbf{v}}$ give rise to vertices of $\pi_{\mathbf{v}}(P)$, the shadow of P in direction $\mathbf{v}$. This follows essentially from [Zie, Lemma 7.11], which shows that the normal fan of $\pi_{\mathbf{v}}(P)$ coincides with the restriction of the normal fan of P to $H_{\mathbf{v}}$.

Thus P has a shadow with at least $\Omega\left(\log(n)/\log\left(\log(n)\right)\right)$ vertices. Since this can be done for each bounded 3-polytope with n vertices, we conclude

$$\mathfrak{s}_b(n) = \min_P \mathfrak{s}(P) = \Omega\left(\frac{\log(n)}{\log\left(\log(n)\right)}\right). \qquad \square$$

4. Moser's unbounded shadow problem

In this section, we will determine the shadow number for unbounded polyhedra. There are two results. In Proposition 4.1 we give a lower bound showing $\mathfrak{s}(P_n) \geq 3$ for $n \geq 3$. In Proposition 4.2 we will construct a sequence of unbounded polyhedra P_n, for all $n \geq 4$, having n vertices and n faces and whose shadow number is $\mathfrak{s}(P_n) = 3$, giving an upper bound for $\mathfrak{s}_u(n)$. [2] Both results together establish Theorem 3.

[2] We are grateful to an anonymous reviewer who suggested this example to improve our original $\mathcal{O}(1)$ upper bound from 5 to 3.

4.1. Unbounded shadow problem: Lower bound. The following proposition gives a lower bound for the unbounded shadow number function.

Proposition 4.1. *The unbounded n-vertex shadow number $\mathfrak{s}_u(n)$ for 3-dimensional convex polyhedra satisfies*

$$\mathfrak{s}_u(n) \geq 3$$

for all $n \geq 3$ (and $\mathfrak{s}_u(1) = 1$ and $\mathfrak{s}_u(2) = 2$).

Proof. Let P be an unbounded polyhedron with at least 3 vertices. Let p_1 and p_2 be two vertices of P connected by an edge e, and p_3 a third vertex. We consider a vector $\mathbf{u}$ normal to a supporting plane for e, and a vector $\mathbf{w}$ normal to a supporting plane for p_3 that is not orthogonal to e. Finally, we take a vector $\mathbf{v}$ orthogonal to $\mathbf{u}$ and $\mathbf{w}$.

Recall that a face F of P is preserved under the orthogonal projection $\pi_{\mathbf{v}}$ along the vector $\mathbf{v}$ if one of the supporting planes for F has an outer normal vector orthogonal to $\mathbf{v}$ (by preserved we mean that $\pi_{\mathbf{v}}(F)$ is a face of $\pi_{\mathbf{v}}(P)$).

Hence, e is preserved, and therefore so are p_1 and p_2. Moreover the images of p_1 and p_2 under $\pi_{\mathbf{v}}$ are different since $\mathbf{v}$ is not parallel to e. Moreover, p_3 is also preserved by construction. We conclude that the orthogonal projection $\pi_{\mathbf{v}}(P)$ has at least three vertices.

For $n = 1$ and $n = 2$, the proof is straightforward. □

4.2. Unbounded shadow problem: Upper bound. The following construction gives an upper bound for unbounded shadow number function.

Proposition 4.2. *The unbounded n-vertex shadow number $\mathfrak{s}_u(n)$ for 3-dimensional convex polyhedra satisfies*

$$\mathfrak{s}_u(n) \leq 3$$

for all n.

Proof. It suffices to show, for each $n \geq 4$, that there is an unbounded pointed convex polyhedron with n vertices P_n whose shadow number $\mathfrak{s}(P_n)$ is 3. For $n \geq 4$, consider the convex polyhedral cone

$$Q_n := \left\{x \in \mathbb{R}^3 \ : \ \langle x, w_k \rangle \leq 0, \ \text{ for } 0 \leq k \leq n-1\right\},$$

where $w_k := \left(\cos\left(\frac{2\pi k}{n-1}\right), \sin\left(\frac{2\pi k}{n-1}\right), -1\right)$ and $\langle \cdot, \cdot \rangle$ denotes the standard scalar product. This is a cone over a regular $(n-1)$-gon. It has a single vertex at the origin and $n-1$ (unbounded) facets. Now we stack a vertex on top of each of these $n-1$ facets. That is, for each facet we add a point that is slightly beyond

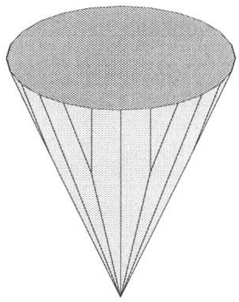

FIGURE 4.1
An instance of P_n, for $n = 9$

it and beneath the planes defining the remaining facets, and take the convex hull. We obtain an unbounded polyhedron with n vertices (the origin plus $n-1$ stacking points), and $3(n-1)$ (unbounded) facets (see Figure 4.1). One explicit realization is the following polyhedron P_n:

$$P_n := \left\{x \in \mathbb{R}^3 \ : \langle x, w_k\rangle \leq 1 \text{ and } \langle x, \tfrac{2}{3}w_k + \tfrac{1}{3}w_{k\pm1}\rangle \leq 0; \text{ for } 0 \leq k \leq n-1\right\}.$$

The shadow number of P_n is at most 3. Indeed, let $\pi : \mathbb{R}^3 \to \mathbb{R}^2$ be a linear projection. If $\pi(Q_n)$ does not cover the whole plane then it is a two-dimensional cone pointed at the origin and bounded by the image of two of the rays of Q_n, which are also rays of P_n. Besides the origin, only the vertices of P_n stacked to facets incident to these rays can appear as vertices of the shadow $\pi(P_n)$. Moreover, for each of the two sides, only one of the two neighboring stacked vertices can be visible: They cannot lie both outside the shadow of Q_n, as the segment between both intersects the interior of Q_n (because each ray of Q_n is preserved by the stacking operation). $\square$

5. The unbounded silhouette span problem

In this final section, we consider silhouettes of possibly unbounded polyhedra, and determine the asymptotics of the unbounded silhouette span function $\mathfrak{s}_u^*(n)$. We will show that the asymptotic growth rate of $\mathfrak{s}(n)$ and $\mathfrak{s}_b^*(n)$ are of the same order by reducing the unbounded case to the bounded case using a projective transformation.

As before, the proof of Theorem 4 will be split in two parts, by providing matching upper and lower bounds. The upper bound follows from the trivial inequality $\mathfrak{s}_u^*(n) \leq \mathfrak{s}_b^*(n)$ and Corollary 3.1:

Corollary 5.1. *The unbounded n-vertex silhouette span number $\mathfrak{s}_u^*(n)$ for 3-dimensional convex polyhedra satisfies*

$$\mathfrak{s}_u^*(n) = \mathcal{O}\left(\frac{\log(n)}{\log\big(\log(n)\big)}\right).$$

Proposition 5.2. *The unbounded n-vertex silhouette span number $\mathfrak{s}_u^*(n)$ for 3-dimensional convex polyhedra satisfies*

$$\mathfrak{s}_u^*(n) = \Omega\left(\frac{\log(n)}{\log\big(\log(n)\big)}\right).$$

Proof. This lower bound holds for polytopes by Proposition 3.2, so we concentrate on unbounded polyhedra.

Let P be an unbounded polyhedron with $n > 0$ vertices (which is therefore pointed). After a suitable rotation, we might assume that all its facets have a normal vector with a negative third coordinate, and hence for $M \in \mathbb{R}$ large enough the plane H_{-M}, defined by $H_{-M} = \{(x, y, z) \ : \ z = -M\}$, avoids P. We will take some very large $M \gg 0$ with $H_{-M} \cap P = \varnothing$ and consider the projective transformation

$$\begin{aligned} \phi \ : \ \mathbb{R}^3 \smallsetminus H_{-M} &\to \mathbb{R}^3 \smallsetminus H_M \\ (x, y, z) &\mapsto \frac{(x, y, z)}{1 + \frac{z}{M}}. \end{aligned}$$

sending H_{-M} to infinity (see [Zie, Appendix 2.6] for a brief introduction to projective transformations in the context of polyhedra).

It maps bijectively $\mathbb{R}^3 \smallsetminus H_{-M}$ to $\mathbb{R}^3 \smallsetminus H_M$. The closure of the image of the polyhedron P is the (bounded) polytope Q bounded by the inequalities inherited from P via ϕ together with the new inequality $z \le M$, which supports a face F of Q. F is the image of the "face at infinity" of P, and $\phi(P) = Q \setminus F$.

By Proposition 3.2 we can find a direction $\mathbf{v} = (v_1, v_2, v_3)$ such that the shadow of Q in direction $\mathbf{v}$ has at least $\Omega\big(\log(n)/\log\big(\log(n)\big)\big)$ vertices. Since small perturbations do not decrease the shadow number, we can assume that $v_3 \neq 0$.

When $v_3 \neq 0$, lines in direction $\mathbf{v}$ are mapped by ϕ^{-1} to lines through the point $p = (\frac{-Mv_1}{v_3}, \frac{-Mv_2}{v_3}, -M)$. Consequently, lines in direction $\mathbf{v}$ through a point of $\phi(P) = Q \setminus F$ are mapped by ϕ^{-1} bijectively to lines through p and a point in P. In fact, $\phi(C_p(P) \smallsetminus p)$ is easily seen to be the one-sided cylinder $(Q + \mathbf{v}\mathbb{R}) \cap \{z \le M\}$. Hence, the shadow of Q in direction $\mathbf{v}$ has the same number of vertices as the silhouettes of P from p. This can be seen explicitly by noting than the image of the silhouette $C_p(P) \cap H_{-\frac{M}{2}}$ under the projective transformation ϕ is the polygon $(Q + \mathbf{v}\mathbb{R}) \cap H_{-M}$, together with the fact that (admissible) projective transformations do not change the combinatorial type.

Thus, the silhouette span of P is $\Omega\left(\frac{\log(n)}{\log\left(\log(n)\right)}\right)$. $\square$

A. On the upper bound for Moser's bounded shadow problem

This appendix is devoted to an alternative direct proof of Corollary 3.1, much simpler than the one in [CEG, Section 5.2], but that applies only to the shadow problem and not to the silhouette span problem. The proof will be based on a construction that will be given in terms of polygonal subdivisions. A *polyhedral subdivision* $\mathcal{E}$ of $\mathbb{R}^d$ is a finite set of d-polyhedra (called *regions*), whose union is $\mathbb{R}^d$ and such that the intersection of any two is a common face.

An important point in the proof is to be able to certify that the subdivisions we use arise from a 3-dimensional polytope, which is the polytope we seek to construct, in a way that reverses the procedure used in the proof of Proposition 3.2.

Definition 5. We say that a polyhedral subdivision $\mathcal{E}$ with n regions is *liftable* if it can be obtained from a polytope P with n vertices, by intersecting the normal fan of P with the unit 2-sphere $\mathbb{S}^2$ and centrally projecting the open lower hemisphere to the plane.

We will repeatedly use three operations. The first pair are classical, based on Steinitz's $\Delta - Y$ operations, and correspond to the polytope operations of stacking and truncating; the third is a combination of both these operations.

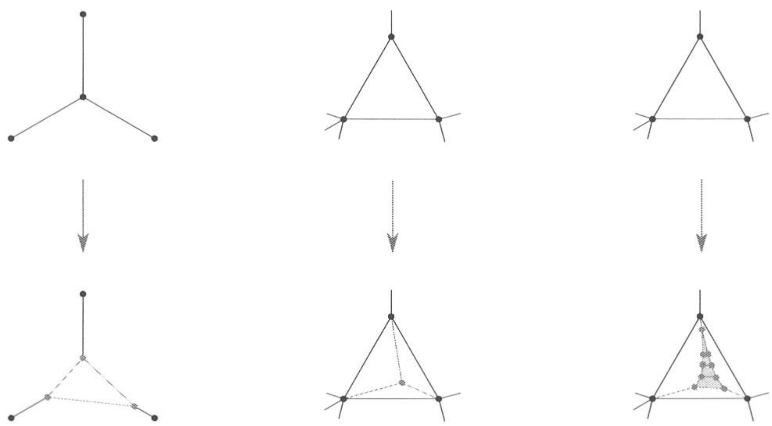

FIGURE A.1
Examples of truncating, stacking and unzipping. The shadowed regions form the spine of the unzipping, which is of length 4.

Definition 6. Let $\mathcal{E}$ be polyhedral subdivision of $\mathbb{R}^2$.

(1) Let v be a degree-3 vertex with neighbors v_1, v_2, v_3. *Truncating* v consists in choosing a point v_i' in the interior of each of the edges (v, v_i) and adding to $\mathcal{E}$ the triangle with vertices v_1', v_2', v_3' (and intersecting the remaining regions with the closure of its complement).

(2) Let T be a triangular region with vertices v_1, v_2, v_3. *Stacking* onto T corresponds to adding a vertex v in the interior of T and substituting T by the three triangles obtained by joining v with an edge of T.

(3) Let T be a triangular region with vertices v_1, v_2, v_3. *Unzipping* T towards v_i is an operation that consists in first stacking onto T and then successively truncating the newly created vertex that is connected to v_i. Its *length* is the number of truncations, and the regions created with the truncations are the *spine*.

See Figure A.1 for an example.

Lemma A.1. *Truncating and stacking, and hence also unzipping, preserve liftability.*

Proof. This is well known and we omit its proof, see [Zie, Section 4.2]. □

The whole construction will consist in successively applying these operations in such a way that at each iteration the new cells are so small that their intersection pattern with lines can be controlled.

We call a set of planar points in *general position* if no three are collinear.

Lemma A.2. *Let S be a subset of the vertices of a subdivision of $\mathbb{R}^2$ that are in general position. Then the vertices of S can be truncated in such a way that no line intersects three of the newly created regions.*

Proof. From the general position assumption there is some $\delta > 0$ such that any line through two points in S stays at distance at least δ from any third point. Hence, there exists an $\varepsilon > 0$ such that any line that goes through two points, each at distance at most ε from a different point of S, stays at distance at least ε from the remaining points of S. The claim follows from the fact that the truncation regions can be arbitrarily small around the truncated points. □

Lemma A.3. *Let T be a triangular region of a subdivision, ℓ a line through one of its vertices v that intersects the interior of T, and $\varepsilon > 0$ a real. Then T can be unzipped towards v in such a way that for every line ℓ' that intersects at least three regions of the spine, the angle between ℓ and ℓ' is at most ε.*

This can be done even when one forces the new vertices to be in general position with respect to a given point configuration.

Proof. Start by stacking with a point v' on ℓ. Notice that the truncations can be made with very thin triangles, in such a way that the spine is sufficiently close to the edge (v, v') in Hausdorff distance. If the pieces have a long enough diameter with respect to the distance of the spine to the edge, then any vector whose endpoints belong two non-consecutive pieces of the spine will form a very small angle with (v, v'). In particular, the line spanned by these points can be forced to be arbitrarily close to the line ℓ.

The last claim follows from the freedom in the choice of the truncation points (the starting line ℓ might have to be perturbed before starting if the configuration has points on it). □

We are ready for the proof of Corollary 3.1.

Proof of Corollary 3.1. We will start by constructing a polyhedral subdivision of the plane with n regions such that no line can intersect more than $\mathcal{O}\left(\log(n)/\log\left(\log(n)\right)\right)$ of them. A sketch of the construction is depicted in Figure A.2.

The starting point of the construction is a regular simplex, inscribed on the unit sphere with one vertex at the south pole $(0, 0, -1)$. We consider the subdivision $\mathcal{E}_0$ obtained by centrally projecting the lower hemisphere of the intersection of its normal fan with $\mathbb{S}^2$. It consists of a bounded triangular region T_0 and three

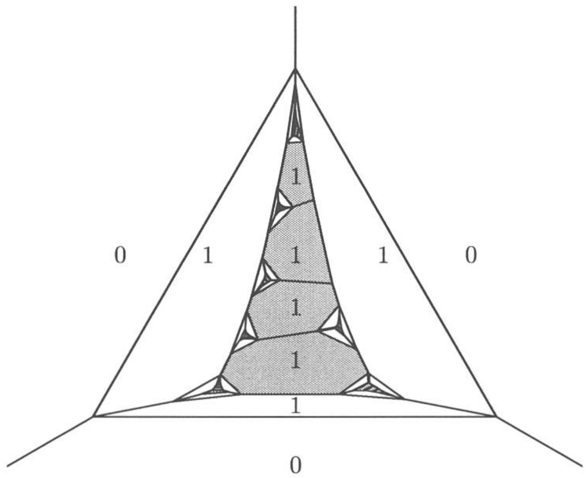

FIGURE A.2
A schema of the construction in the proof of Corollary 3.1, with $\ell = 7$ and $k = 2$. Numbers indicate the level of the regions (unnumbered regions are at level 2), and spine regions are shadowed.

unbounded regions. We say that these 4 regions are at level 0. Note that $\mathcal{E}_0$ is a liftable subdivision.

The triangle T_0 will be unzipped at length $t-3$, for some $t \geq 5$ that will be defined later, in such a way that all the points are in general position. Then we will truncate t of the $2t-5$ newly created vertices on the spine, in such a way that that no line intersects three of the newly created regions, using Lemma A.2. The new regions are at level 1 and T_0 is their predecessor.

For i from 1 to k (k will also be defined later), we will repeat this operation on all the triangles at level i (there are t of them for each triangle at level $i-1$). This is done as follows. We process the triangles at level i one by one. First we select a line through one of its vertices whose direction forms an angle of at least 2ε with all the lines chosen until now (in this and previous levels). This can be done by choosing a set of well-separated candidate directions beforehand, one for each region that will have to be unzipped, and setting ε accordingly. We apply then Lemma A.3 to unzip this triangle at length $t-3$ in such a way that any line through two of its non-consecutive spine regions must form an angle of at most ε with its line (and hence cannot intersect two non-consecutive spine regions of one of the previous spines); while keeping all new vertices in general position.

Except for the last iteration $i = k$, once this is done we choose t among the new spine vertices in each triangle, and we truncate them in such a way that no line intersects three of these newly created regions, using Lemma A.2. These new triangular regions are at level $i+1$ and their predecessor is the triangle at level i that contained them.

Observe that, when unzipping, each triangle at level i is replaced by t new regions at level $i+1$ ($t-3$ of which are spine regions and 3 are non-spine). Then we create t triangles at level $i+1$ by truncating the spine vertices (when $i < k$). This way, the number of regions at level i is 3 for $i = 0$ and t^i for $1 \leq i \leq k$. That is, the total number of regions is

$$n = 2 + \frac{t^{k+1} - 1}{t - 1},$$

and therefore $k \leq \log_t(n)$.

We compute now the maximal number of regions that can be intersected by a line. By construction, if a line intersects more than 2 regions of a spine, then it cannot intersect more than two regions from any other spine. Hence, except for maybe one spine where it can go through at most $t-3 = \mathcal{O}(t)$ regions, it intersects at most 2 regions from the remaining spines. We count these $\mathcal{O}(t)$ separately and continue counting as if no line could intersect more than 2 regions of any spine.

Hence, for a triangle at level i, a line can intersect at most 3 non-spine regions and 2 spine regions at level $i+1$. Thus, for each triangle, there are at most 5 regions that have it as predecessor that intersect any given line. For each level $i \geq 1$, no line can intersect more than two triangles at level i (because we used Lemma A.2). Since there are k levels ≥ 1, this amounts for at most $10 \cdot k$ regions intersected by any single line. And there are at most 3 regions at level 0. These are $\mathcal{O}(k)$ regions that can be intersected in addition to the at most $\mathcal{O}(t)$ regions in a single spine. Hence, a line crosses at most $\mathcal{O}(t+k) = \mathcal{O}\big(t + \log_t(n)\big)$ regions.

Taking $t = \left\lfloor \frac{\log(n)}{\log\left(\log(n)\right)} \right\rfloor$ gives that at most

$$\mathcal{O}\left(\frac{\log(n)}{\log\left(\log(n)\right)}\right)$$

regions are intersected by any line. Note that any large enough value of n can be attained by this construction just by taking $t = \lfloor \log(n)/\log\left(\log(n)\right) \rfloor$, $k = \lceil \log_t(n) \rceil$, and adjusting the length at which the triangles are unzipped at the last iteration.

We are ready to reverse the steps in the proof of Proposition 3.2 to construct a polytope from the resulting subdivision $\mathcal{E}_k$. Indeed, since all the operations were liftable by Lemma A.1, we can lift $\mathcal{E}_k$ to the sphere to obtain the lower hemisphere of the normal fan of a 3-polytope with n vertices. There are only three regions of the normal fan of P intersecting the upper hemisphere, from the original simplex, and intersecting cells of the lower hemisphere with linear planes is equivalent to intersecting $\mathcal{E}_k$ with lines. Hence, no plane through the origin can intersect more than $\mathcal{O}\left(\log(n)/\log\left(\log(n)\right)\right)$ regions of the normal fan of P, and hence the shadow number of P is at most

$$\mathfrak{s}(P) = \mathcal{O}\left(\frac{\log(n)}{\log\left(\log(n)\right)}\right). \qquad \square$$

Acknowledgments. We are indebted to an anonymous reviewer who suggested the example in Section 4, which allowed for improving our original $\mathcal{O}(1)$ upper bound for the unbounded shadow number from 5 to 3.

Work of the first author was partially supported by NSF grants DMS-1100373, DMS-1401224 and DMS-1701576.

Some work of the second author was done while at the National University of Singapore. He thanks them for support to work with the first author in Michigan.

A. Padrol thanks the support of the grant ANR-17-CE40-0018 of the French National Research Agency ANR (project CAPPS), as well as the program PEPS JC 2017 of the INSMI (CNRS).

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(*Reçu le 4 octobre 2018*)

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