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Boundary effects on the magnetic Hamiltonian dynamics in two dimensions

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Abstract. We study the Hamiltonian dynamics of a charged particle submitted to a pure magnetic field in a two-dimensional domain. We provide conditions on the magnetic field in a neighbourhood of the boundary to ensure the confinement of the particle. We also prove a formula for the scattering angle in the case of radial magnetic fields.

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1. Introduction

1.1. Magnetic Hamiltonian dynamics. This article is concerned with the dynamics of a charged particle in a smooth bounded domain $\Omega \subset \mathbb{R}^2$ in the presence of a non homogeneous magnetic field $\mathbf{B}$. The motion of a particle of charge e and mass m under the action of the Lorentz force can be expressed by Newton's equation

$$m\ddot{q} = e\dot{q} \times \mathbf{B}\,, \tag{1.1}$$

where $q = (q_1, q_2, q_3)^T \in \mathbb{R}^3$. To simplify our discussion, we assume that $e = 1$ and $m = 1$. The vector field $\mathbf{B}$, defined on Ω, is assumed to be smooth and to satisfy the Maxwell equation $\nabla\cdot\mathbf{B} = 0$. For our target problem in two dimensions, we suppose that $\mathbf{B}$ is perpendicular to the plane $\mathbb{R}^2$, i.e., $\mathbf{B}(q) = \big(0, 0, b(q)\big)$. This assumption forces particles lying in the $\mathbb{R}^2$ plane and whose initial velocities are in the plane to stay in this same plane for all time. Since a vector field in $\mathbb{R}^3$ can be identified with a 2-form, we write the magnetic field as $\mathbf{B} = b(q)\mathrm{d}q_1 \wedge \mathrm{d}q_2$. Then, if there is a 1-form $\mathbf{A} = A_1\mathrm{d}q_1 + A_2\mathrm{d}q_2$ such that $\mathrm{d}\mathbf{A} = \mathbf{B}$, we can write (1.1) in Hamiltonian form. Consider, for all $(q, p) \in \mathbb{R}^2 \times \mathbb{R}^2$,

$$\mathcal{H}(q,p) = \frac{\|p - \mathbf{A}(q)\|^2}{2}, \tag{1.2}$$

where $\|.\|$ denotes the Euclidean norm on $\mathbb{R}^2$.

The matrix representing the right cross product with $\mathbf{B}$ in the canonical basis is

$$M_{\mathbf{B}} = J_{\mathbf{A}}^T - J_{\mathbf{A}},$$

where $J_{\mathbf{A}}$ is the Jacobian matrix of $\mathbf{A}$. Hence Newton's equation (1.1) becomes

$$\ddot{q} = M_{\mathbf{B}}\dot{q},$$

so that

$$\frac{d}{dt}(\dot{q} + \mathbf{A}(q)) = J_{\mathbf{A}}^T \dot{q}.$$

By introducing the momentum variable $p = \dot{q} + \mathbf{A}(q)$, we see that $\mathcal{H}(q,p) = \frac{1}{2}\|\dot{q}\|^2$ is the kinetic energy of the system, and (q,p) evolves according to the Hamiltonian flow associated with $\mathcal{H}$:

$$\begin{cases} \dot{q} = \partial_p \mathcal{H}(q,p) \\ \dot{p} = -\partial_q \mathcal{H}(q,p) \end{cases}. \tag{1.3}$$

We shall always assume that $q \mapsto b(q)$ is locally Lipschitz-continuous, ensuring that the system (1.3) has a unique local maximal solution, thanks to the Cauchy–Lipschitz theorem. Then, the vector potential $\mathbf{A}$ will always be chosen to be C^1-smooth.

1.2. Two questions. From now on, we call b the magnetic field and it is identified with the 2-form

$$b(q_1,q_2)\mathrm{d}q_1 \wedge \mathrm{d}q_2 = \mathrm{d}\left(A_1\mathrm{d}q_1 + A_2\mathrm{d}q_2\right).$$

This article addresses two classical dynamical problems: confinement and scattering.

– *Confinement.* Consider a charged particle in the magnetized region Ω. A natural question is the following: "Will the particle reach the boundary in finite time?" We will provide a precise answer to this question, depending on the behaviour of the magnetic field at the boundary and on the initial conditions. Our results will improve recent results by Martins in [Mar]. In particular, we will see that, even if the magnetic field is infinite at the boundary, some trajectories can escape from Ω. This kind of (open) problems is mentioned in [CVT, Section 1.4].

– *Scattering*. Consider a charged particle outside the magnetized region Ω. Before it reaches the region Ω, the trajectory is a straight line. If it enters the region Ω, does the particle escape from it in finite time? And, if it does so, what is the deviation angle between the ingoing and outgoing directions? We will explicitly answer these questions in the case of radial magnetic fields and when Ω is a disc. In this case, the angular momentum commutes with the Hamiltonian and allows a reduction to a one degree of freedom system.

For both problems, we provide numerical illustrations of our results.

These questions have intrinsic physical motivations. Their answers allow a better understanding of the classical dynamics of charged particles in magnetic fields. The description of the classical trajectories has also many applications, for instance, at the quantum level. The quantum aspect of the trapped trajectories can be related to the essentially self-adjoint character of the magnetic Laplacian (see [CVT, NN1, NN2, RS]). It is also a key point to describe the spectrum/resonances of magnetic Laplacians. As far as the authors know, whereas the description of the magnetic dynamics has allowed to estimate the spectrum of magnetic Laplacians (see [RN, HKRN]), no result seems to exist to estimate their resonances near the real axis. Investigating the trapped trajectories is a necessary step in this direction.

In the regime of large magnetic field and small energy, a special treatment of the confinement problem can be done and takes advantage of the near-integrable structure of the Hamiltonian dynamics, either via Birkhoff normal form [RN], or KAM theorems [Cas]. On the contrary, our results here will give more explicit initial conditions and allow regimes where the guiding center motion is not necessarily meaningful.

1.3. Organization of the article. The article is organized as follows. In Section 2, we state our main results about confinement and scattering. Section 3 is devoted to the proofs.

2. Statements

2.1. Confinement problem.

2.1.1. Tubular coordinates. In order to state our results, it is convenient to introduce tubular coordinates near the boundary of Ω, following the analysis of [Mar].

We assume that the connected components of $\partial\Omega$ are C^2-smooth closed curves without self-intersections. Let $\mathcal{C}$ be a connected component of $\partial\Omega$. It can be parametrized by its arc length $\gamma : \mathbb{R}/L\mathbb{Z} \to \mathcal{C}$ where L is the length of $\mathcal{C}$.

There exists $\delta > 0$ such that

$$\psi : \begin{cases} (0,\delta) \times \mathbb{R}/L\mathbb{Z} \to \Omega_{\mathcal{C}}(\delta) \\ \quad (n,s) \qquad \mapsto \gamma(s) + nN(s) = q \end{cases} \tag{2.1}$$

is a smooth diffeomorphism. $N(s)$ denotes the inward pointing normal at $\gamma(s)$ and

$$\Omega_{\mathcal{C}}(\delta) = \{q \in \Omega : \mathrm{d}(x,\mathcal{C}) < \delta\}\,.$$

Note that

$$\mathbf{B} = b(q)\mathrm{d}q_1 \wedge \mathrm{d}q_2 = b\big(\psi(n,s)\big)\big(1 - n\kappa(s)\big)\mathrm{d}s \wedge \mathrm{d}n\,, \tag{2.2}$$

where $\kappa(s)$ is the signed curvature of $\mathcal{C}$ at $\gamma(s)$. In these coordinates, we can write

$$\mathbf{A} = A_n(n,s)\mathrm{d}n + A_s(n,s)\mathrm{d}s$$

with A_n, A_s defined on $(0,\delta) \times \mathbb{R}/L\mathbb{Z}$ such that

$$\frac{\partial A_s}{\partial n} - \frac{\partial A_n}{\partial s} =: B(n,s) = -b\big(\psi(n,s)\big)\big(1 - n\kappa(s)\big)\,. \tag{2.3}$$

Via the tubular coordinates, we can define the symplectic change of coordinates

$$\Psi : \begin{cases} (0,\delta) \times \mathbb{R}/L\mathbb{Z} \times \mathbb{R}^2 \to \Omega_{\mathcal{C}}(\delta) \times \mathbb{R}^2 \\ \qquad (n,s,p_n,p_s) \mapsto \Big(\psi(n,s), \big((\mathrm{d}\psi)^{-1}_{(n,s)}\big)^{\mathrm{T}}(p_n,p_s)\Big) = (q,p) \end{cases}, \tag{2.4}$$

where we have explicitly $p = \big(1 - n\kappa(s)\big)^{-1} p_s\gamma'(s) + p_n N(s)$.

The Hamiltonian takes the form (see Lemma A.1):

$$H(n,s,p_n,p_s) = \frac{1}{2}\big(p_n - A_n(n,s)\big)^2 + \frac{\big(p_s - A_s(n,s)\big)^2}{2\big(1 - \kappa(s)n\big)^2}\,. \tag{2.5}$$

2.1.2. General confinement theorems. We can now state our confinement results. Our first theorem provides a sufficient condition on $\mathbf{B}$ so that no trajectory can escape from Ω.

Theorem 2.1. *For every connected component $\mathcal{C}$ of $\partial\Omega$, we assume that*

$$\lim_{n\to 0}\left|\int_n^{\delta_{\mathcal{C}}}\int_0^{L_{\mathcal{C}}} B(\eta,\xi)\mathrm{d}\xi\mathrm{d}\eta\right| = +\infty\,, \tag{2.6}$$

and that there exists $M_{\mathcal{C}} \geq 0$ such that, for all $(n,s) \in (0,\delta_{\mathcal{C}}) \times \mathbb{R}/L_{\mathcal{C}}\mathbb{Z}$,

$$\left|B(n,s) - \frac{1}{L_{\mathcal{C}}}\int_0^{L_{\mathcal{C}}} B(n,\xi)\mathrm{d}\xi\right| \leq M_{\mathcal{C}}\,. \tag{2.7}$$

Then the magnetic Hamiltonian dynamics is complete (i.e., no solution of (1.3)*, starting in Ω, reaches $\partial\Omega$ in finite time).*

Of course, given a starting point $q \in \Omega$, only the components $\mathcal{C}$ that bound the connected component of q in Ω need to be taken into account. Actually, there is a more quantitative version of the previous theorem.

Theorem 2.2. *Consider a connected component $\mathcal{C}$ of $\partial\Omega$. Let*

$$K = \sup_{s\in\mathbb{R}/L\mathbb{Z}} |\kappa(s)|, \quad K' = \sup_{s\in\mathbb{R}/L\mathbb{Z}} |\kappa'(s)|.$$

We assume that, for some $\epsilon \in (0,1)$, δ satisfies $0 < \delta \leq \epsilon/K$. We assume that there exists $M \geq 0$ such that, for all $(n,s) \in (0,\delta) \times \mathbb{R}/L\mathbb{Z}$,

$$\left| B(n,s) - \frac{1}{L}\int_0^L B(n,\xi)\mathrm{d}\xi \right| \leq M\,. \tag{2.8}$$

Consider $T > 0$ and $q(t) = \psi(n(t), s(t))$ a trajectory contained in $\Omega_{\mathcal{C}}(\delta)$ for $t \in [0,T]$ with energy H_0. Let

$$f(n) = -\frac{1}{L}\int_n^\delta \int_0^L B(\eta,\xi)\mathrm{d}\xi\mathrm{d}\eta \tag{2.9}$$

and assume that

$$\liminf_{n\to 0} |f(n)| > C\big(T, q(0), \dot{q}(0)\big)\,, \tag{2.10}$$

where

$$C\big(T, q(0), \dot{q}(0)\big) = \left| \dot{s}(0)\big[1 - \kappa\big(s(0)\big)n(0)\big] + \int_{n(0)}^\delta \int_0^L B(\eta,\xi)\mathrm{d}\xi\mathrm{d}\eta \right| + \sqrt{2H_0}(1+\epsilon) + \left(M\sqrt{2H_0} + \frac{2H_0 K'\delta}{1-\epsilon} \right) T\,.$$

Let g[1] *be a continuous and strictly decreasing function such that*

$$\lim_{n\to 0} g(n) = \liminf_{n\to 0} |f(n)|, \quad g \leq |f| \quad \text{on } [0,\delta]\,.$$

Then, g takes the value $C(T, q(0), \dot{q}(0))$ and, for all $t \in [0,T)$,

$$n(t) > g^{-1}\Big(C\big(T, q(0), \dot{q}(0)\big)\Big)\,. \tag{2.11}$$

[1] Such a function g always exists.

Remark 2.1. Theorems 2.1 and 2.2 are improvements of [Mar, Theorems 1&2]. They tell us that a particle in Ω never reaches the boundary of Ω. In [Mar], it is assumed that $\partial_s B$ is integrable:

$$\sup_{s\in C} \int_0^N \left|\partial_s B(m,s)\right| \mathrm{d}m < +\infty\,, \tag{2.12}$$

and the question of removing this assumption was explicitly mentioned as important (op. cit., Section 3). Our theorems give a partially positive answer to this question, thus allowing for magnetic fields having wilder tangential behaviors.

– Theorem 2.1 generalizes [Mar, Theorem 1] by replacing the integrability assumption by (2.7). This allows in particular to consider a magnetic field (on the unit disc) of the form

$$B(n,s) = \frac{1}{n} + \sin\left(\frac{\chi(s)}{n}\right),$$

where χ is a smooth function supported in $(-\pi,\pi)$ such that $\chi'(0)\neq 0$ and $\chi(0) = 0$. For this magnetic field, it is easy to check that (2.12) is not satisfied. In fact, the C^∞ smoothness is actually not required; in order to draw Figure 1, we took, for simplicity, a small perturbation of $\chi(s) = \arcsin\big(\sin(s)\big)$.

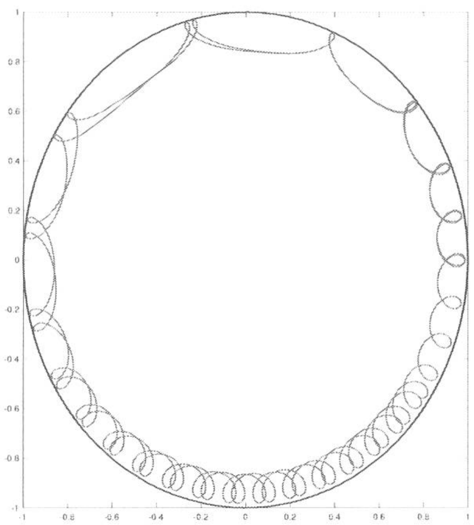

Figure 1

A trajectory obtained with a magnetic field on the unit disc that is strong near the boundary with a non-integrable tangential derivative: $B(q) = \frac{1}{1-\sqrt{q_1^2+q_2^2}} + \sin\left(\frac{\arcsin(q_2)}{1-\sqrt{q_1^2+q_2^2}}\right) + 5q_1^3 - 7q_2\,.$

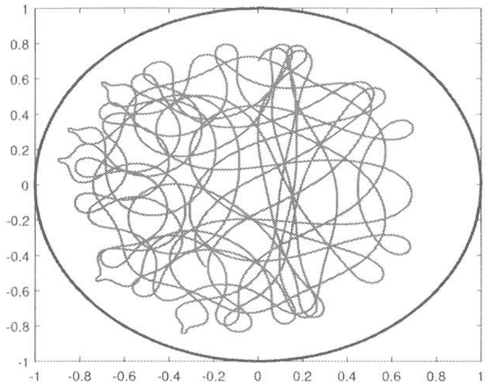

FIGURE 2
A trajectory obtained with a magnetic field on the unit disc that strongly oscillates near the boundary:

$$B(q)=\frac{\frac{1}{2}-\sin\left(\frac{1}{1-\sqrt{q_1^2+q_2^2}}\right)}{\left(1-\sqrt{q_1^2+q_2^2}\right)^2}+10q_1-2q_1^2-10q_2^2\,.$$

- An explicit lower bound for the escaping time of a magnetized region is given in [Mar, Theorem 2] in the case when

$$B(n,s)=\frac{M}{n^\alpha}+h(n,s)\,,\qquad \alpha\geq 1\,. \tag{2.13}$$

where $M\neq 0$ and h is bounded and smooth in $\Omega_C(\delta)$, and so that (2.12) holds. Theorem 2.2 implies [Mar, Theorem 2], and also provides an explicit lower bound for magnetic fields that are not in the form (2.13), see Figure 2 where the magnetic field changes sign infinitely many times.
- Note that, at the quantum level, a magnetic field (on the unit disc D) like

$$B(n,s)=\frac{2+\sin s}{n^2}\,,\qquad n=1-\sqrt{q_1^2+q_2^2}\,,\quad s\in\mathbb{R}/2\pi\mathbb{Z}\,, \tag{2.14}$$

is confining (i.e., the magnetic Laplacian acting on $\mathcal{C}_0^\infty(D)$ is essentially self-adjoint), see [CVT]. Nevertheless, this magnetic field does not satisfy our assumption (2.7) and thus we can not establish the classical confinement with our method.

2.1.3. Confinement results in the radial case. When $\Omega=D(0,1)$ and when B is radial, the dynamics is completely integrable, and hence can be entirely described by a one degree of freedom Hamiltonian; concerning the confinement problem, this of course leads to stronger results.

Proposition 2.3. *Let $q(t) = (q_1(t), q_2(t))$ be a solution to* (1.3) *starting at $t = 0$ from inside the unit disc. If the initial data $(q(0), \dot{q}(0))$ satisfies either* (H1) *or* (H2) *below:*

(H1)

$$\liminf_{r\to 1^-} \left| \frac{1}{2\pi} \int_{\|q(0)\| \leq \|q\| \leq r} B(q)\mathrm{d}q - \det\big(q(0), \dot{q}(0)\big) \right| > \|\dot{q}(0)\|, \tag{2.15}$$

(H2)

$$\liminf_{r\to 1^-} \left| \frac{1}{2\pi} \int_{\|q(0)\| \leq \|q\| \leq r} B(q)\mathrm{d}q - \det\big(q(0), \dot{q}(0)\big) \right| = \|\dot{q}(0)\|, \tag{2.16}$$

and

$$\limsup_{r\to 1^-} \frac{\left| \frac{1}{2\pi} \int_{\|q(0)\| \leq \|q\| \leq r} B(q)\mathrm{d}q - \det\big(q(0), \dot{q}(0)\big) \right| - \|\dot{q}(0)\|}{r-1} < 0, \tag{2.17}$$

then the solution exists for all $t \geq 0$, and there exists $\eta \in [0,1)$ such that

$$\forall t \geq 0, \quad \|q(t)\| < \eta. \tag{2.18}$$

One can find situations where none of the hypothesis of Proposition 2.3 hold and the trajectory can be arbitrarily close to the boundary (see Figure 3: this unusual behavior can be explained by a critical point of the radial Hamiltonian at $r = 1$, see (2.21)).

If the magnetic field is L^1-integrable near the boundary of Ω, we can prove that there exist trajectories escaping from Ω in finite time. In particular, even if the magnetic field is infinite at the boundary, the confinement is not ensured.

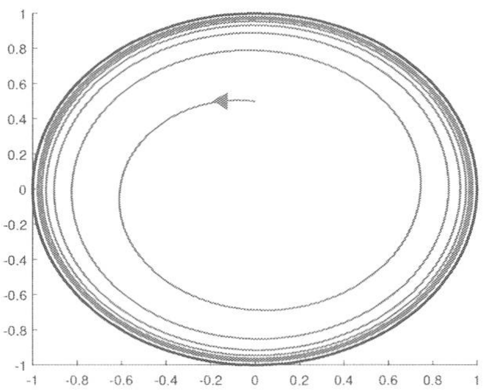

FIGURE 3
$B(r) = e^{-r} - \frac{2}{r}$

Proposition 2.4. *When*

$$\limsup_{r\to 1^-}\left|\int_{D(0,r)} B(q)\mathrm{d}q\right| < +\infty\,, \tag{2.19}$$

there exists a trajectory starting in Ω *and reaching the boundary in finite time.*

Of course, even under assumption (2.19), some trajectory may be confined, depending on initial conditions (see Figure 4 where the simulations are performed with $B(r) = \ln^2(1-r)$).

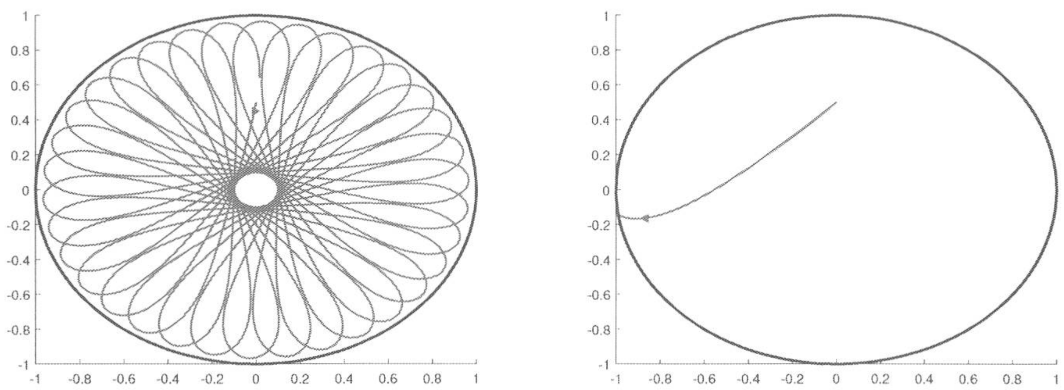

FIGURE 4
$B(r) = \ln^2(1-r)$: The particle is confined or not.

2.2. Scattering in the radial case. Let us now describe our scattering result in the radial case. We assume that $\mathbf{B}_{|\Omega}$ admits a locally Lipschitz extension in a neighbourhood of Ω.

In polar coordinates, we have

$$\mathbf{B} = B(r)r\mathrm{d}r \wedge \mathrm{d}\theta = \mathrm{d}\big(G(r)\mathrm{d}\theta\big)\,,$$

where

$$G(r) = \int_0^r \tau B(\tau)\mathrm{d}\tau\,.$$

Via the symplectic change of coordinates

$$\begin{aligned}
&\mathbb{R}_+^* \times \mathbb{R}/2\pi\mathbb{Z} \times \mathbb{R}^2 \to (D\setminus\{0\})\times\mathbb{R}^2\\
&\qquad (r,\theta,p_r,p_\theta) \mapsto \left(r\cos\theta, r\sin\theta, \cos\theta p_r - \frac{\sin\theta}{r}p_\theta, \sin\theta p_r + \frac{\cos\theta}{r}p_\theta\right)\\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad = (q,p)\,,
\end{aligned}$$

the Hamiltonian becomes

$$\tilde{H}(r,\theta,p_r,p_\theta) = \frac{p_r^2}{2} + \frac{\big(p_\theta - G(r)\big)^2}{2r^2}, \tag{2.20}$$

In particular, the angular momentum p_θ is constant along the flow and we consider the reduced one dimensional Hamiltonian on $T^*\mathbb{R}_+^*$

$$H(r,p_r) := \frac{p_r^2}{2} + V(r), \qquad V(r) := \frac{\big(p_\theta - G(r)\big)^2}{2r^2}, \tag{2.21}$$

where $V \in C^1(\mathbb{R}_+^*)$. We notice that (see, for example, Lemma A.1)

$$v_r = p_r, \qquad v_\theta = r^{-1}\big(p_\theta - G(r)\big),$$

where v_r and v_θ are the classical radial and tangential components of the velocity v.

We consider a charged particle with energy H_0 arriving into the disk with velocity v_1. In particular, $H_0 = \frac{1}{2}\|v_1\|^2$. If the particle escapes from the disc with velocity v_2 (see Figure 5), we have $\|v_2\| = \|v_1\|$, and a natural question is to compute the (scattering) angle between these two vectors. Let $\omega \in (-\pi, \pi]$ be the oriented angle between v_1 and v_2.

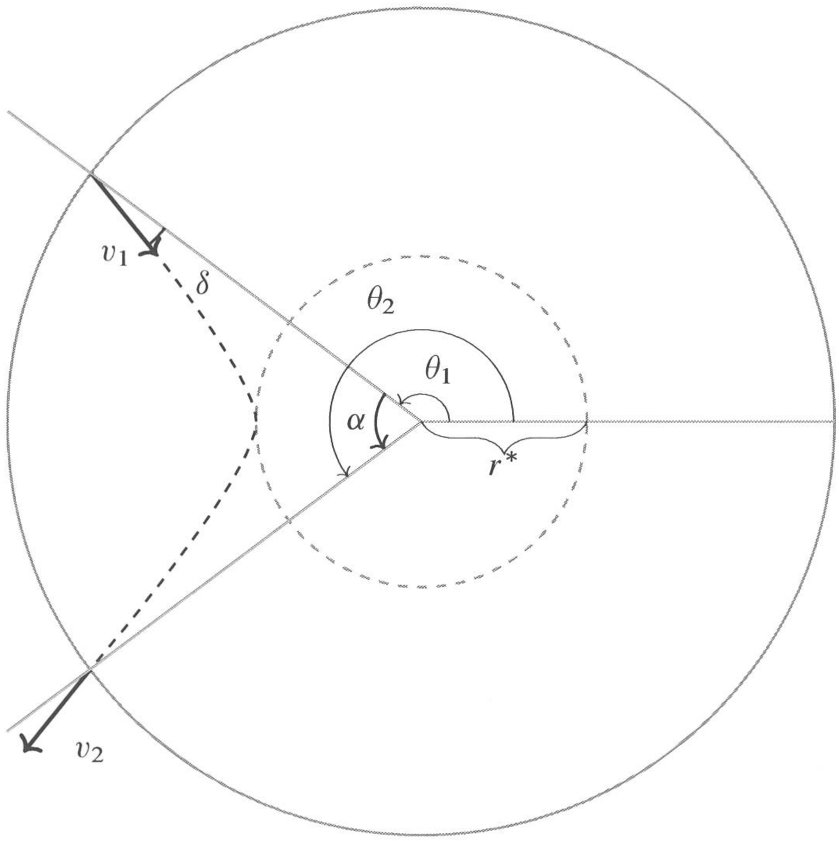

FIGURE 5
The scattering arrows

Theorem 2.5. *Consider a trajectory starting on $\partial\Omega$, with velocity $v_1 \neq 0$ and entering Ω. This means that either $v_r < 0$, or $v_r = 0$ and $\frac{B(1)}{v_\theta} < -1$. We define δ as the angle between the inward pointing normal and v_1.*

We also assume

(i) *either that the equation $V(r) = H_0$ has a solution for $r \in (0,1)$ and that the closest solution to 1, denoted by r^*, satisfies $V'(r^*) < 0$.*

(ii) *or, only when $p_\theta = 0$, that the equation $V(r) = H_0$ has no solution.*

Then the trajectory escapes from Ω in finite time with velocity v_2, and we can compute the scattering angle $\omega \mod 2\pi$:

(i) *either the trajectory does not pass through the origin and*

$$\omega = \alpha - \pi + 2\delta\,,$$

where

$$\alpha = 2\int_{r^*}^{1} \frac{p_\theta - G(r)}{r\sqrt{2H_0 r^2 - \big(p_\theta - G(r)\big)^2}}\mathrm{d}r\,, \tag{2.22}$$

(ii) *or the trajectory passes through the origin (in this case $p_\theta = 0$) and*

$$\omega = \alpha + 2\delta\,,$$

where

$$\alpha = 2\int_{0}^{1} \frac{-G(r)}{r\sqrt{2H_0 r^2 - G(r)^2}}\mathrm{d}r\,. \tag{2.23}$$

3. Proofs

3.1. Proof of Theorems 2.1 and 2.2. To reach the boundary, the particle has to be close to a connected component $\mathcal{C}$ of $\partial\Omega$. Thus, we can assume that, for all $t \in [0, T)$,

$$q(t) \in \Omega_{\mathcal{C}}(\delta)\,.$$

Modifying the vector potential corresponds to a symplectic transformation of the form $(q,p) \mapsto \big(q, p + \mathrm{d}S(q)\big)$, for some smooth function S, and hence does not modify the trajectory of the particle. Thus, we consider the function

$$\alpha(n,s) = \frac{s}{L}\int_0^L B(n,\xi)\mathrm{d}\xi - \int_0^s B(n,\xi)\mathrm{d}\xi\,.$$

Notice that $\alpha(n,\cdot)$ is L-periodic. Recalling (2.9) and letting $\mathbf{A} = \alpha(n,s)\mathrm{d}n + f(n)\mathrm{d}s$, we have $\mathbf{B} = \mathrm{d}\mathbf{A}$.

By (2.5), the corresponding Hamiltonian is

$$H(n,s,p_n,p_s) = \frac{\big(p_n - \alpha(n,s)\big)^2}{2} + \frac{\big(p_s - f(n)\big)^2}{2\big(1-\kappa(s)n\big)^2}.$$

Concerning Hamilton's equations, we have in particular

$$\dot{n} = p_n - \alpha(n,s)\,, \qquad \dot{p}_s = \tilde{B}(n,s)\dot{n} - \frac{\big(p_s - f(n)\big)^2}{\big(1-\kappa(s)n\big)^3}\kappa'(s)n\,,$$

where

$$\tilde{B}(n,s) = \frac{1}{L}\int_0^L B(n,\xi)\mathrm{d}\xi - B(n,s)\,.$$

We recall that, for all $t \in [0,T)$, $H\big(n(t),s(t),p_n(t),p_s(t)\big) = H_0$. We get

$$\begin{aligned} |\dot{n}| &\leq \sqrt{2H_0} \\ |p_s - f(n)| &\leq \sqrt{2H_0}(1+\epsilon) \\ \left|\frac{\big(p_s - f(n)\big)^2}{\big(1-\kappa(s)n\big)^3}\kappa'(s)n\right| &\leq \frac{2H_0K'\delta}{1-\epsilon}\,, \end{aligned} \tag{3.1}$$

where in the last estimates we have used the notation of Theorem 2.2 and in particular $|\kappa|n \leq K\delta \leq \epsilon$. With our assumption (2.8) on $\tilde{B}(n,s)$, we find, for all $t \in [0,T)$,

$$|p_s(t)| \leq |p_s(0)| + \left(M\sqrt{2H_0} + \frac{2H_0K'\delta}{1-\epsilon}\right)T\,,$$

and thus

$$\big|f\big(n(t)\big)\big| \leq |p_s(t)| + \big|p_s(t) - f\big(n(t)\big)\big| \leq C\big(T,q(0),\dot{q}(0)\big)\,, \tag{3.2}$$

with

$$C\big(T,q(0),\dot{q}(0)\big) = |p_s(0)| + \sqrt{2H_0}(1+\epsilon) + \left(M\sqrt{2H_0} + \frac{2H_0K'\delta}{1-\epsilon}\right)T\,.$$

If the trajectory reaches the boundary at $t = T$, then

$$\lim_{t\to T} n(t) = 0\,.$$

This, with (3.2) and (2.6), gives a contradiction. This proves Theorem 2.1.

Now, consider a function g as in Theorem 2.2. We have, for all $t \in [0,T)$,

$$g\big(n(t)\big) \leq \big|f\big(n(t)\big)\big| \leq C\big(T,q(0),\dot{q}(0)\big)\,.$$

From (2.10), we have $\lim_{n\to 0} g(n) > C\big(T,q(0),\dot{q}(0)\big)$; hence g must take the value $C\big(T,q(0),\dot{q}(0)\big)$ and the conclusion follows.

3.2. Proof of Proposition 2.3. Let us recall (2.21). The assumptions of Proposition 2.3 can be written in terms of V.

(H1) If

$$\liminf_{r\to 1^-} V(r) > H_0\,, \tag{3.3}$$

we consider $\eta = \sup\{x \in (0,1) : V(x) = H_0\} \in (0,1)$. Consider a trajectory $\big(q(t), p(t)\big)$ with $q(0) \in D(0,1)$. We can assume that $q(0) \neq 0$. Let T be the maximal time of existence in $D(0,1)$. By energy conservation, we have, for all $t \in [0,T)$,

$$V\big(r(t)\big) \leq H_0\,,$$

so that $r(t) \leq \eta$.

Note that (3.3) means

$$\liminf_{r\to 1^-} |G(r) - p_\theta| > \sqrt{2H_0}\,.$$

Using the usual complex coordinate in the plane $\mathbb{R}^2$, we can write $\dot q = \Big(\dot r + i\dot\theta r\Big) e^{i\theta}$ and thus

$$\det\big(q(t), \dot q(t)\big) = r^2(t)\dot\theta(t) = p_\theta - G\big(r(t)\big)\,.$$

Finally, we notice that $\|\dot q(0)\| = \sqrt{2H_0}$ and write

$$G(r) - p_\theta = G(r) - G\big(r(0)\big) - \big[p_\theta - G\big(r(0)\big)\big]\,,$$

which gives (2.15).

(H2) If

$$\liminf_{r\to 1^-} V(r) = H_0\,, \tag{3.4}$$

and

$$\limsup_{r\to 1^-} \frac{V(r) - H_0}{r-1} < 0\,,$$

then we must again have

$$\sup\big\{x \in (0,1) : V(x) = H_0\big\} < 1\,,$$

and we can proceed as above.

3.3. Proof of Proposition 2.4. Consider $p_\theta = 0$. Let $|V|_\infty := \sup_{r\in(0,1)}|V(r)|$. By assumption, $|V|_\infty < +\infty$.

Let $r(0) \in (0,1)$ and choose $p_r(0) > 0$ such that $p_r^2(0) = 2(|V|_\infty - V(r(0))) + v^2$, with $v > 0$. Since, for all $t \in [0,T)$,

$$\frac{p_r^2(t)}{2} + V\big(r(t)\big) = \frac{p_r^2(0)}{2} + V\big(r(0)\big),$$

we get $\dot{r}(t) = p_r(t) \geq v$ so that

$$r(t) \geq vt + r(0).$$

The escape time is at most $t = \frac{1-r(0)}{v}$.

3.4. Proof of Theorem 2.5. We distinguish between the cases $p_\theta = 0$ and $p_\theta \neq 0$.

3.4.1. Case when $p_\theta \neq 0$. In this case, $\lim_{r\to 0} V(r) = +\infty$; hence, due to energy conservation, the trajectory does not approach the origin.

(i) Assume that $p_r(0) < 0$. We have $V(1) < H_0$ and we can consider the right most turning point $r^* \in (0,1)$. By definition $V(r^*) = H_0$, and necessarily $V'(r^*) \leq 0$.

If $V'(r^*) < 0$, it is easy to check that r reaches r^* in finite time, say $t = t^*$. This time is given by

$$t^* = \int_{r^*}^{1} \frac{\mathrm{d}r}{\sqrt{2\big(H_0 - V(r)\big)}}.$$

By symmetry, the escape time is $2t^*$. Since $\dot{\theta} = \frac{p_\theta - G(r)}{r^2}$, we have

$$\begin{aligned}\theta(t^*) - \theta(0) &= \int_0^{t^*} \frac{p_\theta - G(r)}{r^2}\mathrm{d}t = \int_0^{t^*} \frac{\big(p_\theta - G(r)\big)\dot{r}}{r^2 p_r}\mathrm{d}t \\ &= \int_0^{t^*} -\frac{\big(p_\theta - G(r)\big)\dot{r}}{r^2\sqrt{2\big(H_0 - V(r)\big)}}\mathrm{d}t,\end{aligned}$$

so that

$$\theta(t^*) - \theta(0) = \int_{r^*}^{1} \frac{p_\theta - G(r)}{r^2\sqrt{2\big(H_0 - V(r)\big)}}\mathrm{d}r.$$

By symmetry, we have

$$\theta(2t^*) - \theta(0) = 2\int_{r^*}^{1} \frac{p_\theta - G(r)}{r^2\sqrt{2\big(H_0 - V(r)\big)}}\mathrm{d}r.$$

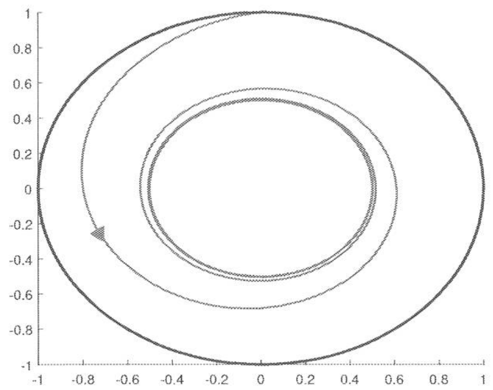

FIGURE 6
$B(r) = e^{-r} - \frac{2}{r}$

If $V'(r^*) = 0$, $(r^*, 0)$ is a critical point of the Hamiltonian and we get that r reaches r^* in infinite time (see Figure 6).

(ii) Assume that $p_r(0) = 0$ Then $V(1) = H_0$. By assumption (the trajectory enters $D(0,1)$), we have $V'(1) \geq 0$, i.e., $\big(p_\theta - G(1)\big)B(1) + \big(p_\theta - G(1)\big)^2 \leq 0$. If $V'(1) = 0$, the particle sits at a fixed point of the Hamiltonian system, and hence $r(t) \equiv 1$ is constant. If $V'(1) > 0$, it enters $D(0,1)$ and the discussion is the same as previously.

3.4.2. Case when $p_\theta = 0$. In this case, since $G(0) = 0$, $V(r) = \frac{1}{2r^2}G(r)^2$ admits a continuous extension at $r = 0$.

(i) Assume that $p_r(0) < 0$. We have $V(1) < H_0$. The existence of r^* such that $V(r^*) = H_0$ is not ensured. If $V(r) < H_0$ on $[0,1]$, the particle reaches $r = 0$ in finite time $t = t^*$:

$$t^* = \int_0^1 \frac{\mathrm{d}r}{\sqrt{2\big(H_0 - V(r)\big)}} .$$

We get, by symmetry,

$$\theta(2t^*) - \theta(0) = 2\int_0^1 \frac{-G(r)}{r^2\sqrt{2\big(H_0 - V(r)\big)}}\mathrm{d}r + \pi .$$

If there exists $r^* \in (0,1)$ such that $V(r^*) = H_0$, the trajectory does not reach the origin and the discussion is the same as in the case $p_\theta \neq 0$.

(ii) Assume that $p_r(0) = 0$. The discussion is the same as when $p_\theta \neq 0$.

3.4.3. Scattering angle. We can now end the proof of Theorem 2.5. In terms of complex numbers, we can write

$$v_1 = \big(v_r(0) + i v_\theta(0)\big)e^{i\theta_1}\,, \qquad v_2 = \big(-v_r(0) + i v_\theta(0)\big)e^{i\theta_2}\,.$$

The scattering angle is

$$\theta_2 - \theta_1 + \operatorname{Arg}\left(\frac{-v_r(0) + i v_\theta(0)}{v_r(0) + i v_\theta(0)}\right).$$

Since δ is the argument of $-v_r(0) + i v_\theta(0)$, the scattering angle is

$$\theta_2 - \theta_1 - \pi + 2\delta\,.$$

A. Tubular coordinates

Lemma A.1. *We write* $\mathbf{A} = A_1 \mathrm{d}q_1 + A_2 \mathrm{d}q_2$. *With* (2.1), *we have*

$$\mathbf{A} = A_n \mathrm{d}n + A_s \mathrm{d}s\,, \qquad \tilde{A} = (A_n, A_s)^{\mathrm{T}} = (\mathrm{d}\psi)^{\mathrm{T}}(A_1, A_2)^{\mathrm{T}}\,.$$

We have

$$\begin{aligned} H(n,s,p_n,p_s) &= \mathcal{H} \circ \Psi(n,s,p_n,p_s) \\ &= \frac{\big(p_n - A_n(n,s)\big)^2}{2} + \frac{\big(p_s - A_s(n,s)\big)^2}{2\big(1-\kappa(s)n\big)^2}\,. \end{aligned} \tag{A.1}$$

Moreover, $v_n = p_n - A_n(n,s)$ *and* $v_s = \big(1 - n\kappa(s)\big)^{-1}(p_s - A_s)$ *are the normal and tangential component of* v.

Proof. We write

$$\begin{aligned} 2H(q,p) = \|p - A\|^2 &= \big\|(\mathrm{d}\psi^{-1})^{\mathrm{T}}(\tilde{p} - \tilde{A})\big\|^2 \\ &= \big\langle (\mathrm{d}\psi^{-1})(\mathrm{d}\psi^{-1})^{\mathrm{T}}(\tilde{p} - \tilde{A}), \tilde{p} - \tilde{A}\big\rangle, \end{aligned}$$

with $\tilde{p} = (p_n, p_s)^{\mathrm{T}}$. Note that

$$(\mathrm{d}\psi^{-1})^{\mathrm{T}} = \big[N(s)\ , (1 - n\kappa(s))\gamma'(s)\big]. \tag{A.2}$$

We get

$$(\mathrm{d}\psi^{-1})(\mathrm{d}\psi^{-1})^{\mathrm{T}} = \begin{pmatrix} 1 & 0 \\ 0 & \big(1 - n\kappa(s)\big)^{-2} \end{pmatrix},$$

which gives (A.1). Concerning the velocity v, since $q = \gamma(s) + nN(s)$ and thanks to the Frenet–Serret formula $N'(s) = -\kappa(s)\gamma'(s)$, we have

$$v = \dot{s}\big(1 - n\kappa(s)\big)\gamma'(s) + \dot{n}N(s) =: v_s\gamma' + v_n N\,,$$

and thus we get the result from (A.1) by using the Hamilton equations $\dot{s} = \big(1 - \kappa(s)n\big)^{-2}(p_s - A_s)$ and $\dot{n} = p_n - A_n$. □

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